Lectures on classical optics

Part I

Contents

Maxwell's Equations

E = electric field
H = magnetic field

D = dielectric displacement
B = magnetic induction

P = polarisation
M = magnetisation

$D = \varepsilon_0 E + P$

$B = \mu_0 H + M$

$S = E \times H$ Poynting vector

$u = \frac{1}{2}\left(DE + BH\right)$ electromagnetic energy density

$\rho = \rho_w + \rho_M$ electric charge density $\rho_P = -\nabla P$

$\eta = \eta_w + \eta_M$ magnetic charge density $\eta_M = -\nabla M$

$j = j_w + j_P + j_M$ electric current density $j_P = \dot{P}, \quad j_M = \frac{1}{\mu_0} \nabla \times M$

$\ell = \ell_w + \ell_P + \ell_M$ magnetic current density $\ell_M = \dot{M}, \quad \ell_P = -\frac{1}{\varepsilon_0} \nabla \times P$

$$\nabla D = \rho_w$$

$$\nabla B = \eta_w$$

$$\nabla \times E = -\dot{B} - \ell_w$$

$$\nabla \times H = \dot{D} + j_w$$

$$\nabla E = \frac{1}{\varepsilon_0}\rho$$

$$\nabla H = \frac{1}{\mu_0}\eta$$

$$\nabla \times D = -\frac{1}{c^2}\dot{H} - \varepsilon_0 \ell$$

$$\nabla \times B = \frac{1}{c^2}\dot{E} + \mu_0 j$$

$$\nabla S + H\dot{B} + E\dot{D} + Ej_w + H\ell_w = 0$$

Experiment: no true magnetic charges or currents

$$\eta_w = 0 \,, \quad \ell_w = 0$$

Maxwell ‘s equations are extremely general. They describe classical Electromagnetic phenomena in any frequency range in any materials

In the next few pages we will sucessively specialize to the domain of linear optics of monochromatic light fields in isotropic dielectric media

Verify energy conservation:

$$\nabla S = \nabla(E \times H) = H(\nabla \times E) - E(\nabla \times H)$$

$$= H(-\dot{B} - \ell_w) - E(\dot{D} + j_w) = -H\dot{B} - H\ell_w - E\dot{D} - E j_w$$

Monochromatic radiation fields in insulators

$\rho_w = 0 \;, \;\; j_w = 0$

$$E(r,t) = \frac{1}{\sqrt{2}} \left(E(r)\, e^{-i\omega t} + E(r)^{*} e^{i\omega t} \right)$$

$$B(r,t) = \frac{1}{\sqrt{2}} \left(B(r)\, e^{-i\omega t} + B(r)^{*} e^{i\omega t} \right)$$

analog for D, H

Maxwell

$$\nabla \times E = i\omega B$$
$$\nabla \times H = -i\omega D$$
$$\nabla D = \nabla B = 0$$

Time averaged quantities

$$A(r,t) = \frac{1}{\sqrt{2}}\left(A(r)\, e^{-i\omega t} + A(r)^{*} e^{i\omega t}\right), \quad T \equiv 2\pi/\omega : \; \left\langle A(r,t)\right\rangle \equiv \frac{1}{T}\int_T dt \; A(r,t)$$

$$\Rightarrow \quad \left\langle A(r,t)\otimes B(r,t)\right\rangle = \mathrm{Re}\left(A(r)\otimes B(r)^{*}\right)$$

Examples:

1. time-averaged intensity: $\left\langle E(r,t)^2\right\rangle = \mathrm{Re}\left(E(r)\, E(r)^{*}\right)$

2. time-averaged Poynting vector:

$$S(r,t) = E(r,t) \times H(r,t) = \frac{1}{2}\left(E(r)\, e^{-i\omega t} + E(r)^{*} e^{i\omega t}\right)\times\left(H(r)\, e^{-i\omega t} + H(r)^{*} e^{i\omega t}\right)$$

$$= \frac{1}{2}\left(E(r) \times H(r)^{*} + E(r)^{*} \times H(r)\right) + \frac{1}{2}\left(E(r) \times H(r)\, e^{-2i\omega t} + E(r)^{*} \times H(r)^{*} \; e^{2i\omega t}\right)$$

$$\Rightarrow \quad \left\langle S(r,t)\right\rangle = \mathrm{Re}\left(E(r)\times H(r)^{*}\right)$$

From non-linear to linear optics

P, M induced by E, H

$$P = \varepsilon_0 \chi\, E \qquad M = \mu_0 \xi\, H \quad \Rightarrow \quad D = \varepsilon\, E\,, \ \varepsilon = \varepsilon_0\,(1+\chi) \qquad B = \mu\, H\,, \ \mu = \mu_0\,(1+\xi)$$

$\chi = \chi(E,\omega)$, $\xi = \xi(H,\omega)$ dimensionless complex tensors

(electric und magnetic susceptibility)

$$\xi(H,\omega) = \xi^{(1)}(\omega)$$

$$\chi(E,\omega) = \chi^{(1)}(\omega) + \sum_{\nu_1 \in \{1,2,3\}} \chi^{(2)}_{\nu_1}(\omega)\, E_{\nu_1} + \sum_{\nu_1,\nu_2 \in \{1,2,3\}} \chi^{(3)}_{\nu_1,\nu_2}(\omega)\, E_{\nu_1} E_{\nu_2} + \dots$$

linear Optics

non-linear optics: frequency-conversion

non-linear optics: Four wave mixing, Kerr-effect

$$\vec{E}^* \vec{P} = \varepsilon_0 \sum_{\nu\mu} \sum_{\nu_1 \in \{1,2,3\}} \left[\chi^{(2)}_{\nu_1}(\omega)\right]_{\nu\mu} E^*_\nu E_{\nu_1} E_\mu \qquad \vec{E}^* \vec{P} = \varepsilon_0 \sum_{\nu\mu} \sum_{\nu_1,\nu_2 \in \{1,2,3\}} \left[\chi^{(3)}_{\nu_1,\nu_2}(\omega)\right]_{\nu\mu} E^*_\nu E_{\nu_1} E_{\nu_2} E_\mu$$

$\xi^{(1)}(\omega)$, $\chi^{(1)}(\omega)$, $\chi^{(2)}_{\nu_1}(\omega)$, $\chi^{(3)}_{\nu_1,\nu_2}(\omega)$, ... complex matrices

Monochromatic radiation fields in linear isotropic media

$\varepsilon = \varepsilon(\omega,r),\ \mu = \mu(\omega,r)$ complex scalars with $\left|\nabla\mu / \mu\right| << \left|k\right|,\ k^2 \equiv \omega^2\ \mu\varepsilon$

assume that μ does not change on the length scale of the optical wavelength

a. $\nabla \times E = i\omega\ B$

b. $\nabla \times \frac{1}{\mu} B = -i\omega\ \varepsilon\ E$

$$\Delta E + k^2 E + \frac{\nabla\mu}{\mu} \times (\nabla \times E) = 0$$

Herman Helmholtz 1821-1894

$$-\Delta E = \nabla\left(\nabla E\right) - \Delta E = \nabla \times \left(\nabla \times E\right) \overset{(a.)}{=} i\omega\ \nabla \times B$$

$$\underset{(b.)}{=} \omega^2 \varepsilon\mu\ E - i\omega\mu\left(\nabla \tfrac{1}{\mu}\right) \times B = k^2 E - \mu\left(\nabla \tfrac{1}{\mu}\right) \times \left(\nabla \times E\right)$$

$$= k^2 E + \left(\frac{\nabla\mu}{\mu}\right) \times \left(\nabla \times E\right)$$

$$\Rightarrow \Delta E + k^2 E \approx 0$$

Helmholtz equation

Linear isotropic dielectrica

$\varepsilon = \varepsilon(\omega)$, $\mu = \mu_0$ complex scalars

$$\Rightarrow \quad \begin{aligned} \nabla \times E &= i\omega B \\ \nabla \times \tfrac{1}{\mu_0} B &= -i\omega\, \varepsilon\, E \end{aligned} \quad \Rightarrow \quad \begin{aligned} \Delta E + k^2 E &= 0 \\ k^2 &\equiv \mu_0 \varepsilon\, \omega^2 \end{aligned}$$

We define three observable quantities characterizing the propagation of light in linear isotropic dielectrica

Definitions

refractive index:

$$n \equiv c\sqrt{\mu_0 \varepsilon} = n_{Re} + i n_{Im} = \sqrt{1+\chi} \approx 1 + \frac{1}{2}\chi$$

and hence $$n^2 = c^2 \mu_0 \varepsilon = c^2 \frac{k^2}{\omega^2}$$

phase velocity:

$$\tilde{c} \equiv \frac{c}{n_{Re}}$$

absorption coefficient:

$$\kappa \equiv 2\,\frac{\omega}{c}\, n_{Im}$$

Example: travelling wave $\vec{E} = \vec{E}_0\, e^{i\vec{k}\vec{r}}$

$$\vec{\nabla}\vec{E} = 0 \quad \Rightarrow \quad \vec{k}\,\vec{E} = 0$$

$$\vec{\nabla} \times \vec{E} = i\omega\,\vec{B} \quad \Rightarrow \quad \vec{B} = \frac{1}{\omega}\vec{k} \times \vec{E}, \quad \vec{k}\,\vec{B} = 0$$

$$\Delta E + k^2\,E = 0, k^2 = \mu_0 \varepsilon \omega^2 \quad \Rightarrow \quad \vec{k}^2 = k^2 = \mu_0 \varepsilon\,\omega^2 = \frac{\omega^2}{c^2} n^2$$

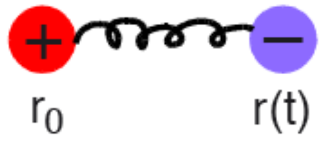
Absorption along direction of propagation

$\vec{k} = k\,\hat{u}$, $\hat{u} \equiv$ real unit vector, $k =$ complex scalar $\Rightarrow$ $k = \frac{\omega}{c} n$

$$\vec{k}\vec{r} = k\,\hat{u}\vec{r} = \frac{\omega}{c} n\,\hat{u}\vec{r} = \frac{\omega}{c}\left(n_{Re} + i n_{Im}\right)\hat{u}\vec{r}, \qquad \tilde{c} \equiv \frac{c}{n_{Re}}, \quad \kappa \equiv 2\,\frac{\omega}{c}\,n_{Im}$$

$$\vec{E} = \vec{E}_0\,\exp\left(i\frac{\omega}{c}\,n_{Re}\,\hat{u}\vec{r}\right)\exp\left(-\frac{\omega}{c}\,n_{Im}\,\hat{u}\vec{r}\right) = \vec{E}_0\,\exp\left(i\frac{\omega}{\tilde{c}}\,r\right)\exp\left(-\frac{\kappa}{2}\,r\right)$$

Lorentz-model for polarization



Hendrik A. Lorentz
1853 - 1928

$$m\left(\ddot{q}+\gamma_0\dot{q}+\omega_0^2 q\right) = e\,E(t) \qquad q(t) = r(t) - r_0 \ll \lambda$$

$$E(t) = \frac{1}{\sqrt{2}}\left(E_0\, e^{-i\omega t} + E_0^*\, e^{i\omega t}\right),\; q(t) = \frac{1}{\sqrt{2}}\left(q_0\, e^{-i\omega t} + q_0^*\, e^{i\omega t}\right)$$

$$\Rightarrow \text{ induced dipole moment: } eq_0 = \frac{e^2}{\sqrt{2}\, m}\,\frac{1}{\left(\omega_0^2 - \omega^2 - i\gamma_0\omega\right)}\,E_0 \overset{!}{=} \tilde{\alpha}\,E_0$$

Atomic polarizability

unit = $\varepsilon_0 \cdot$ volume

Multiple resonances:

$$\tilde{\alpha} = \frac{e^2}{\sqrt{2}\, m}\sum_n \frac{f_n}{\left(\omega_n^2 - \omega^2 - i\gamma_n\omega\right)}\;,\quad Z = \sum_n f_n \text{ number of nuclear charges}$$

Oscillator strengths f_n result from quantum mechanical consideration

Macroscopic susceptibility: $\tilde{\chi} = \frac{1}{\varepsilon_0} \bar{n}\, \tilde{\alpha}$, $\bar{n}$ = particle density

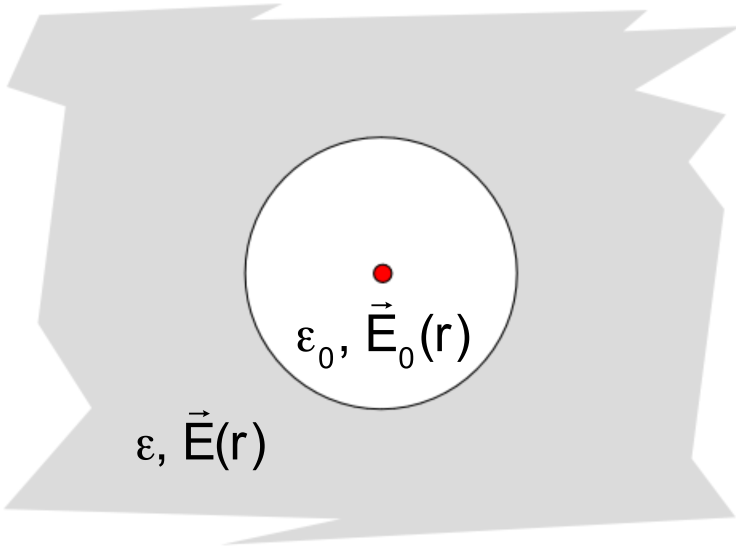
Macroscopic polarization: $\vec{P}(r) = \bar{n}(r) \; e\, \vec{q}_0 = \bar{n}(r)\, \tilde{\alpha}\, E_0(r) = \varepsilon_0 \; \tilde{\chi}\, \vec{E}_0(r)$

particle density — Dipole moment per particle

For small particle densites $\bar{n}(r)$, $\vec{E}_0(r)$ may be viewed as the macroscopic electric field at the position of the atom. In this case $\tilde{\chi} = \chi$ is the electric Susceptibility of Maxwell ‘s equations.

Case of large particle densites

Electrostatics: $\vec{E}_0(r) = \dfrac{3\varepsilon}{2\varepsilon+\varepsilon_0}\vec{E}(r)$



$$\Rightarrow \quad \vec{P} = \varepsilon_0\ \tilde{\chi}\ \vec{E}_0 = \frac{3\varepsilon_0\varepsilon}{2\varepsilon+\varepsilon_0}\tilde{\chi}\ \vec{E}$$

$$\vec{P} = \varepsilon_0\ \chi\ \vec{E} = \left(\varepsilon-\varepsilon_0\right)\vec{E}$$

together $\quad \tilde{\chi} = \dfrac{\left(2\varepsilon+\varepsilon_0\right)\left(\varepsilon-\varepsilon_0\right)}{3\varepsilon_0\varepsilon} = \dfrac{(2\chi+3)\ \chi}{3(\chi+1)}$ $\quad$ Clausius Mossotti formula

For $\chi \ll 1$ approximate $\quad \tilde{\chi} \approx \dfrac{\chi}{1+\chi}, \chi \approx \dfrac{\tilde{\chi}}{1-\tilde{\chi}}$ $\quad$ and hence

for a single resonance

$$\chi \approx \frac{\tilde{\chi}}{1-\tilde{\chi}} = \frac{\beta}{\omega_0^2-\beta-\omega^2-i\gamma_0\omega}, \quad \beta \equiv \frac{e^2\overline{n}}{\sqrt{2}m\varepsilon_0}$$

large density yields decrease of resonance frequency

Physical interpretation

Binding of electrons to the nucleus is decreased by dipole-dipole interactions

Index of refraction

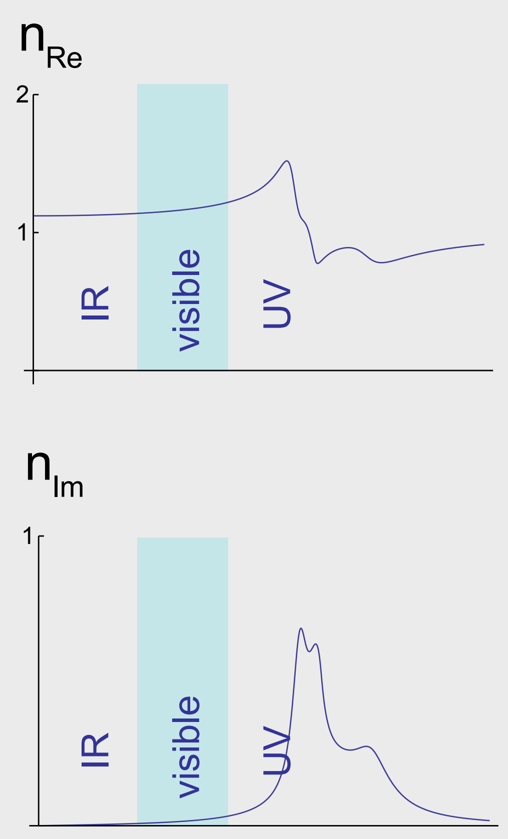
$$n_{Re} - 1 \approx \tfrac{1}{2}\mathrm{Re}(\chi) = \frac{\tfrac{1}{2}\beta\left(\omega_0^2 - \beta - \omega^2\right)}{\left(\omega_0^2 - \beta - \omega^2\right)^2 + \gamma^2\omega^2}$$

$$n_{Im} \approx \tfrac{1}{2}\mathrm{Im}(\chi) = \frac{\tfrac{1}{2}\beta\gamma\omega}{\left(\omega_0^2 - \beta - \omega^2\right)^2 + \gamma^2\omega^2}$$

Absorption coefficient

$$\kappa = 2\, n_{Im}\frac{\omega}{c} = \frac{\omega}{c}\frac{\beta\gamma\omega}{\left(\omega_0^2 - \beta - \omega^2\right)^2 + \gamma^2\omega^2}$$

Example with several resonances in the UV



Classical Description of Light Forces

Lorentz Model:

r_0 $r(t)$

dipole moment: $P(t) = e\,(r(t) - r_0)$

Oscillating electron at position $r(t) = r_0 + e^{-1}\,P(t)$ and proton at position r_0 experience time-averaged Coulomb-force:

$$F_C = \left\langle e\,E(r(t),t) - e\,E(r_0,t) \right\rangle \approx \left\langle (P\nabla)E \right\rangle$$

$$\langle A \rangle \equiv \frac{1}{T}\int_0^T A(t)\,dt$$

$$\vec{E}(\vec{r} + \delta\vec{r}\,) = E(\vec{r}\,) + (\delta\vec{r}\,\vec{\nabla})\vec{E} \;+\; O(\delta r^2)$$

Induced dipole $P(t) = (\,r(t) - r_0\,)\,e$ yields time-averaged Lorentz-force:

∇ acts on E only

$$F_L = \left\langle \frac{\partial}{\partial t} P \times B \right\rangle = -\left\langle P \times \frac{\partial}{\partial t} B \right\rangle = \left\langle P \times (\nabla \times E) \right\rangle = \left\langle \nabla(PE) - (P\nabla)E \right\rangle$$

integration by parts, bondary terms vanish because P,B periodic in t

a x (b x c) = b(ac) - c(ab)

Total Force: $F = F_C + F_L = \left\langle \nabla(PE) \right\rangle$

$\Rightarrow$ Same expression as known for static dipoles in static electric fields

Consider Harmonic Field: $E(r,t) = \frac{1}{\sqrt{2}} \left(E(r)\, e^{i\omega t} + E(r)^*\, e^{-i\omega t} \right)$

$P(t) = \frac{1}{\sqrt{2}} \left(P\, e^{i\omega t} + P^*\, e^{-i\omega t} \right)$ $\qquad$ $E(r), P$ = complex vectors

$$\Rightarrow \qquad F = \frac{1}{2} \Big(\nabla(P E^*) + \nabla(P^* E) \Big)$$

Express complex polarization P by means of polarizability tensor $\alpha(E)$: $P = \varepsilon_0\, \alpha(E)\, E$

$\alpha(E)$ = complex 3x3 Matrix

Choose basis such that $\alpha(E)$ diagonal, with $\alpha_{\nu\nu} \equiv \alpha_\nu + i\,\beta_\nu$, $E_\nu \equiv \sqrt{\frac{I_\nu}{\varepsilon_0}}\; e^{-i\psi_\nu}$

$$F = \frac{1}{2} \sum_{\nu=1}^{3} \alpha_\nu \nabla I_\nu - \sum_{\nu=1}^{3} \beta_\nu\, I_\nu\, \nabla \psi_\nu \qquad (*)$$

dipole force $\qquad$ radiation pressure

Ü $\quad$ Verify expression (*)

Exc: Detailed calculation of force

$$F = \frac{1}{2}\left(\nabla(PE^*) + \nabla(P^*E)\right) = \frac{1}{2}\sum_{n=1}^{3} P_n\nabla E_n^* + P_n^*\nabla E_n = \frac{\varepsilon_0}{2}\sum_{n=1}^{3}\sum_{m=1}^{3} \alpha_{nm}E_m\nabla E_n^* + \alpha_{nm}^*E_m^* \nabla E_n$$

$$= \frac{\varepsilon_0}{2}\sum_{n=1}^{3} \alpha_{nn}E_n\nabla E_n^* + \alpha_{nn}^*E_n^* \nabla E_n = \frac{\varepsilon_0}{2}\sum_{n=1}^{3} \left(\alpha_n + i\beta_n\right)E_n\nabla E_n^* + \left(\alpha_n - i\beta_n\right)E_n^* \nabla E_n$$

$$= \frac{\varepsilon_0}{2}\sum_{n=1}^{3} \alpha_n \left(E_n\nabla E_n^* + E_n^*\nabla E_n\right) + i\,\beta_n\left(E_n\nabla E_n^* - E_n^* \nabla E_n\right) = \frac{\varepsilon_0}{2}\sum_{n=1}^{3} \alpha_n\nabla\left(E_nE_n^*\right) + i\,\beta_n\left(E_n\nabla E_n^* - E_n^* \nabla E_n\right)$$

$$E_n = \sqrt{\frac{I_n}{\varepsilon_0}}e^{-i\psi_n} \quad \Rightarrow \quad \begin{aligned} E_n\nabla E_n^* + E_n^*\nabla E_n &= \frac{1}{\varepsilon_0}\nabla I_n \\ E_n\nabla E_n^* - E_n^*\nabla E_n &= 2i\,\nabla\psi_n\frac{I_n}{\varepsilon_0} \end{aligned}$$

$$\Rightarrow \quad F = \frac{1}{2}\sum_{n=1}^{3}\alpha_n\nabla I_n - \sum_{n=1}^{3}\beta_n\, I_n\nabla\psi_n$$

Example: linear polarization along z-axis $\quad E = \hat{z}\sqrt{\frac{I(x,y,z)}{\varepsilon_0}}\; e^{-i\psi(x,y,z)}$

$I(x,y,z)$ energy density, $\psi(x,y,z)$ local phase

$$\Rightarrow \quad F = \tfrac{1}{2}\,\alpha_z \nabla I - \beta_z\, I\, \nabla\psi$$

Ü Consider the Poynting – vector $S(r,t) = E(r,t)\times H(r,t)$

$\Rightarrow \langle S(r,t)\rangle = \mathrm{Re}\left[S(r)\right],\; S(r) \equiv E(r)\times H^{*}(r)$ Verify expression !

$$\nabla\times E = -i\omega B,\; \nabla\times B = i\frac{\omega}{c^2}E \;\Rightarrow\; S(r) = \frac{1}{\mu_0}E\times B^{*} = \frac{-i}{\mu_0\omega}E\times\left(\nabla\times E^{*}\right)$$

Exc: Assume spatially constant polarization

$$E(r) = \hat{e}\, f(r),\; \hat{e}\hat{e}^{*} = 1,\; \nabla E = 0 \;\Rightarrow\; S(r) = \frac{-i}{\mu_0\omega} f\,\hat{e}\times\left(\nabla f^{*}\times\hat{e}\right) = \frac{-i}{\mu_0\omega} f\,\nabla f^{*}$$

use $f(r) = \sqrt{I(r)/\varepsilon_0}\; e^{-i\psi(r)}$ and $\nabla E = 0 \;\Rightarrow\; S = \frac{c^2}{\omega}\left(I\nabla\psi - \frac{i}{2}\nabla I\right)$

$F_{Rad} \sim \beta S$ and hence with $\nabla S = 0$ one gets

Therefore $\nabla F_{Rad} \sim \nabla(\beta S) = S\nabla\beta + \beta\nabla S = S\nabla\beta$

$\nabla F_{Rad} = 0$ if $\nabla\beta = 0$ $\Rightarrow$ trapping with radiation pressure requires $\nabla\beta \neq 0$

Ü Discuss the following two special cases

Plane travelling wave $E = \sqrt{\frac{I_0}{\varepsilon_0}}\ \hat{z}\, e^{ikx}\ ,\ \ k = \frac{\omega}{c}$ reell

Plane standing wave wave $E = \sqrt{\frac{I_0}{\varepsilon_0}}\ \hat{z}\cos(kx)\ ,\ \ k = \frac{\omega}{c}$ reell

How do these simple cases justify the terms „radiation pressure“ and“ dipole force“?

Exc: special cases

Plane travelling wave $(**)\ \Rightarrow\ \vec{F} = -\beta_z\, I\vec{\nabla}\psi\ = \frac{\omega}{c}\beta_z\, I\,\hat{x},\qquad \frac{\omega}{c}\beta_z = \frac{\omega}{c}\mathrm{Im}(\chi)\ \approx\ 2\frac{\omega}{c}n_{Im}\ =\ \kappa$

Plane standing wave $(**)\ \Rightarrow\ \vec{F} = \tfrac{1}{2}\alpha_z\ \vec{\nabla}I = \tfrac{1}{2}\alpha_z I_0\ \frac{\omega}{c}\hat{x}\sin(2kx),\qquad \tfrac{1}{2}\alpha_z\ = \tfrac{1}{2}\chi\ \approx\ n_{Re} - 1$

Eikonal equation

Consider scalar Helmholtz equation $\Delta\psi + k^2\psi = 0$ with real $k = k(r) = \frac{\omega}{c} n(r)$

Write $\psi = A\, e^{iS}$ with real functions $A(r), S(r)$

$$\Rightarrow \qquad 0 = \left(k^2 - \left(\nabla S\right)^2\right) A + \Delta A$$

$$0 = 2\nabla A \nabla S + A \Delta S$$

$$\nabla\psi = \left(\nabla A + i A \nabla S\right) e^{iS}$$

$$\Delta\psi = \left(\Delta A + i \nabla A \nabla S + i A \Delta S\right) e^{iS} + i\nabla S\left(\nabla A + i A \nabla S\right) e^{iS}$$

$$0 = \Delta\psi + k^2\psi = \left(\Delta A - \left(\nabla S\right)^2 A + k^2 A\right) e^{iS} + \left(2\nabla A \nabla S + A \Delta S\right) i e^{iS}$$

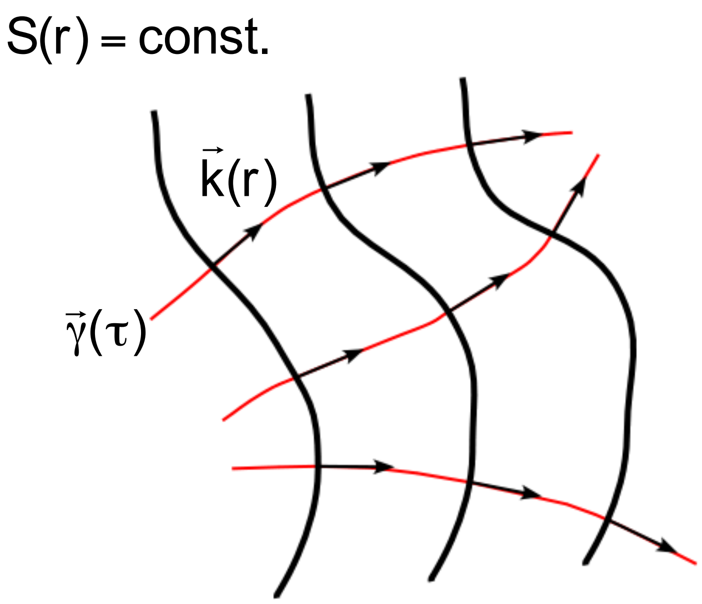
Upon the assumption that A varies little over an optical wavelength apply Eikonal-approximation $\Delta A << k^2 A$ to obtain

Eikonal equation $\qquad k^2 = \left(\nabla S\right)^2$

Light rays

Light rays are integral curves of the vector field $\vec{k}(r) \equiv \vec{\nabla} S(r)$, i.e.,

solutions to the equation $$\frac{d}{d\tau}\vec{\gamma}(\tau) = \vec{\nabla} S(\vec{\gamma}(\tau))$$



Fermat's principle

Pierre de Fermat 1607-1657

Any light ray between points P and Q yields

an extremum of the integral $\int_\gamma n(r)dr$

Consider a path $\vec{\gamma} + z\,\vec{\eta}$ where $\vec{\gamma}$ is an admissible light ray and

$\vec{\eta}(\tau_1) = \vec{\eta}(\tau_2) = 0,\ \vec{\gamma}(\tau_1) = P,\ \vec{\gamma}(\tau_2) = Q$

Eikonal equation $n = \frac{c}{\omega}k = \frac{c}{\omega}|\nabla S|$

$$\int_{\gamma+z\eta} n(r)dr = \int_{\tau_1}^{\tau_2} d\tau\ n(\vec{\gamma}+z\vec{\eta})\left|\dot{\vec{\gamma}} + z\,\dot{\vec{\eta}}\right| = \frac{c}{\omega}\int_{\tau_1}^{\tau_2} d\tau\left|\vec{\nabla}S(\vec{\gamma}+z\vec{\eta})\right|\left|\dot{\vec{\gamma}} + z\,\dot{\vec{\eta}}\right|$$

$$\frac{d}{dz}\left(\int_{\gamma+z\eta} n(r)dr\right)_{z=0} = \frac{c}{\omega}\int_{\tau_1}^{\tau_2} d\tau\,\frac{d}{dz}\left(\left|\vec{\nabla}S(\vec{\gamma}+z\vec{\eta})\right|\left|\dot{\vec{\gamma}} + z\,\dot{\vec{\eta}}\right|\right)_{z=0}$$

$$\frac{d}{dz}\left(\left|\vec{\nabla}S(\vec{\gamma}+z\vec{\eta})\right|\left|\dot{\vec{\gamma}} + z\ \dot{\vec{\eta}}\right|\right)_{z=0} = \frac{1}{\left|\vec{\nabla}S(\vec{\gamma})\right|}\vec{\nabla}S(\vec{\gamma})\frac{d}{dz}\left(\vec{\nabla}S(\vec{\gamma}+z\vec{\eta})\right)_{z=0}\left|\dot{\vec{\gamma}}\right| + \left|\vec{\nabla}S(\vec{\gamma})\right|\frac{1}{\left|\dot{\vec{\gamma}}\right|}\ \dot{\vec{\gamma}}\ \dot{\vec{\eta}}$$

$$= \dot{\vec{\gamma}}\ \frac{d}{dz}\left(\vec{\nabla}S(\vec{\gamma}+z\vec{\eta})\right)_{z=0} + \vec{\nabla}S(\vec{\gamma})\ \dot{\vec{\eta}}$$

For the second line we use the fact that γ is an admissible light ray:

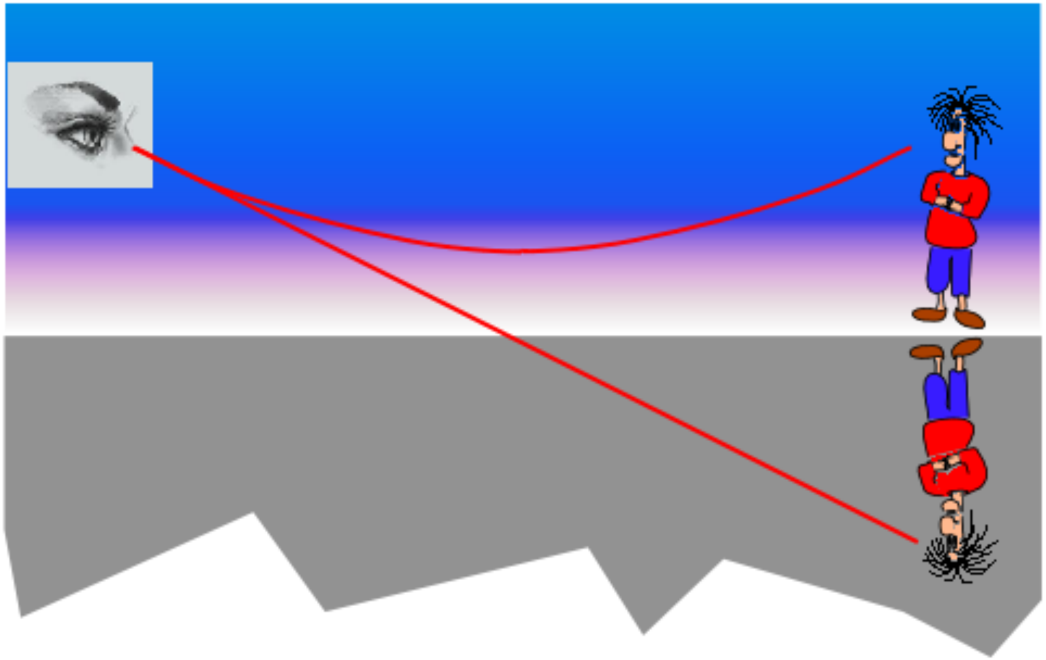
$$\frac{d}{d\tau}\vec{\gamma}(\tau) = \vec{\nabla}S(\vec{\gamma}(\tau))$$

$$\frac{\omega}{c}\frac{d}{dz}\left(\int_{\gamma+z\eta} n(r)dr\right)_{z=0} = \int_{\tau_1}^{\tau_2} d\tau\frac{d}{dz}\left(\left|\vec{\nabla}S(\vec{\gamma}+z\vec{\eta})\right|\left|\dot{\vec{\gamma}} + z\ \dot{\vec{\eta}}\right|\right)_{z=0}$$

$$= \int_{\tau_1}^{\tau_2} d\tau\ \ \dot{\vec{\gamma}}\ \frac{d}{dz}\left(\vec{\nabla}S(\vec{\gamma}+z\vec{\eta})\right)_{z=0} + \int_{\tau_1}^{\tau_2} d\tau\ \ \vec{\nabla}S(\vec{\gamma})\ \dot{\vec{\eta}}$$

$$= \int_{\tau_1}^{\tau_2} d\tau\ \ \dot{\vec{\gamma}}\ \left(\vec{\eta}\vec{\nabla}\right)\vec{\nabla}S(\vec{\gamma}) - \int_{\tau_1}^{\tau_2} d\tau\ \left(\frac{d}{d\tau}\vec{\nabla}S(\vec{\gamma})\right)\vec{\eta}$$ partial integration

$$= \int_{\tau_1}^{\tau_2} d\tau\ \ \dot{\vec{\gamma}}\ \left(\vec{\eta}\vec{\nabla}\right)\vec{\nabla}S(\vec{\gamma}) - \int_{\tau_1}^{\tau_2} d\tau\ \ \dot{\vec{\gamma}}\left(\vec{\eta}\vec{\nabla}\right)\vec{\nabla}S(\vec{\gamma}) = 0$$



Boundary conditions at dielectric interfaces

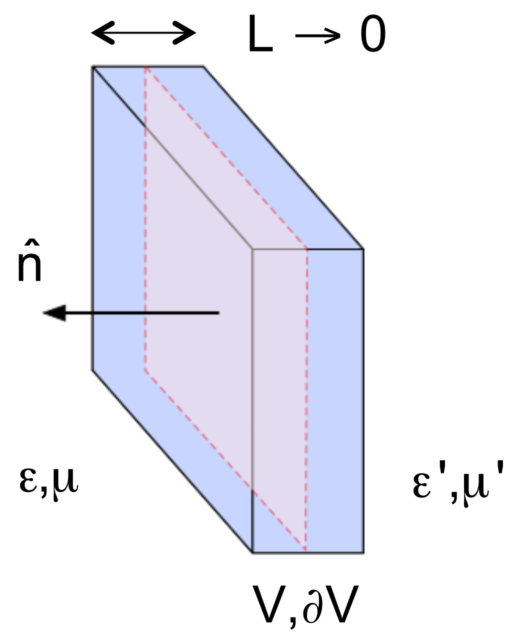
Normal components of $\vec{D} = \varepsilon\vec{E}$, $\vec{B}$ are continuous

Maxwell's equations $\vec{\nabla}\vec{B} = \vec{\nabla}\vec{D} = 0$

$$\Rightarrow \quad \vec{\nabla}\vec{f} = \text{bounded}, \quad \vec{f} \in \{\vec{B},\vec{D}\}$$

$$\Rightarrow \quad 0 = \lim_{L\to 0} \int_V \vec{\nabla}\vec{f}\, dV = \lim_{L\to 0} \int_{\partial V} \vec{f}\; d\vec{a} = da\; \hat{n}\left(\vec{f} - \vec{f}\,'\right)$$

$$\Rightarrow \quad \begin{aligned} \left(\varepsilon\vec{E} - \varepsilon'\vec{E}'\right)\hat{n} &= 0 \\ \left(\vec{B} - \vec{B}'\right)\hat{n} &= 0 \end{aligned}$$



Boundary conditions at dielectric interfaces

Tangential components of $\vec{E}, \vec{H} = \frac{1}{\mu}\vec{B}$ are continuous

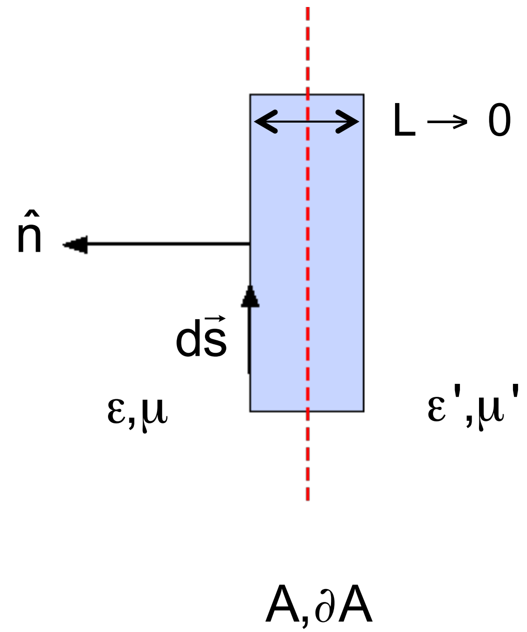
Maxwell's equations $\vec{\nabla}\times\vec{E} = i\omega\vec{B},\ \vec{\nabla}\times\vec{H} = -i\omega\vec{D}$

$$\Rightarrow \quad \vec{\nabla}\times\vec{f} = \text{bounded},\ \vec{f}\in\left\{\vec{E}, \tfrac{1}{\mu}\vec{B}\right\}$$

$$\Rightarrow \quad 0 = \lim_{L\to 0}\int_A \vec{\nabla}\times\vec{f}\, d\vec{a} = \lim_{L\to 0}\int_{\partial A}\vec{f}\, d\vec{s} = \left(\vec{f}-\vec{f}\,'\right) d\vec{s}$$

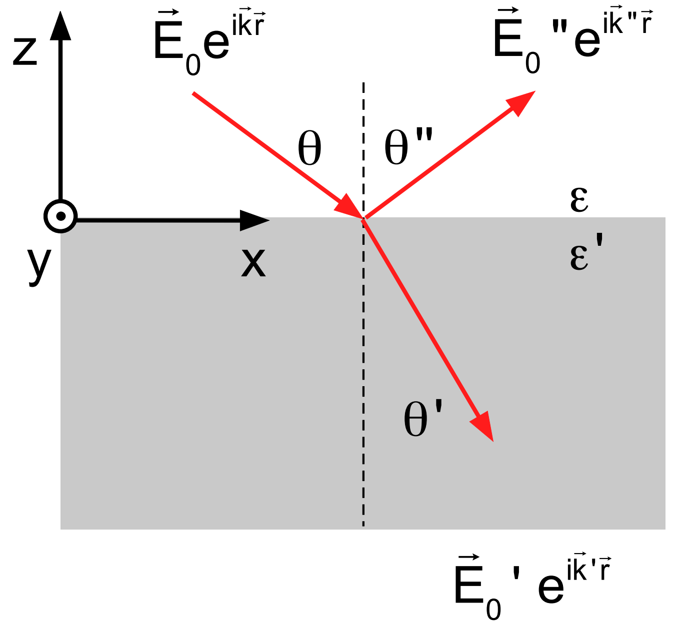
$$\Rightarrow \quad \left(\vec{E}-\vec{E}\,'\right)d\vec{s} = 0$$

$$\left(\frac{1}{\mu}\vec{B}-\frac{1}{\mu'}\vec{B}\,'\right)d\vec{s} = 0$$



Transmission and reflection at dielectric interfaces

$\mu = \mu' = \mu_0,\ k_y = 0$



0.

$$\tan(\theta) = -\frac{k_x}{k_z}$$

$$\sin(\theta) = \frac{k_x}{\sqrt{k_x^2 + k_z{}^2}} \qquad \tan(\theta') = -\frac{k_x'}{k_z'}$$

$$\sin(\theta') = \frac{k_x'}{\sqrt{k_x^2{}' + k_z^2{}'}} \qquad \tan(\theta'') = \frac{k_x''}{k_z''}$$

1. Tangential (x) components of electric field are continuous at z=0

$$E_{0,x}\, e^{i\vec{k}\vec{r}} + E_{0,x}''\, e^{i\vec{k}''\vec{r}} = E_{0,x}'\, e^{i\vec{k}'\vec{r}}$$

$$\Rightarrow \quad k''_y = k'_y = 0, \quad k''_x = k'_x = k_x$$

2. $$\frac{\omega}{c} n = \sqrt{k_x^2 + k_z^2} = \sqrt{k_x^2 + k''^2_z} \quad \Rightarrow \quad k''_z = -k_z, \quad \theta'' = \theta$$

3. $\frac{\omega}{c} n' = \sqrt{k_x^2 + k_z'^2} \overset{(0.)}{\Rightarrow} \frac{n'}{n} = \frac{\sqrt{k_x^2 + k_z'^2}}{\sqrt{k_x^2 + k_z^2}} = \frac{\sin(\theta)}{\sin(\theta')}$ Snell's law

4. Calculation of perpendicular component E_y

$$\vec{B} = \frac{1}{\omega} \vec{k} \times \vec{E} = \frac{1}{\omega} \begin{pmatrix} -k_z E_y \\ k_x E_z - k_z E_x \\ k_x E_y \end{pmatrix}$$

E_y and B_x (wg. $\mu = \mu' = \mu_0$) are continuous at z=0 $\Rightarrow$ $k_z E_y + k_z'' E_y'' = k_z' E_y'$, $E_y + E_y'' = E_y'$

$$\Rightarrow \quad E_y' = E_y \frac{2k_z}{k_z + k_z'} = E_y \frac{2\sin(\theta')\cos(\theta)}{\sin(\theta' + \theta)}$$

$$E_y'' = E_y \frac{k_z - k_z'}{k_z + k_z'} = E_y \frac{\sin(\theta' - \theta)}{\sin(\theta' + \theta)}$$

$$\frac{2k_z}{k_z + k_z'} = \frac{2}{1 + k_z'/k_z} \overset{(0.)}{=} \frac{2}{1 + \tan(\theta)/\tan(\theta')} = \frac{2\tan(\theta')}{\tan(\theta') + \tan(\theta)} = \frac{2\sin(\theta')\cos(\theta)}{\sin(\theta' + \theta)}$$

multiply $\cos(\theta)\cos(\theta')/\cos(\theta)\cos(\theta')$

$$\frac{k_z - k_z'}{k_z + k_z'} = \frac{1 - k_z'/k_z}{1 + k_z'/k_z} \overset{(0.)}{=} \frac{1 - \tan(\theta)/\tan(\theta')}{1 + \tan(\theta)/\tan(\theta')} = \frac{\tan(\theta') - \tan(\theta)}{\tan(\theta') + \tan(\theta)} = \frac{\sin(\theta' - \theta)}{\sin(\theta' + \theta)}$$

5. Sagital components E_x, E_z

(*) $$\vec{E} = -\frac{c^2}{n^2}\frac{1}{\omega}\,\vec{k}\times\vec{B} = -\frac{c^2}{n^2}\frac{1}{\omega}\begin{pmatrix} -k_z B_y \\ k_x B_z - k_z B_x \\ k_x B_y \end{pmatrix}$$

use $$\frac{k_z}{k_z'} = \frac{\tan(\theta')}{\tan(\theta)}, \quad \frac{n'}{n} = \frac{\sin(\theta)}{\sin(\theta')}$$

$$\frac{k_z}{k_z'}\left(\frac{n'}{n}\right)^2 = \frac{\tan(\theta')}{\tan(\theta)}\frac{\sin^2(\theta)}{\sin^2(\theta')} = \frac{\sin(\theta)\cos(\theta)}{\cos(\theta')\sin(\theta')} = \frac{\sin(2\theta)}{\sin(2\theta')}$$

E_x and B_y are continuous at z=0

1st line of (*) and $k_z'' = k_z$

$$\Rightarrow \quad E_x = \frac{c^2}{\omega\, n^2} k_z B_y = \frac{c^2}{\omega\, n^2} k_z \left(B_y' - B_y''\right) = \frac{k_z}{k_z'}\left(\frac{n'}{n}\right)^2 E_x' + E_x'', \quad E_x = E_x' - E_x''$$

B_y continuous

$$\Rightarrow \quad E_x' = \frac{2}{1+\frac{k_z}{k_z'}\left(\frac{n'}{n}\right)^2} E_x = \frac{2\sin(2\theta')}{\sin(2\theta)+\sin(2\theta')} E_x$$

and with

$$0 = \vec{k}\vec{E} = \vec{k}'\vec{E}'$$

$$k_x = k'_x$$

$$0 = k_y = k'_y$$

obtain

$$E_z' = \frac{2\frac{k_z}{k_z'}}{1+\frac{k_z}{k_z'}\left(\frac{n'}{n}\right)^2} E_z = \frac{2\sin(2\theta)}{\sin(2\theta)+\sin(2\theta')} E_z$$

6. Using 5. with $E_x'' = E_x' - E_x$ yields

$$E_x'' = \frac{1 - \frac{k_z}{k_z'}\left(\frac{n'}{n}\right)^2}{1 + \frac{k_z}{k_z'}\left(\frac{n'}{n}\right)^2} E_x = \frac{\tan(\theta' - \theta)}{\tan(\theta' + \theta)} E_x$$

use

$$\frac{k_z}{k_z'}\left(\frac{n'}{n}\right)^2 = \frac{\sin(2\theta)}{\sin(2\theta')}, \quad \frac{\sin(2\theta') - \sin(2\theta)}{\sin(2\theta') + \sin(2\theta)} = \frac{\tan(\theta' - \theta)}{\tan(\theta' + \theta)}$$

and with

$$0 = \vec{k}\vec{E} = \vec{k}''\vec{E}''$$

$$k_z = -k_z''$$

$$k_x = k_x''$$

$$0 = k_y = k_y''$$

$$E_z'' = -\frac{1 - \frac{k_z}{k_z'}\left(\frac{n'}{n}\right)^2}{1 + \frac{k_z}{k_z'}\left(\frac{n'}{n}\right)^2} E_z = \frac{\tan(\theta - \theta')}{\tan(\theta' + \theta)} E_z$$

7a. Amplitude transmission coefficients

$$t_\nu^{(12)} \equiv \frac{E_{2,\nu}}{E_{1,\nu}}, \; \nu \in \{x,y,z\}$$

$$t_x^{(12)} = \frac{2}{1+\frac{k_z^{(1)}}{k_z^{(2)}}\left(\frac{n_2}{n_1}\right)^2}$$

$$t_x^{(21)} = \frac{k_z^{(1)}}{k_z^{(2)}}\left(\frac{n_2}{n_1}\right)^2 t_x^{(12)}$$

$$t_y^{(12)} = \frac{2k_z^{(1)}}{k_z^{(1)}+k_z^{(2)}}$$

$$t_y^{(21)} = \frac{k_z^{(2)}}{k_z^{(1)}} t_y^{(12)}$$

$$t_z^{(12)} = \frac{2\frac{k_z^{(1)}}{k_z^{(2)}}}{1+\frac{k_z^{(1)}}{k_z^{(2)}}\left(\frac{n_2}{n_1}\right)^2}$$

$$t_z^{(21)} = \frac{k_z^{(2)}}{k_z^{(1)}}\left(\frac{n_2}{n_1}\right)^2 t_z^{(12)}$$

7b. Amplitude reflection coefficients

$$r_\nu^{(12)} \equiv \frac{E''_{1,\nu}}{E_{1,\nu}}, \ \nu \in \{x,y,z\}$$

$$r_x^{(12)} = \frac{1 - \frac{k_z^{(1)}}{k_z^{(2)}}\left(\frac{n_2}{n_1}\right)^2}{1 + \frac{k_z^{(1)}}{k_z^{(2)}}\left(\frac{n_2}{n_1}\right)^2}$$

$$r_y^{(12)} = \frac{k_z^{(1)} - k_z^{(2)}}{k_z^{(1)} + k_z^{(2)}}$$

$$r_x^{(12)} = -r_x^{(21)} = -r_z^{(12)} = r_z^{(21)}, \ r_y^{(12)} = -r_y^{(21)}$$

$$r_z^{(12)} = -\frac{1 - \frac{k_z^{(1)}}{k_z^{(2)}}\left(\frac{n_2}{n_1}\right)^2}{1 + \frac{k_z^{(1)}}{k_z^{(2)}}\left(\frac{n_2}{n_1}\right)^2}$$

8. Note

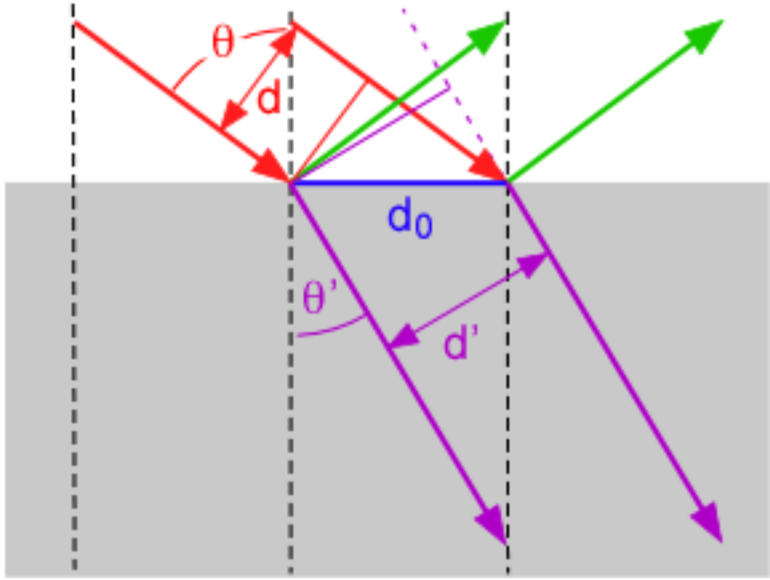
$$t_\nu^{(12)} t_\nu^{(21)} - r_\nu^{(12)} r_\nu^{(21)} = 1, \nu \in \{x,y,z\}$$

9. Poynting vector, power, transmission, reflection (assume real $\vec{k}, \vec{k}'$)

Poynting vector $$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}^* = \frac{1}{\mu_0 \omega} \vec{E} \times \left(\vec{k} \times \vec{E}^* \right) = \frac{1}{\mu_0 \omega} \left| \vec{E} \right|^2 \vec{k}$$

Power P proportional to $\left|\vec{S}\right| d$ with

$$d = d_0 \frac{k_z}{\left|\vec{k}\right|} = d_0 \cos(\theta), \; d' = d_0 \frac{k_z'}{\left|\vec{k}'\right|} = d_0 \cos(\theta')$$



Hence
$$P \propto \left|\vec{E}\right|^2 \left|\vec{k}\right| d = \left|\vec{E}\right|^2 k_z$$
$$P' \propto \left|\vec{E}'\right|^2 \left|\vec{k}'\right| d' = \left|\vec{E}'\right|^2 k_z'$$
$$P'' \propto \left|\vec{E}''\right|^2 \left|\vec{k}\right| d = \left|\vec{E}''\right|^2 k_z$$

Power transmission coefficient $$T \equiv \frac{P'}{P} = \frac{\left|\vec{E}'\right|^2 k'_z}{\left|\vec{E}\right|^2 k_z}$$

Power reflection coefficient $$R \equiv \frac{P''}{P} = \frac{\left|\vec{E}''\right|^2}{\left|\vec{E}\right|^2}$$

10.1 Power transmission for parallel polarization $0 = E_x = E_z$

$$T_p \equiv \frac{|E_y'|^2 \, k'_z}{|E_y|^2 \, k_z} \overset{(4.)}{=} \frac{4k_z k_z'}{(k_z + k_z')^2} = \frac{\sin(2\theta')\sin(2\theta)}{\sin^2(\theta' + \theta)}$$

(0.), (1.)

$$\frac{4k_z k_z'}{(k_z + k_z')^2} = \frac{4k_z / k_z'}{(k_z / k_z' + 1)^2} = \frac{4\tan(\theta')/\tan(\theta)}{(1 + \tan(\theta')/\tan(\theta))^2}$$

$$= \frac{4\tan(\theta')\tan(\theta)}{(\tan(\theta) + \tan(\theta'))^2} = \frac{\sin(2\theta')\sin(2\theta)}{\sin^2(\theta' + \theta)}$$

$$R_p \equiv \frac{|E_y''|^2}{|E_y|^2} = \frac{(k_z - k_z')^2}{(k_z + k_z')^2} = \frac{\sin^2(\theta' - \theta)}{\sin^2(\theta' + \theta)}$$

(0.), (1.)

$$\frac{(k_z - k_z')^2}{(k_z + k_z')^2} = \frac{(k_z / k_z' - 1)^2}{(k_z / k_z' + 1)^2} = \frac{(1 - \tan(\theta')/\tan(\theta))^2}{(1 + \tan(\theta')/\tan(\theta))^2}$$

$$= \frac{(\tan(\theta) - \tan(\theta'))^2}{(\tan(\theta) + \tan(\theta'))^2} = \frac{\sin^2(\theta - \theta')}{\sin^2(\theta + \theta')}$$

$$\Rightarrow \quad R_p + T_p = 1$$

Snell's law: If $\theta \to 0$ and hence $k_x = k_x' \to 0$: $\frac{k_z}{k_z'} \to \frac{n}{n'}$ and hence

$$T_p \to \frac{4n'n}{(n + n')^2}, \quad R_p \to \frac{(n - n')^2}{(n + n')^2}$$

10.2 Power transmission for sagittal polarization $0 = E_y$

(5.) and $|E_x|^2 = \frac{k_z^2}{k_x^2}|E_z|^2,\; |E'_x|^2 = \frac{k'^2_z}{k_x^2}|E'_z|^2$ $\qquad \left(\frac{n'}{n}\right)^2 = \frac{k_x^2 + k'^2_z}{k_x^2+k_z^2}$ Snell's law

$$T_s \equiv \frac{\left(|E_x'|^2 + |E_z'|^2\right) k'_z}{\left(|E_x|^2 + |E_z|^2\right) k_z} = \frac{4}{\left(1+\frac{k_z}{k_z'}\left(\frac{n'}{n}\right)^2\right)^2} \frac{\left(1+\left|\frac{k_x}{k_z'}\right|^2\right) k_z'}{\left(1+\left|\frac{k_x}{k_z}\right|^2\right) k_z} = \frac{4\left(\frac{k_z}{k_z'}\right)\left(\frac{n'}{n}\right)^2}{\left(1+\frac{k_z}{k_z'}\left(\frac{n'}{n}\right)^2\right)^2} = \frac{\sin(2\theta')\sin(2\theta)}{\sin^2(\theta'+\theta)\cos^2(\theta'-\theta)}$$

$$R_s \equiv \frac{|E_x''|^2 + |E_z''|^2}{|E_x|^2 + |E_z|^2} = \frac{\left(1-\frac{k_z}{k_z'}\left(\frac{n'}{n}\right)^2\right)^2}{\left(1+\frac{k_z}{k_z'}\left(\frac{n'}{n}\right)^2\right)^2} = \frac{\tan^2(\theta'-\theta)}{\tan^2(\theta'+\theta)}$$

use

$$0 = \vec{k}\vec{E} = \vec{k}'\vec{E}'$$

$$0 = k_y = k'_y, k_x = k'_x$$

$$\left(\frac{n'}{n}\right)^2 = \frac{k_x^2 + k'^2_z}{k_x^2+k_z^2}$$

$$\frac{\sin(2\theta') - \sin(2\theta)}{\sin(2\theta') + \sin(2\theta)} = \frac{\tan(\theta'-\theta)}{\tan(\theta'+\theta)}$$

$$\Rightarrow \quad R_s + T_s = 1$$

Snell's law:

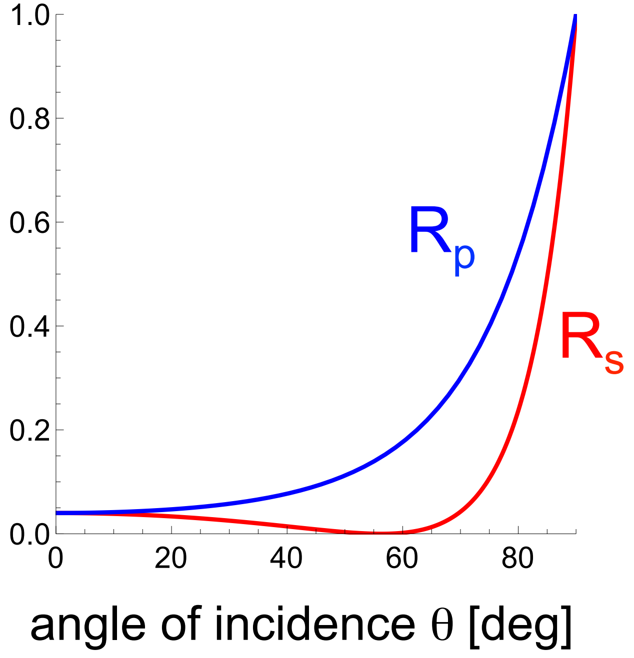
If $\theta \to 0$ and hence $k_x = k_x' \to 0$: $\frac{k_z}{k_z'} \to \frac{n}{n'}$ and hence $T_s \to \frac{4nn'}{(n+n')^2}$, $R_s \to \frac{(n-n')^2}{(n+n')^2}$

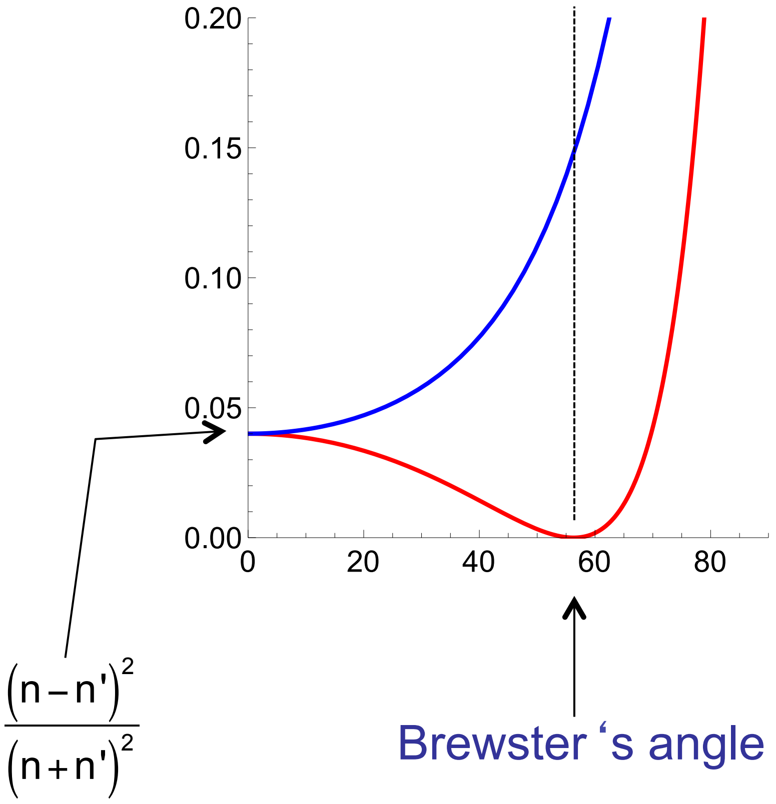
Brewster 's angle

Snell's law:

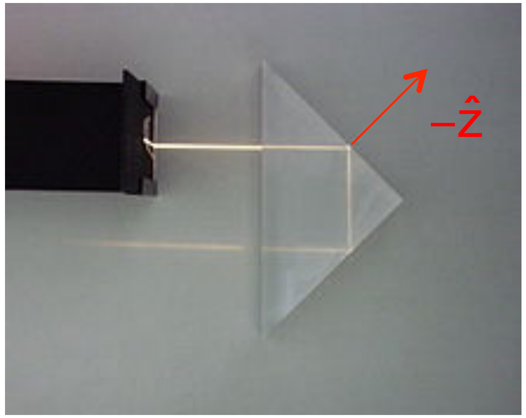
$$\tan(\theta) = \frac{n'}{n} \Rightarrow \tan(\theta) = \frac{\sin(\theta)}{\sin(\theta')} \Rightarrow \theta + \theta' = \pi / 2 \Rightarrow R_s = \frac{\tan^2(\theta' - \theta)}{\tan^2(\theta' + \theta)} = 0$$

define Brewster angle $\theta_B = \mathrm{Arctan}\left(\frac{n'}{n}\right)$





Total internal reflection



$$\left(\frac{n'}{n}\right)^2 = \frac{k_x^2 + k'^2_z}{k_x^2 + k_z^2} = \frac{k'^2_z}{k_x^2 + k_z^2} + \frac{k_x^2}{k_x^2 + k_z^2} = \frac{k'^2_z}{k_x^2 + k_z^2} + \sin^2(\theta)$$

Snell's law:

$$\Rightarrow \quad k'^2_z = \left(k_x^2 + k_z^2\right)\left(\left(\frac{n'}{n}\right)^2 - \sin^2(\theta)\right) = k_z^2 \frac{1}{\cos^2(\theta)}\left(\left(\frac{n'}{n}\right)^2 - \sin^2(\theta)\right)$$

$$n > n',\ \sin(\theta) > \frac{n'}{n} \quad \Rightarrow \quad k'^2_z < 0 \quad \text{with} \quad \frac{k'_z}{k_z} = i\,\frac{1}{\cos(\theta)}\sqrt{\sin^2(\theta) - \left(\frac{n'}{n}\right)^2}$$

$$\theta_c = \arcsin\left(\frac{n'}{n}\right) \qquad k'_z = \frac{i}{z_e} \quad \Rightarrow \quad e^{ik'_z z} = e^{-z/z_e}$$

Transmitted field decays exponentially with e^{-1}-length $\quad z_e \equiv \dfrac{\cos(\theta)}{k_z\sqrt{\sin^2(\theta) - \left(\frac{n'}{n}\right)^2}}$

reflected field

$$\frac{E_x''}{E_x} = -\frac{E_z''}{E_z} \underset{(6.)}{=} \frac{1+ik_z z_e\left(\frac{n'}{n}\right)^2}{1-ik_z z_e\left(\frac{n'}{n}\right)^2} = \frac{1-(k_z z_e)^2\left(\frac{n'}{n}\right)^4 + 2ik_z z_e\left(\frac{n'}{n}\right)^2}{1+(k_z z_e)^2\left(\frac{n'}{n}\right)^4}$$

use

$$\frac{k_z}{k'_z} = -i\,k_z z_e$$

$$\frac{E_y''}{E_y} \underset{(4.)}{=} -\frac{1+ik_z z_e}{1-ik_z z_e} = -\frac{1-(k_z z_e)^2 + 2ik_z z_e}{1+(k_z z_e)^2}$$

phase of reflected field

use $z = |z| e^{i\phi} \Rightarrow \tan(\phi) = \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}$

use

$$\tan(x) = \frac{2\tan(\frac{x}{2})}{1-\tan^2(\frac{x}{2})}$$

$$\tan(\phi_x) = \tan(\phi_z) = \frac{2k_z z_e\left(\frac{n'}{n}\right)^2}{1-(k_z z_e)^2\left(\frac{n'}{n}\right)^4} \Rightarrow \tan(\frac{\phi_x}{2}) = \tan(\frac{\phi_z}{2}) = k_z z_e\left(\frac{n'}{n}\right)^2$$

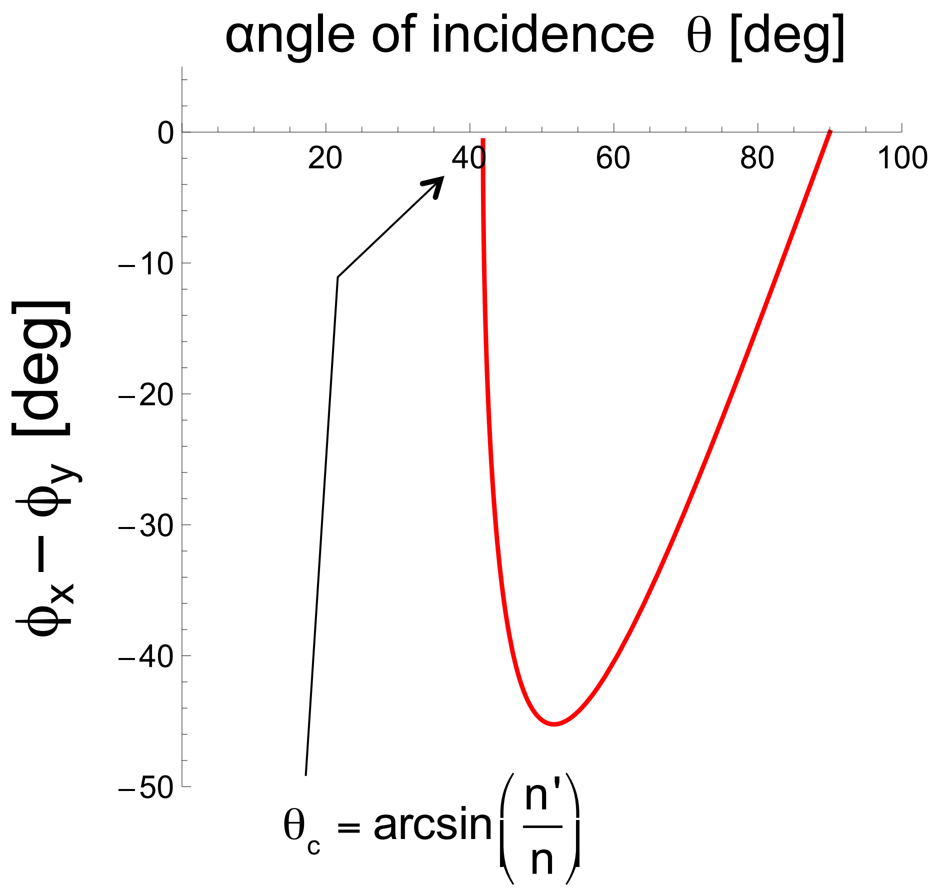
$$\tan(\phi_y) = \frac{2k_z z_e}{1-(k_z z_e)^2} \Rightarrow \tan(\frac{\phi_y}{2}) = k_z z_e$$

phase difference

use $\tan(x-y) = \dfrac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$

$$\tan\left(\frac{\phi_x}{2}-\frac{\phi_y}{2}\right) = \frac{k_z z_e\left(\left(\frac{n'}{n}\right)^2-1\right)}{1+\left(k_z z_e\right)^2\left(\frac{n'}{n}\right)^2}$$

$$k_z z_e = \frac{\cos(\theta)}{\sqrt{\sin^2(\theta)-\left(\frac{n'}{n}\right)^2}}$$



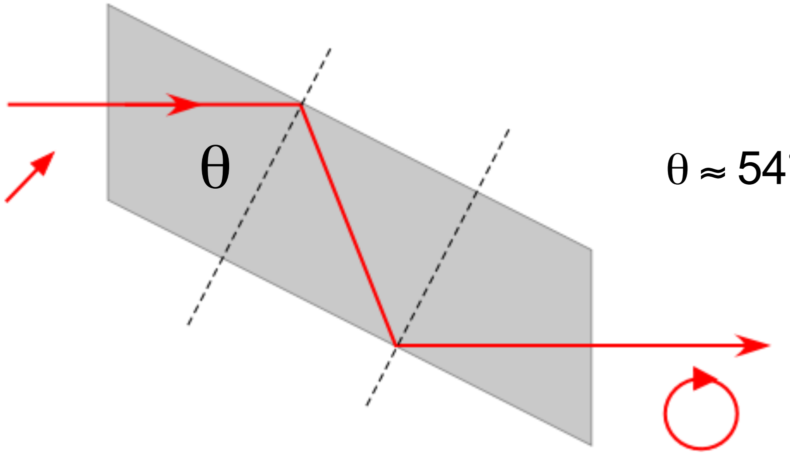
amplitude of reflected field

$$\frac{\left|E_x''^2\right|}{\left|E_x^2\right|} = \frac{\left|E_z''^2\right|}{\left|E_z^2\right|} = \frac{\left|1-\frac{k_z}{k_z'}\left(\frac{n'}{n}\right)^2\right|^2}{\left|1+\frac{k_z}{k_z'}\left(\frac{n'}{n}\right)^2\right|^2} = \frac{\left|1+ik_z z_e\left(\frac{n'}{n}\right)^2\right|^2}{\left|1-ik_z z_e\left(\frac{n'}{n}\right)^2\right|^2} = 1, \quad \left|\frac{E_y''}{E_y}\right|^2 = \left|\frac{1+ik_z z_e}{1-ik_z z_e}\right|^2 = 1$$

Applicaton: Fresnel rhomb –> making circularly polarized light

Each total internal reflection contributes 45° phase shift.

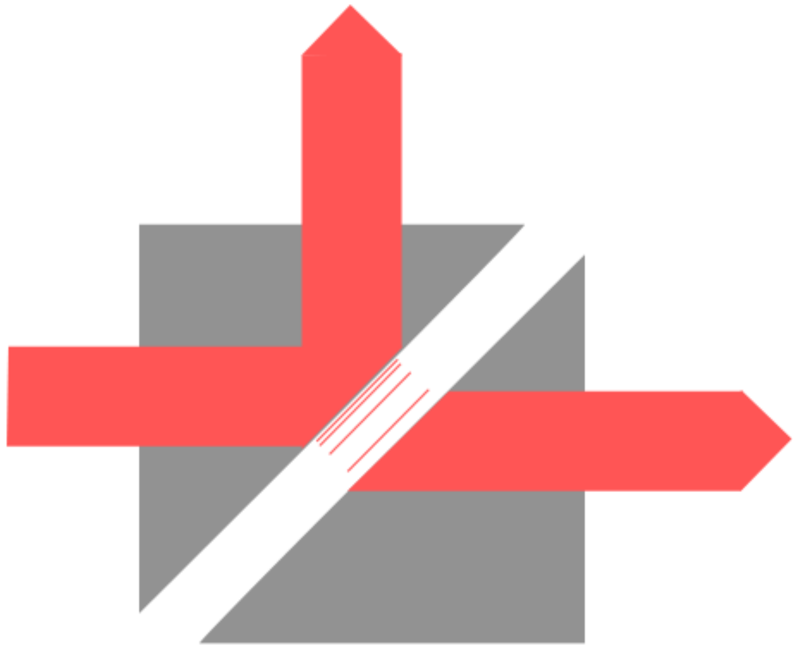
Augustin Jean Fresnel
1788 - 1827



$\theta \approx 54^\circ$ for $n = 1.5$

Frustrated total internal reflection

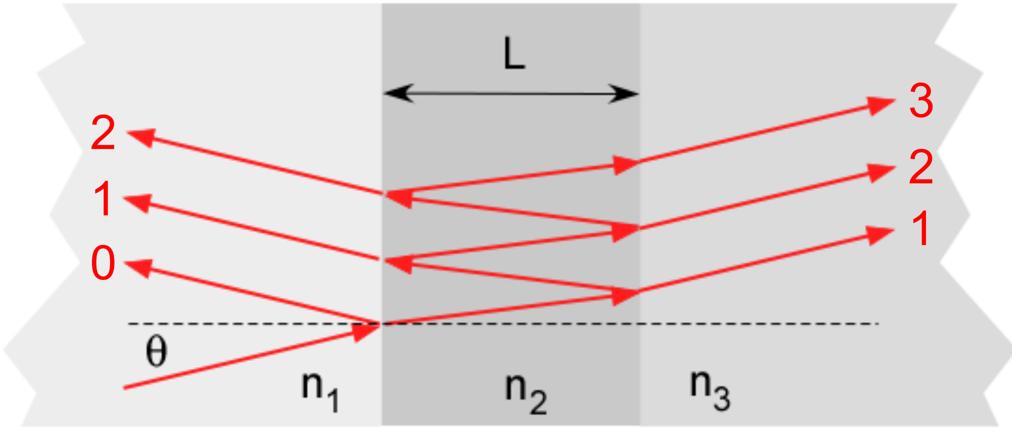
evanescent wave enters into second prism yielding transmission across the gap:



Multiple beam interference

Two dielectric interfaces

$\phi \equiv kL$



	reflection	transmission
$\nu + 1$	$t_{12}\, e^{i2\phi} r_{23}\, (r_{21} r_{23}\, e^{2i\phi})^{\nu}\, t_{21}$	$t_{12}\, e^{i\phi}\, (r_{23} r_{21}\, e^{2i\phi})^{\nu}\, t_{23}$
...	...	...
2	$t_{12}\, e^{i2\phi} r_{23}\, (r_{21} r_{23}\, e^{2i\phi})\, t_{21}$	$t_{12}\, e^{i\phi}\, (r_{23} r_{21}\, e^{2i\phi})\, t_{23}$
1	$t_{12}\, e^{i2\phi} r_{23}\, t_{21}$	$t_{12}\, e^{i\phi}\, t_{23}$
0	r_{12}	

reflection

$$\tilde{r} = r_{12} + t_{12}\, t_{21}\, r_{23}\, e^{2i\phi} \sum_{\nu=0}^{\infty} (r_{21}\, r_{23}\, e^{2i\phi})^{\nu} = r_{12} + \frac{t_{12}\, t_{21}\, r_{23}\, e^{2i\phi}}{1 - r_{21}\, r_{23}\, e^{2i\phi}}$$

$$= \frac{r_{12} + r_{23}\, e^{2i\phi}\left(t_{12}\, t_{21} - r_{12}\, r_{21}\right)}{1 - r_{21}\, r_{23}\, e^{2i\phi}} \overset{(8.)}{=} \frac{r_{12} + r_{23}\, e^{2i\phi}}{1 - r_{21}\, r_{23}\, e^{2i\phi}} = \frac{r_{12}\, e^{-i\phi} + r_{23}\, e^{i\phi}}{e^{-i\phi} - r_{21}\, r_{23}\, e^{i\phi}}$$

$$= \frac{(r_{12} + r_{23})\cos(\phi) + i\,(r_{23} - r_{12})\sin(\phi)}{(1 - r_{21} r_{23})\cos(\phi) - i\,(1 + r_{21} r_{23})\sin(\phi)}$$

normal incidence

$$= \frac{(n_1 - n_3)\cos(\phi) + i\left(n_2 - \frac{n_1 n_3}{n_2}\right)\sin(\phi)}{(n_1 + n_3)\cos(\phi) - i\left(n_2 + \frac{n_1 n_3}{n_2}\right)\sin(\phi)}$$

use $r_{12} = \frac{n_1 - n_2}{n_1 + n_2}$

transmission

$$\tilde{t} = t_{12}\, t_{23}\, e^{i\phi} \sum_{\nu=0}^{\infty} (r_{21}\, r_{23}\, e^{2i\phi})^{\nu} = \frac{t_{12}\, t_{23}\, e^{i\phi}}{1 - r_{21}\, r_{23}\, e^{2i\phi}}$$

normal incidence

$$= \frac{2n_1}{(n_1 + n_3)\cos(\phi) - i\left(n_2 + \frac{n_1 n_3}{n_2}\right)\sin(\phi)}$$

use $t_{12} = \frac{2n_1}{n_1 + n_2}$

Example: glas window $n_1 = n_3 = 1,\ n_2 = n = 1.5$

$$\tilde{T} \equiv \left|\tilde{t}\right|^2 = \frac{1}{1 + \underbrace{\frac{1}{4}\left(n - \frac{1}{n}\right)^2}_{F} \sin^2(\phi)} \qquad (= 0.17)$$

Airy-formula

use

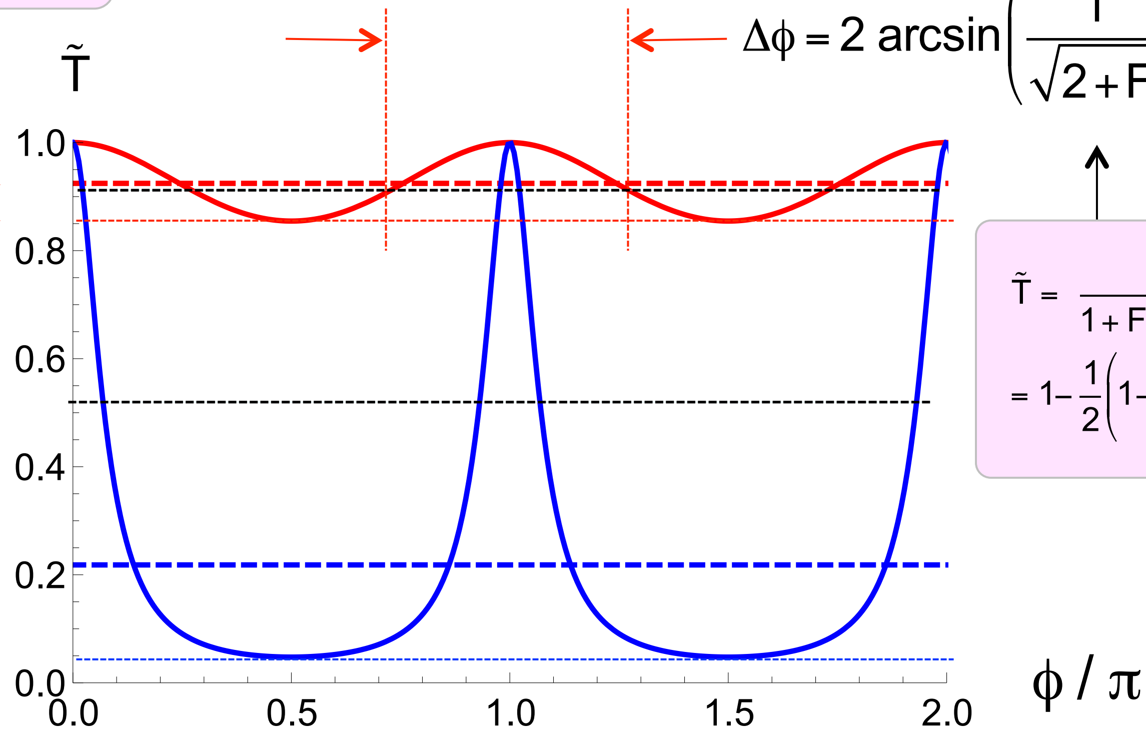
$$\left\langle \tilde{T} \right\rangle_\phi \equiv \tfrac{2}{\pi} \int_0^{\pi/2} d\phi \, \frac{1}{1 + F \sin^2(\phi)}$$

$$\frac{\sqrt{1+F}}{1 + F \sin^2(\phi)} = \frac{d}{d\phi} \arctan\left(\sqrt{1+F}\,\tan(\phi)\right)$$

$$\left\langle \tilde{T} \right\rangle_\phi = \frac{1}{\sqrt{1+F}}$$

$$\frac{1}{1+F}$$

$$\Delta\phi = 2 \arcsin\left(\frac{1}{\sqrt{2+F}}\right)$$



$$\tilde{T} = \frac{1}{1 + F \sin^2(\phi)}$$

$$= 1 - \frac{1}{2}\left(1 - \frac{1}{1+F}\right)$$

Multi-layer systems

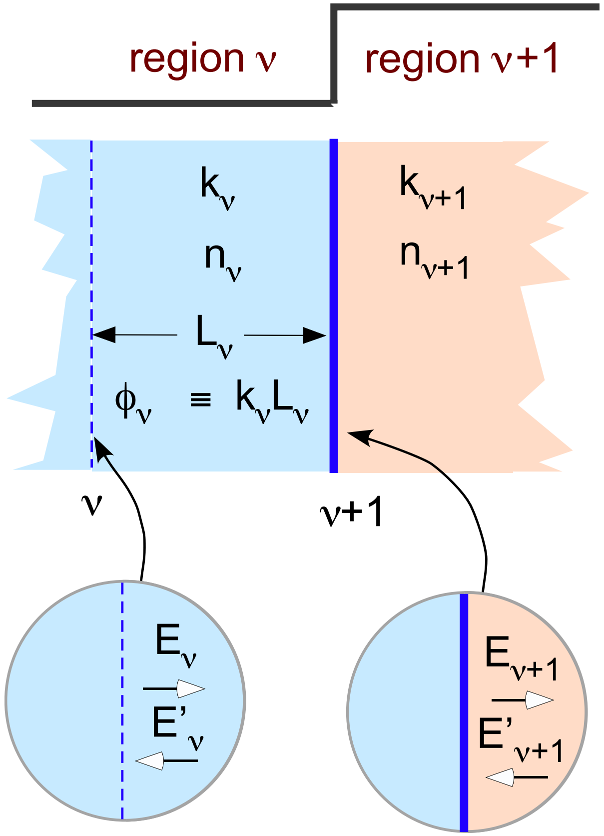
E and B are continuous

$$E_\nu e^{i\phi_\nu} + E'_\nu e^{-i\phi_\nu} = E_{\nu+1} + E'_{\nu+1}$$

$$n_\nu \left(E_\nu e^{i\phi_\nu} - E'_\nu e^{-i\phi_\nu}\right) = n_{\nu+1} \left(E_{\nu+1} - E'_{\nu+1}\right)$$

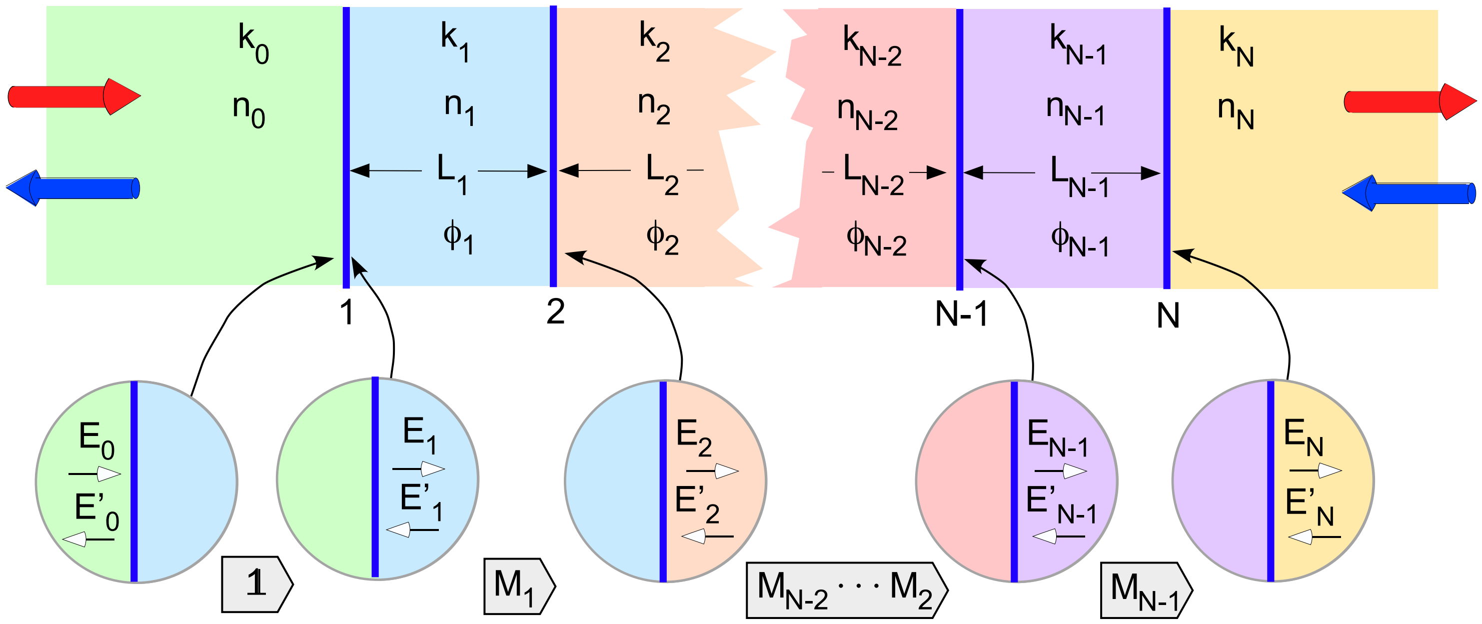
use $\dfrac{k_\nu}{k_{\nu+1}} = \dfrac{n_\nu}{n_{\nu+1}}, \; k'_\nu = -k_\nu$ Snell's law

$$\vec{B} = \frac{1}{\omega} \vec{k} \times \vec{E}$$



$$\Rightarrow \begin{pmatrix} E_{\nu+1} + E'_{\nu+1} \\ n_{\nu+1}\left(E_{\nu+1} - E'_{\nu+1}\right) \end{pmatrix} = \begin{pmatrix} \cos(\phi_\nu) & \dfrac{i}{n_\nu}\sin(\phi_\nu) \\ i\, n_\nu \sin(\phi_\nu) & \cos(\phi_\nu) \end{pmatrix} \begin{pmatrix} E_\nu + E'_\nu \\ n_\nu\left(E_\nu - E'_\nu\right) \end{pmatrix}$$

Transfer matrix $\det(M_\nu) = 1$



$$\begin{pmatrix} E_N + E_N' \\ n_N\left(E_N - E_N'\right) \end{pmatrix} = \underbrace{M_{N-1} \cdots M_1}_{M} \begin{pmatrix} E_0 + E_0' \\ n_0\left(E_0 - E_0'\right) \end{pmatrix}$$

$$\det(M) = \det(M_{N-1}) \cdots \det(M_1) = 1$$

$$E'_N = 0 \quad \Rightarrow \quad \begin{pmatrix} E_N \\ n_N E_N \end{pmatrix} = M \begin{pmatrix} E_0 + E_0' \\ n_0 \left(E_0 - E_0' \right) \end{pmatrix}$$

Devide by E_0

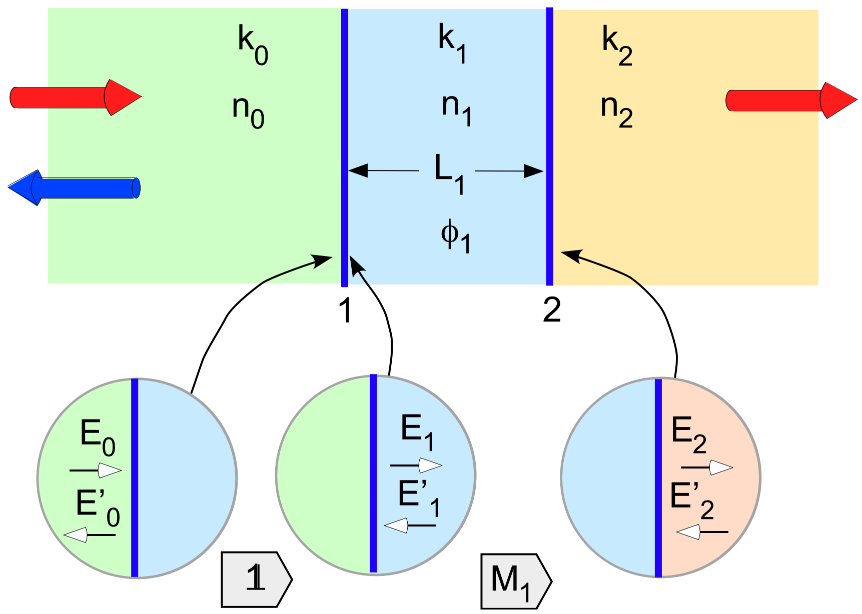
$$\Rightarrow \quad \begin{pmatrix} 1 \\ n_N \end{pmatrix} t = M \begin{pmatrix} 1 + r \\ n_0 \left(1 - r \right) \end{pmatrix}$$

$$r \equiv E_0' / E_0 \ , \quad t \equiv E_N / E_0 \ , \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$r = \frac{n_0 D - n_0 n_N B - n_N A + C}{n_0 D - n_0 n_N B + n_N A - C}$$

$$t = \frac{2 n_0}{n_0 D - n_0 n_N B + n_N A - C}$$

Example N=2:



$$M = \begin{pmatrix} \cos(\phi_1) & \frac{i}{n_1}\sin(\phi_1) \\ i\, n_1 \sin(\phi_1) & \cos(\phi_1) \end{pmatrix}$$

$$r = \frac{(n_0 - n_2)\cos(\phi_1) + i\left(n_1 - \frac{n_0 n_2}{n_1}\right)\sin(\phi_1)}{(n_0 + n_2)\cos(\phi_1) - i\left(n_1 + \frac{n_0 n_2}{n_1}\right)\sin(\phi_1)}$$

$$t = \frac{2n_0}{(n_0 + n_2)\cos(\phi_1) - i\left(n_1 + \frac{n_0 n_2}{n_1}\right)\sin(\phi_1)}$$

cf. page 44-45

Sylvester 's formula

A,B,C,D complex numbers with AD-BC = 1 $\Rightarrow$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^m = \frac{1}{\sin(\theta)} \begin{pmatrix} A\sin(m\theta) - \sin\left((m-1)\,\theta\right) & B\sin(m\theta) \\ C\sin(m\theta) & D\sin(m\theta) - \sin\left((m-1)\,\theta\right) \end{pmatrix}$$

$$\cos(\theta) \equiv \frac{1}{2}(A + D)$$

Proof by induction:

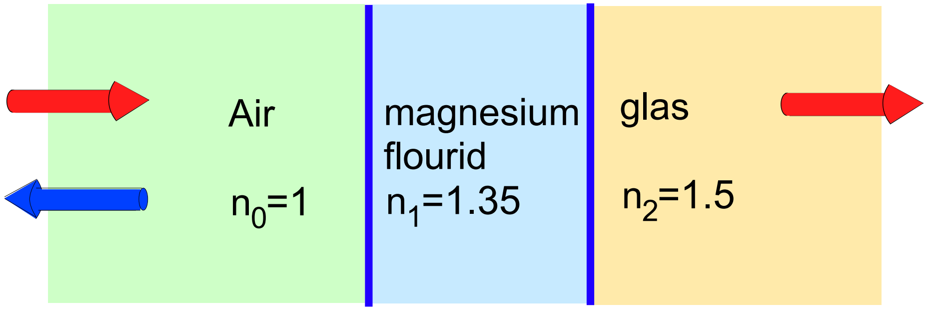
$m = 1$ trivially fullfilled

$m \rightarrow m+1$ use $\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$ with $x = m\theta,\ y = \theta$
accounting for $AD - BC = 1$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{m+1} = \frac{1}{\sin(\theta)} \begin{pmatrix} A\sin(m\theta) - \sin\left((m-1)\,\theta\right) & B\sin(m\theta) \\ C\sin(m\theta) & D\sin(m\theta) - \sin\left((m-1)\,\theta\right) \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$A' = A\left[A\sin(m\theta) - \sin\left((m-1)\,\theta\right)\right] + \left(AD - 1\right)\sin(m\theta) = A\left[\left(A + D\right)\sin(m\theta) - \sin\left((m-1)\,\theta\right)\right] - \sin(m\theta)$$
$$= A\left[2\sin(m\theta)\cos(\theta) - \sin\left((m-1)\,\theta\right)\right] - \sin(m\theta) = A\left[2\sin(m\theta)\cos(\theta) - \sin(m\theta)\cos(\theta) + \cos(m\theta)\sin(\theta)\right] - \sin(m\theta)$$
$$= A\left[\sin(m\theta)\cos(\theta) + \cos(m\theta)\sin(\theta)\right] - \sin(m\theta) = A\sin((m+1)\theta) - \sin(m\theta)$$

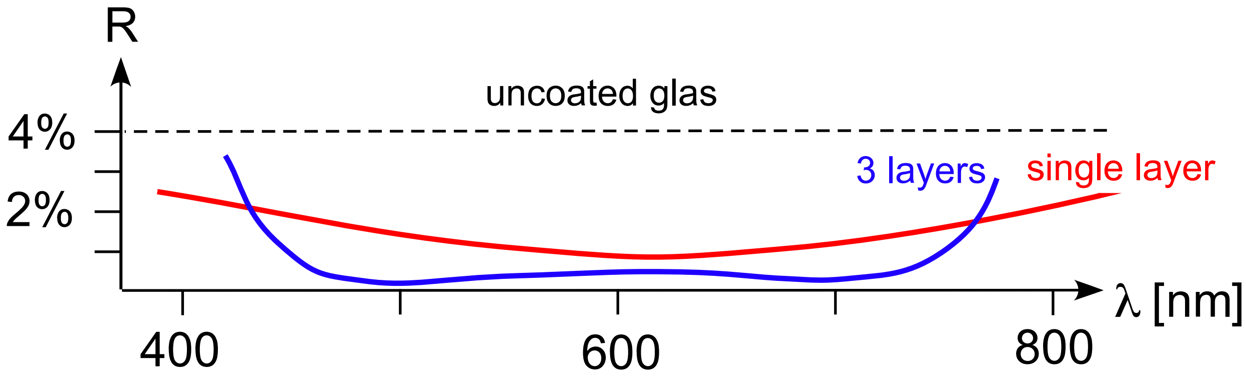
Antireflection coating



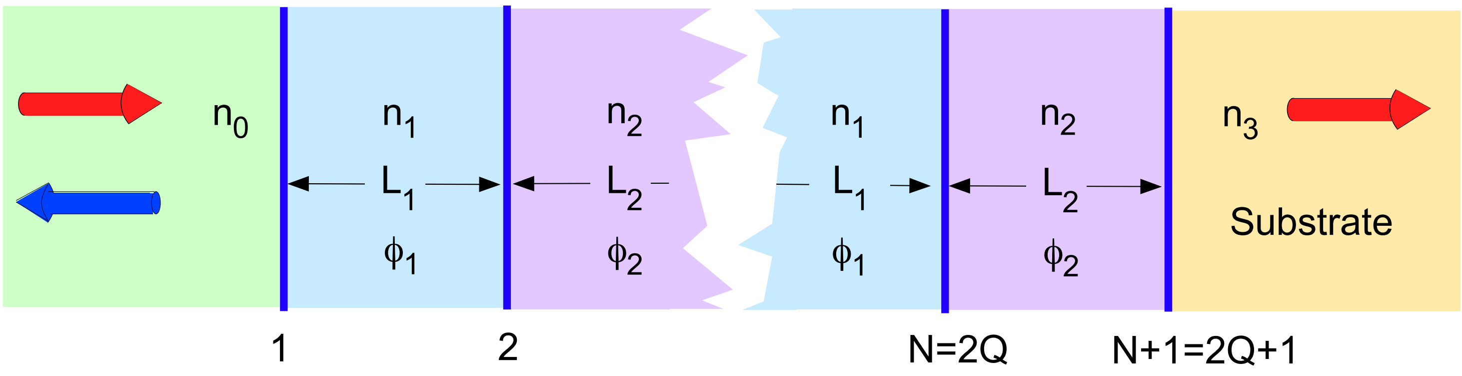
$$r = \frac{(1-n_2)\cos(\phi_1) + i\left(n_1 - \frac{n_2}{n_1}\right)\sin(\phi_1)}{(1+n_2)\cos(\phi_1) - i\left(n_1 + \frac{n_2}{n_1}\right)\sin(\phi_1)}$$

$$\phi_1 = \frac{2\pi}{\lambda} L_1$$

$$\phi_1 = \frac{\pi}{2} \Rightarrow \quad L_1 = \frac{\lambda}{4} \quad \& \quad r = \frac{n_2 - n_1^2}{n_2 + n_1^2}, \quad R = r^2 = \left(\frac{n_2 - n_1^2}{n_2 + n_1^2}\right)^2 \cong 0.01$$



High reflective coatings for laser mirrors



$$\phi_\nu = k_\nu L_\nu = \frac{\pi}{2} \quad \text{for } \nu \in \{1,2\}$$

$$M_{2\nu} = \begin{pmatrix} 0 & \frac{i}{n_2} \\ i\,n_2 & 0 \end{pmatrix}, \quad M_{2\nu-1} = \begin{pmatrix} 0 & \frac{i}{n_1} \\ i\,n_1 & 0 \end{pmatrix} \quad \Rightarrow \quad M_{2\nu} M_{2\nu-1} = \begin{pmatrix} -\frac{n_1}{n_2} & 0 \\ 0 & -\frac{n_2}{n_1} \end{pmatrix}$$

$$\Rightarrow \quad M = \left(M_{2\nu}\, M_{2\nu-1}\right)^Q = \begin{pmatrix} \left(-\frac{n_1}{n_2}\right)^Q & 0 \\ 0 & \left(-\frac{n_2}{n_1}\right)^Q \end{pmatrix}$$

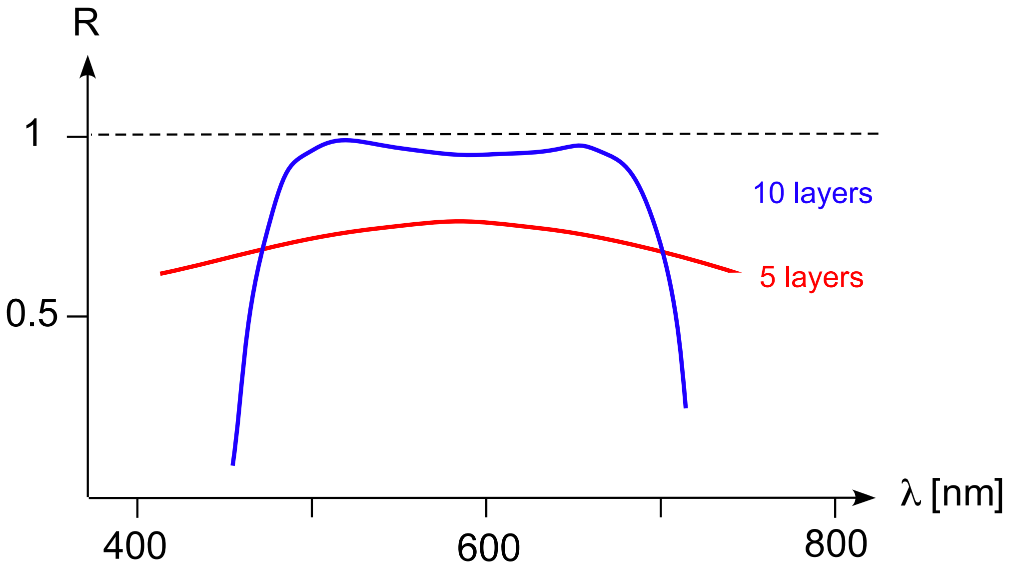
$$r = \frac{n_0 D - n_0 n_N B - n_N A + C}{n_0 D - n_0 n_N B + n_N A - C}$$

$$t = \frac{2n_0}{n_0 D - n_0 n_N B + n_N A - C}$$

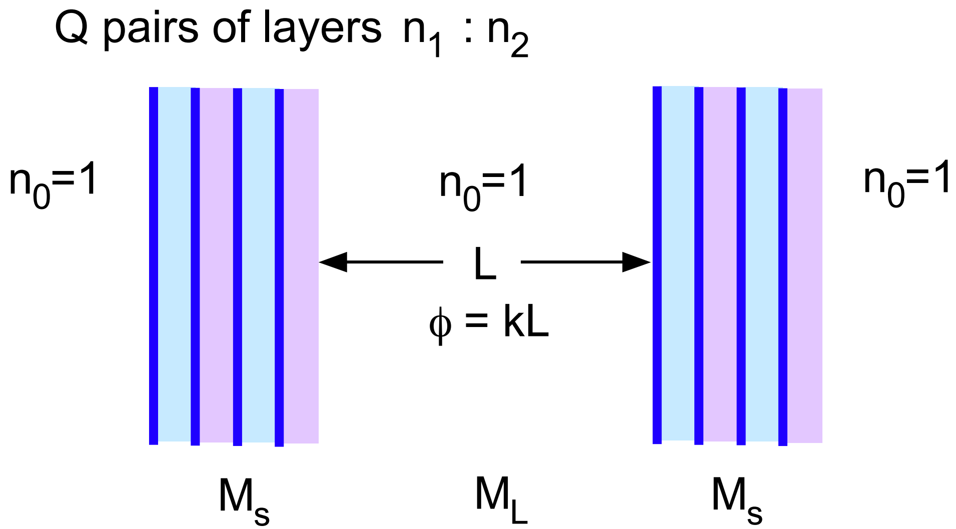
$$\Rightarrow \quad r = \frac{n_0 \left(\frac{n_2}{n_1}\right)^{2Q} - n_3}{n_0 \left(\frac{n_2}{n_1}\right)^{2Q} + n_3}, \quad t = \frac{2n_0 \left(-\frac{n_2}{n_1}\right)^{Q}}{n_0 \left(\frac{n_2}{n_1}\right)^{2Q} + n_3}$$

$$A = \frac{1}{D} = \left(-\frac{n_1}{n_2}\right)^{Q}, \; B = C = 0, \; n_N = n_3$$

$$n_1 \neq n_2 \quad \Rightarrow \quad |r| \xrightarrow[Q \to \infty]{} 1$$



Fabry-Pérot interferometer



Charles Fabry
1867-1945

Alfred Pérot
1863-1925

Transfer matrix of mirrors

$$M_S = \begin{pmatrix} q & 0 \\ 0 & 1/q \end{pmatrix}, \; q \equiv \left(-\frac{n_2}{n_1}\right)^Q \quad \Rightarrow \quad r_{12} = -r_{21} = \frac{1-q^2}{1+q^2}, \; t_{12} = -t_{21} = \frac{2q}{1+q^2}$$

$$R = -r_{12}\, r_{21} \;, \; T = t_{12}\, t_{21}$$

Transfer matrix of propagation between mirrors

$$M_L = \begin{pmatrix} \cos(\phi) & i\sin(\phi) \\ i\sin(\phi) & \cos(\phi) \end{pmatrix}$$

$$M = M_S \; M_L \; M_S \quad \Rightarrow \quad M = \begin{pmatrix} q^2\cos(\phi) & i\sin(\phi) \\ i\sin(\phi) & q^{-2}\cos(\phi) \end{pmatrix}$$

use $\quad r = \dfrac{n_0 D - n_0 n_N B - n_N A + C}{n_0 D - n_0 n_N B + n_N A - C} \qquad t = \dfrac{2 n_0}{n_0 D - n_0 n_N B + n_N A - C}$

$$\Rightarrow \quad \tilde{r} = \frac{\left(q^{-2} - q^2\right)\cos(\phi)}{\left(q^{-2} + q^2\right)\cos(\phi) - 2i\sin(\phi)} \; , \quad \tilde{t} = \frac{2}{\left(q^{-2} + q^2\right)\cos(\phi) - 2i\sin(\phi)}$$

In terms of the mirror reflectivities:

use $\quad r_{12} = -r_{21} = \dfrac{1-q^2}{1+q^2} \; , \; t_{12} = -t_{21} = \dfrac{2q}{1+q^2}$

$$\underset{(*)}{\Rightarrow} \quad \tilde{r} = \frac{r_{12}\left(1 + e^{i2\phi}\right)}{1 + R\, e^{i2\phi}} = \sqrt{\frac{F\cos^2(\phi)}{1 + F\cos^2(\phi)}} \; e^{i\Phi} \; , \quad \Phi \equiv \arctan\left[\frac{1-R}{1+R}\tan(\phi)\right] \; , \quad F \equiv \frac{4R}{T^2}$$

$$\Rightarrow \quad \tilde{t} = \frac{T\, e^{i\phi}}{1 + R\, e^{i2\phi}} = \sqrt{\frac{1}{1 + F\cos^2(\phi)}} \; e^{i\Phi}$$

Example: calculation of (*)

$$\tilde{r} = \frac{\left(q^{-2} - q^{2}\right)\cos(\phi)}{\left(q^{-2} + q^{2}\right)\cos(\phi) - 2i\sin(\phi)} = \frac{\left(1 - q^{4}\right)\frac{1}{2}(e^{i\phi} + e^{-i\phi})}{\left(1+q^{4}\right)\frac{1}{2}(e^{i\phi} + e^{-i\phi}) - q^{2}(e^{i\phi} - e^{-i\phi})}$$

$$= \frac{\left(1-q^{4}\right)(e^{i2\phi}+1)}{\left(1+q^{4}\right)(e^{i2\phi}+1) - 2q^{2}(e^{i2\phi}-1)} = \frac{\left(1-q^{2}\right)\left(1+q^{2}\right)(e^{i2\phi}+1)}{\left(1-2q^{2}+q^{4}\right)e^{i2\phi} + \left(1+2q^{2}+q^{4}\right)}$$

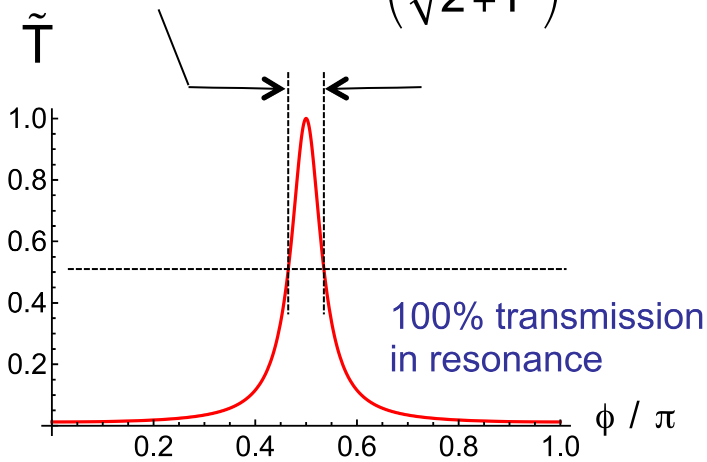
$$= \frac{\left(1-q^{2}\right)\left(1+q^{2}\right)(e^{i2\phi}+1)}{\left(1-q^{2}\right)^{2} e^{i2\phi} + \left(1+q^{2}\right)^{2}} = \frac{r_{12}(e^{i2\phi}+1)}{R\, e^{i2\phi}+1} = \frac{r_{12}(e^{i\phi}+e^{-i\phi})}{R\, e^{i\phi}+e^{-i\phi}} = \frac{r_{12}\, 2\cos(\phi)}{(1+R)\cos(\phi) - i(1-R)\sin(\phi)}$$

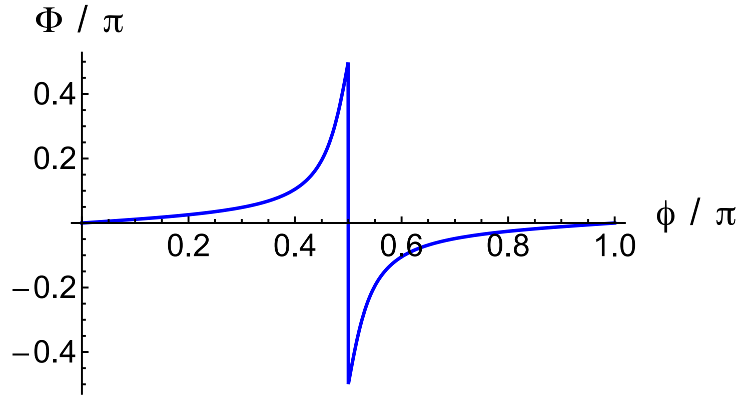
$$= \frac{r_{12}\, 2\cos(\phi)\left[(1+R)\cos(\phi) + i(1-R)\sin(\phi)\right]}{(1-R)^{2} + 4R\cos^{2}(\phi)}$$

$$= \frac{r_{12}\, 2\cos(\phi)}{\left[4R\cos^{2}(\phi) + T^{2}\right]^{1/2}} \quad \arctan\left(\frac{(1-R)}{(1+R)}\tan(\phi)\right) = \sqrt{\frac{F\cos^{2}(\phi)}{1+F\cos^{2}(\phi)}}\; e^{i\Phi}$$

Power and phase of transmitted light: $\tilde{T} = \dfrac{1}{1+F\cos^2(\phi)}\,,\quad \Phi \equiv \arctan\left[\dfrac{1-R}{1+R}\tan(\phi)\right]$

$$\delta\phi = 2\arcsin\left(\frac{1}{\sqrt{2+F}}\right)$$





Free spectral range $\delta\nu_{FSR} = \dfrac{c}{2\pi}\,\delta k_{FSR} = \dfrac{c}{2\pi\,L}\,\delta\phi_{FSR} = \dfrac{c}{2L}$

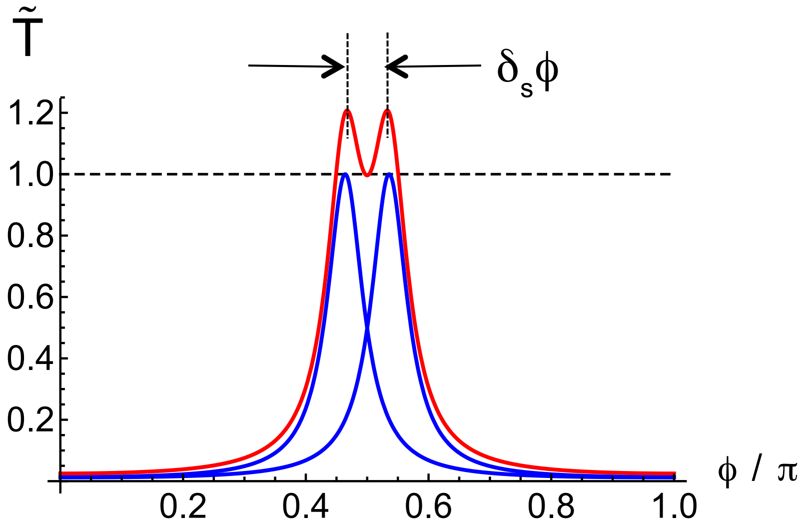
Linewidth FWHM

$$\delta\nu = \frac{c}{2\pi}\,\delta k = \frac{c}{2\pi L}\,\delta\phi = \frac{c}{\pi L}\arcsin\left(\frac{1}{\sqrt{2+F}}\right) = \delta\nu_{FSR}\,\frac{2}{\pi}\arcsin\left(\frac{1}{\sqrt{2+F}}\right)$$

$$\boldsymbol{\mathcal{F}} \equiv \frac{\delta\nu_{FSR}}{\delta\nu} = \left(\frac{2}{\pi}\arcsin\left(\frac{1}{\sqrt{2+F}}\right)\right)^{-1}$$

Finesse $\boldsymbol{\mathcal{F}} \xrightarrow[F\to\infty]{} \dfrac{\pi}{2}\sqrt{F} = \pi\,\dfrac{\sqrt{R}}{1-R}$

Spectral resolution



Assume F >1

$$1 = \frac{1}{1+F\cos^2(\frac{\pi}{2}-\frac{1}{2}\delta_s\phi)} + \frac{1}{1+F\cos^2(\frac{\pi}{2}+\frac{1}{2}\delta_s\phi)} = \frac{2}{1+F\sin^2(\frac{1}{2}\delta_s\phi)}$$

$$\Rightarrow \quad \delta_s\phi = 2\arcsin\left(\frac{1}{\sqrt{F}}\right) \Rightarrow \delta_s\nu = \frac{c}{2\pi L}\,\delta_s\phi = \frac{c}{\pi L}\arcsin\left(\frac{1}{\sqrt{F}}\right) = \delta\nu_{FSR}\,\frac{2}{\pi}\arcsin\left(\frac{1}{\sqrt{F}}\right)$$

F >>1 $\quad \Rightarrow \quad \delta_s\nu = \delta\nu = \delta\nu_{FSR}\,\frac{1}{\boldsymbol{\mathcal{F}}}$

Generalization to resonators made of two mirrors with different real amplitude reflectivities r_a and r_b and absorption losses per roundtrip t^2

$$\Rightarrow \qquad \tilde{r} = \sqrt{\frac{A + F\cos^2(\phi)}{1 + F\cos^2(\phi)}}\, e^{i\Phi_r}$$

$$\tilde{t} = \sqrt{\frac{B}{1 + F\cos^2(\phi)}}\, e^{i\Phi_t}$$

$$A \equiv \left(\frac{r_a - r_m}{1 - r_a r_m}\right)^2, \quad r_m \equiv r_b\, t^2$$

$$F \equiv \frac{4 r_a r_m}{\left(1 - r_a r_m\right)^2}$$

$$B \equiv \left(\frac{t\, t_a t_b}{1 - r_a r_m}\right)^2$$

$$\Phi_r \equiv \arctan\left[\frac{1}{1 + \frac{(r_a - r_m)(1 - r_a r_m)}{2 r_m (1 + r_a^2)\cos^2(\phi)}}\, \frac{1 - r_a^2}{1 + r_a^2} \tan(\phi)\right]$$

$$\Phi_t \equiv \arctan\left[\frac{1 - r_a r_m}{1 + r_a r_m} \tan(\phi)\right]$$

Finesse $\quad \mathcal{F} \equiv \frac{\delta\nu_{FSR}}{\delta\nu} = \left(\frac{2}{\pi} \arcsin\left(\frac{1}{\sqrt{2 + F}}\right)\right)^{-1}$

$$\mathcal{F} \xrightarrow[F \to \infty]{} \pi \frac{\sqrt{r_a r_m}}{1 - r_a r_m}$$

Impedance matching

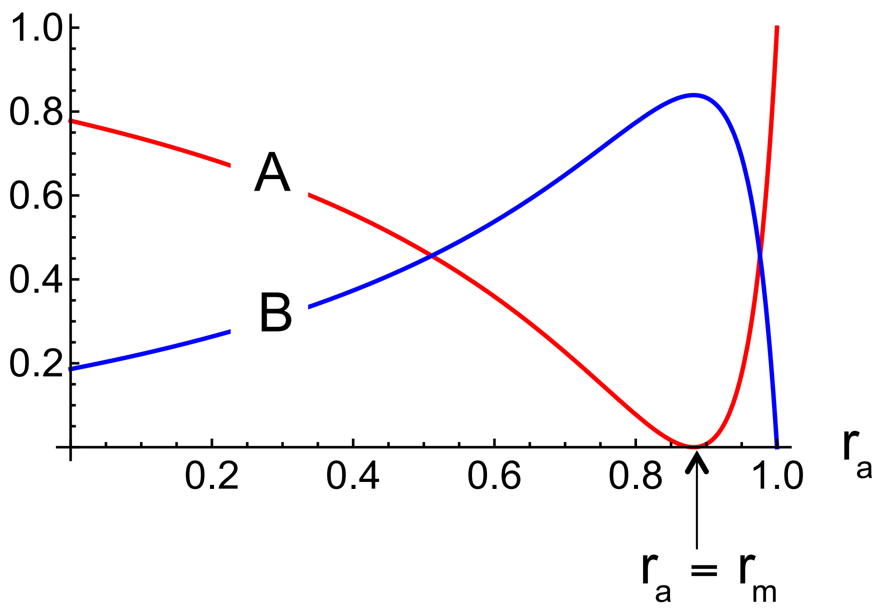
A = resonant power reflectivity

B = resonant power transmission

$$A + B = 1 \text{ if } t = 1$$

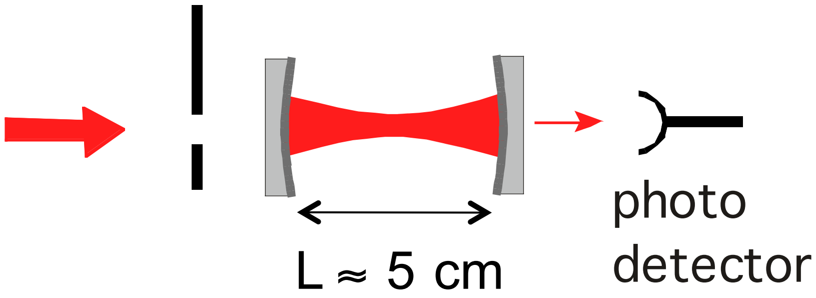
Impedance matching: $$r_a = r_m \Rightarrow A = 0\,,\; B = t^2\frac{\left(1-r_b^2\right)}{\left(1-r_m^2\right)}\,,\; F = \frac{4r_m^2}{\left(1-r_m^2\right)^2}$$

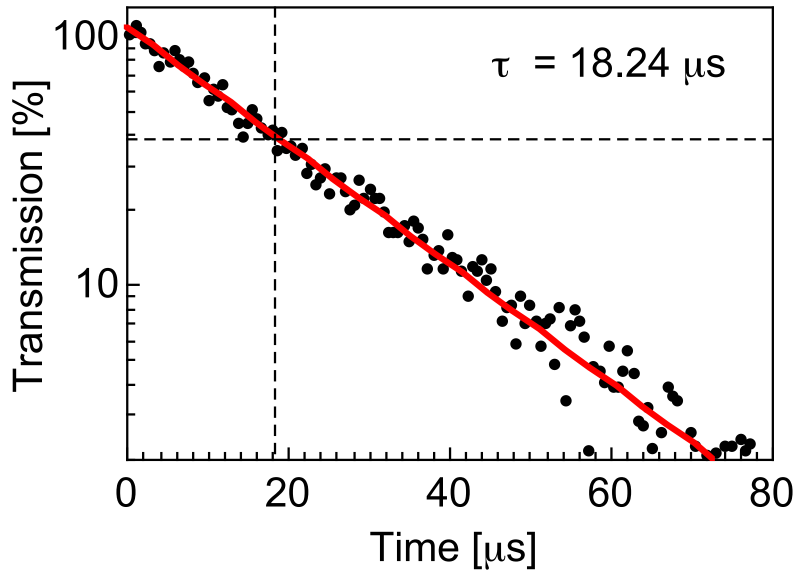
$$\Phi_t = \Phi_r = \arctan\left[\frac{1-r_m^2}{1+r_m^2}\tan(\phi)\right]$$



$$\boldsymbol{F} \xrightarrow[F\to\infty]{} \pi\frac{r_m}{1-r_m^2}$$

Photon storage in a high finesse cavity





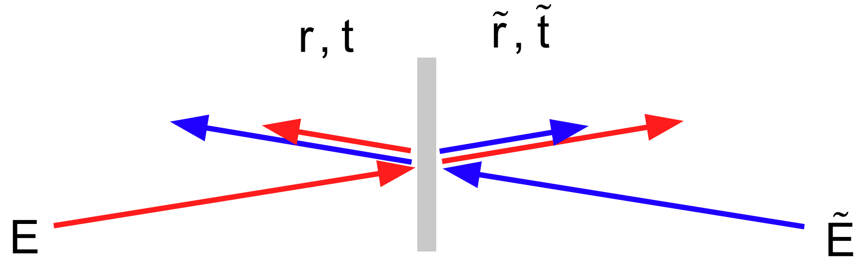
$\mathcal{F} = 340.000\,,\ \delta\nu_{FSR} = 3\,\mathrm{GHz}$

$$\frac{dI}{dt} = -\frac{1}{\tau} I\,,\quad \tau = \text{ Intra-cavity photon lifetime}$$

$$\Rightarrow E(t) = E_0\, e^{\left(i\omega_0 - \frac{1}{2\tau}\right)} \Rightarrow \hat{E}(\omega) \sim \frac{1}{i\left(\omega_0 - \omega\right) - \frac{1}{2\tau}}$$

$$\Rightarrow \left|\hat{E}(\omega)\right|^2 \sim \frac{1}{4\tau^2\left(\omega_0 - \omega\right)^2 + 1} \Rightarrow \text{FWHM-width} \quad \delta\omega = \frac{1}{\tau} = \frac{\delta\omega_{FSR}}{\mathcal{F}}$$

General phase relations for beam splitters



$$\left|tE+\tilde{r}\tilde{E}\right|^2+\left|rE+\tilde{t}\tilde{E}\right|^2=\left|E\right|^2+\left|\tilde{E}\right|^2$$

1) $E=0$ or $\tilde{E}=0 \quad \Rightarrow \quad \left|t\right|^2+\left|r\right|^2=\left|\tilde{r}\right|^2+\left|\tilde{t}\right|^2=1$

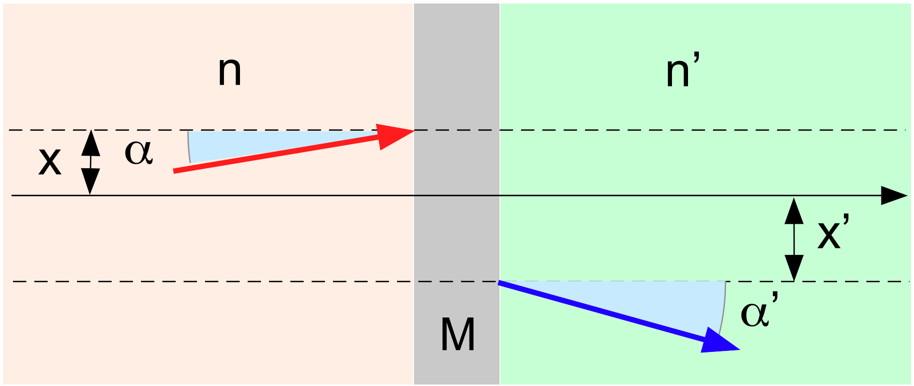
2) $E=\tilde{E}$ or $iE=\tilde{E} \quad \Rightarrow \quad \begin{cases} t\tilde{r}^*+\tilde{t}^*r=0 \\ \left|r\right|=\left|\tilde{r}\right|, \ \left|t\right|=\left|\tilde{t}\right| \end{cases}$

3) $r\equiv\left|r\right|e^{i\phi}, \tilde{r}\equiv\left|\tilde{r}\right|e^{i\tilde{\phi}}, t\equiv\left|t\right|e^{i\psi}, \tilde{t}\equiv\left|\tilde{t}\right|e^{i\tilde{\psi}} \underbrace{\Rightarrow}_{1),2)} e^{i(\phi+\tilde{\phi})}+e^{i(\psi+\tilde{\psi})}=0$

4) $t\tilde{t}-r\tilde{r}=-e^{i(\phi+\tilde{\phi})}=e^{i(\psi+\tilde{\psi})}$

Symmetric beam splitter: $\quad r=\tilde{r}=\left|r\right|e^{i\phi}, t=\tilde{t}=\left|t\right|e^{i\psi} \Rightarrow \phi+\psi=\pi$

Paraxial ray optics



$$\begin{pmatrix} n'\alpha' \\ x' \end{pmatrix} = \begin{pmatrix} f_1(n\alpha, x) \\ f_2(n\alpha, x) \end{pmatrix}$$

Paraxial approximation α, α', x, x' small such that

$$\begin{pmatrix} n'\alpha' \\ x' \end{pmatrix} = \underbrace{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}_{M} \begin{pmatrix} n\alpha \\ x \end{pmatrix}$$

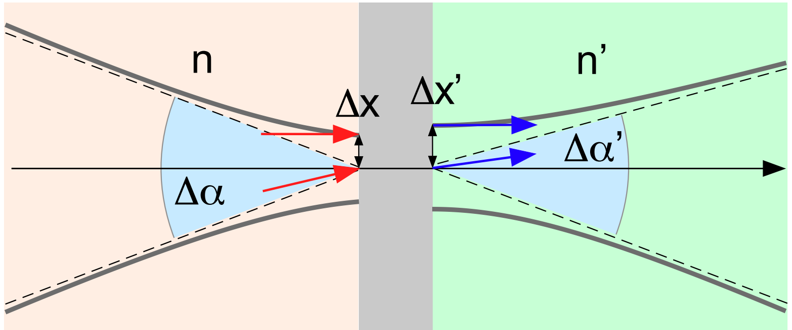
We show now that generally $|\det(M)| = 1$

We first consider the case C=0: $\tilde{M} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \rightarrow \begin{matrix} n'\alpha' = An\alpha + Bx \\ x' = D\,x \end{matrix}$

Consider a locally confined light beam irradiating the system at x=0 along the optical axis. Describe this beam by a superposition of travelling waves with

wavevectors $nk\begin{pmatrix}\alpha \\ 1\end{pmatrix}$ with angles α distributed according to a Gaussian

around $\alpha = 0$ with a width $\Delta\alpha \ll 1$.



Hence

$$E \propto \frac{1}{\sqrt{2\pi}} \int d\alpha\, e^{-\left(\frac{\alpha}{\Delta\alpha}\right)^2} e^{ink\alpha x} e^{inkz}$$

$$= \frac{k\Delta\alpha}{\sqrt{2}} e^{-\left(\frac{x}{\Delta x}\right)^2} e^{inkz} \quad \text{with} \quad \Delta x = \frac{2}{nk\Delta\alpha} \qquad (*)$$

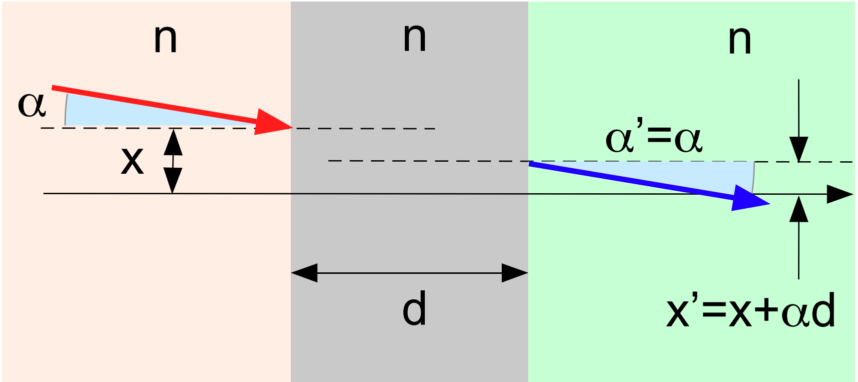
Transmission through the system $\tilde{M}$ at x=0: $n'\Delta\alpha' = n\Delta\alpha A$ (**)

and hence with (*),(**): $\Delta x' \underset{(*)}{=} \frac{2}{n'k\Delta\alpha'} \underset{(**)}{=} \frac{2}{nk\Delta\alpha A}$ (I)

Transmission through Δx : $\Delta x' = D\Delta x \underset{(*)}{=} D\frac{2}{nk\Delta\alpha}$ (II)

(I),(II) $\Rightarrow AD = 1$ and hence $\left|\det(\tilde{M})\right| = 1$

Translation



$$T_d \equiv \begin{pmatrix} 1 & 0 \\ \frac{d}{n} & 1 \end{pmatrix}, \quad \det(T_d) = 1$$

One may transform an arbitrary matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ via a translation T into a matrix of the form $\tilde{M} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ 0 & \tilde{D} \end{pmatrix}$

$$\underbrace{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}_{M} \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ -\frac{C}{D} & 1 \end{pmatrix}}_{T} = \underbrace{\begin{pmatrix} A - \frac{C}{D}B & B \\ 0 & D \end{pmatrix}}_{\tilde{M}}$$

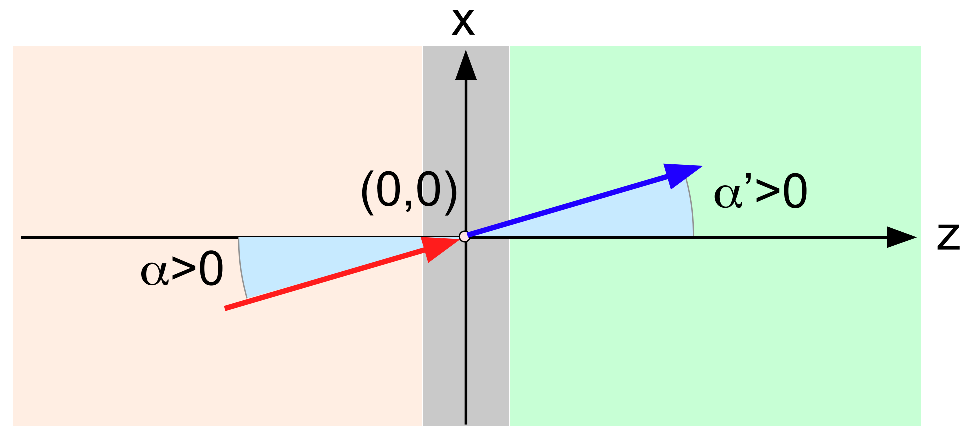
With $\left|\det(\tilde{M})\right| = 1$ follows $\left|\det(M)\right| = 1$

Conventions in paraxial optics

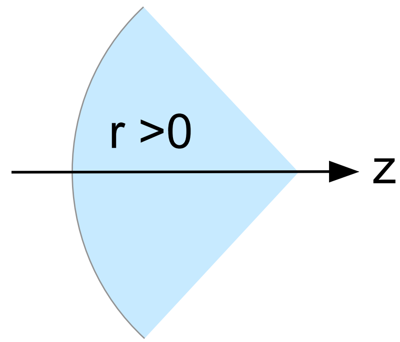
horizontal distances:

vertical distances:

angles:

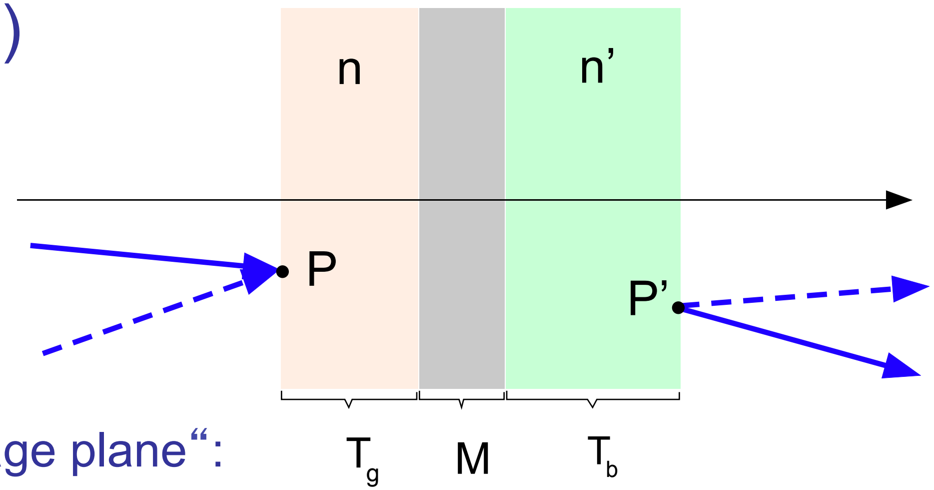


Curvature radii:



Refracting boundaries: indicate in propagation direction of light (z-direction)

Conjugate planes (imaging)



Definition of „object plane“ and „image plane“:

For any point P in the object plane there is a point P‘ in the image plane such that all rays through P intersect P‘.

Hence, for $\tilde{M} \equiv T_b \circ M \circ T_g = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}$ this is $0 = \dfrac{\partial x'}{\partial \alpha} = \tilde{C}$

One calculates

$$\tilde{M} = \begin{pmatrix} A + \frac{g}{n} B & B \\ \frac{b}{n'} A + \frac{g}{n} D + \frac{bg}{n'n} B + C & \frac{b}{n'} B + D \end{pmatrix} \quad \text{with} \quad M \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and hence

$$0 = \tilde{C} \Rightarrow 0 = \frac{b}{n'} A + \frac{g}{n} D + \frac{bg}{n'n} B + C$$

One defines

angle magnification $m_\alpha \equiv \frac{\partial \alpha'}{\partial \alpha} = \frac{n}{n'}\tilde{A} = \frac{n}{n'}A + \frac{g}{n'}B$

lateral magnification $m_x \equiv \frac{\partial x'}{\partial x} = \tilde{D} = \frac{b}{n'}B + D$

$$\det(\tilde{M}) = 1 \quad \Rightarrow \quad m_\alpha m_x = \frac{n}{n'}\tilde{A}\tilde{D} = \frac{n}{n'}\left(1 - \tilde{B}\tilde{C}\right) = \frac{n}{n'}$$

Principle planes

For arbitrary $M \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $\det(M) = 1$ define $p \equiv \frac{n}{B}(1 - A),\ p' \equiv \frac{n'}{B}(1 - D)$

Hence $$\tilde{M} \equiv T_{p'} \circ M \circ T_p = \begin{pmatrix} A + \frac{p}{n}B & B \\ \frac{p'}{n'}A + \frac{p}{n}D + \frac{p'p}{n'n}B + C & \frac{p'}{n'}B + D \end{pmatrix} = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}$$

Focal planes

For arbitrary $M \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $\det(M) = 1$ define $f \equiv -\frac{A}{B}n,\ f' \equiv -\frac{D}{B}n'$

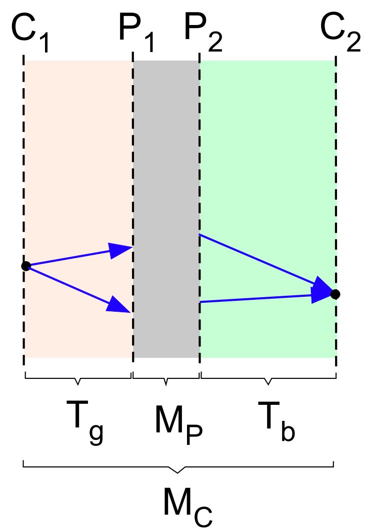
Hence $$\tilde{M} \equiv T_{f'} \circ M \circ T_f = \begin{pmatrix} A + \frac{f}{n}B & B \\ \frac{f'}{n'}A + \frac{f}{n}D + \frac{f'f}{n'n}B + C & \frac{f'}{n'}B + D \end{pmatrix} = \begin{pmatrix} 0 & B \\ -\frac{1}{B} & 0 \end{pmatrix}$$

consequently $\alpha = 0 \Rightarrow x' = 0$ and $x = 0 \Rightarrow \alpha' = 0$

Treatment of general systems $M \equiv M_N \circ M_{N-1} \circ \ldots \circ M_1$

One determines the principle planes. With respect to these planes the system is represented by a matrix of the form

$$M_P = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & -P \\ 0 & 1 \end{pmatrix}$$

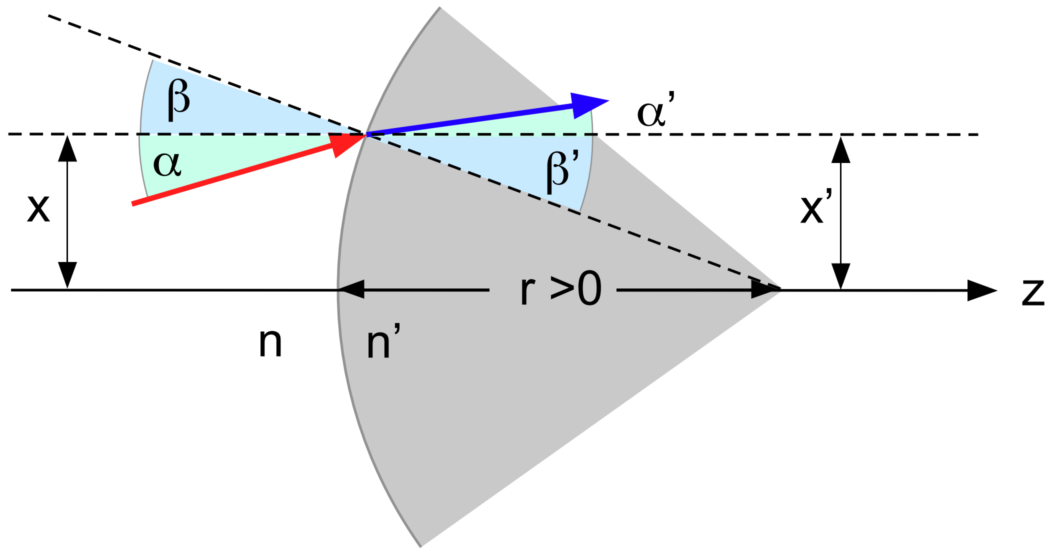


For conjugate planes one gets

$$M_C = \begin{pmatrix} A + \frac{g}{n}B & B \\ \frac{b}{n'}A + \frac{g}{n}D + \frac{bg}{n'n}B + C & \frac{b}{n'}B + D \end{pmatrix} = \begin{pmatrix} 1 - \frac{g}{n}P & -P \\ 0 & 1 - \frac{b}{n'}P \end{pmatrix}$$

$$\frac{b}{n'}A + \frac{g}{n}D + \frac{bg}{n'n}B + C = 0 \quad \Rightarrow \quad \frac{n}{g} + \frac{n'}{b} = P$$

Example: spherical dielectric boundary



Gaussian approximation: $\beta = \beta' = \dfrac{x}{r},\ n(\alpha+\beta) = n'(\alpha'+\beta')$ (Snell)

$$\Rightarrow \quad n'\alpha' = n\alpha + \frac{(n-n')}{r}x$$

$$x' = x$$

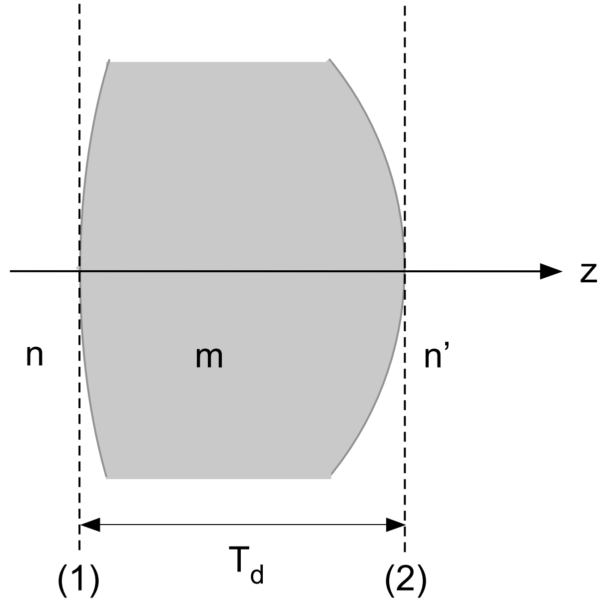
$$\Rightarrow \quad M = \begin{pmatrix} 1 & -K \\ 0 & 1 \end{pmatrix},\ K \equiv \frac{n'-n}{r}$$ = refractive power

Example: thick lens

$$M \equiv M_2 \circ T_d \circ M_1 \,, \quad M_\nu \equiv \begin{pmatrix} 1 & -K_\nu \\ 0 & 1 \end{pmatrix}$$

$$K_1 = \frac{m-n}{r_1}, K_2 = \frac{n'-m}{r_2}, \quad T_d \equiv \begin{pmatrix} 1 & 0 \\ \frac{d}{m} & 1 \end{pmatrix}$$

$$\Rightarrow \quad M = \begin{pmatrix} 1 - K_2 \frac{d}{m} & -K_1 - K_2 + K_1 K_2 \frac{d}{m} \\ \frac{d}{m} & 1 - K_1 \frac{d}{m} \end{pmatrix}$$



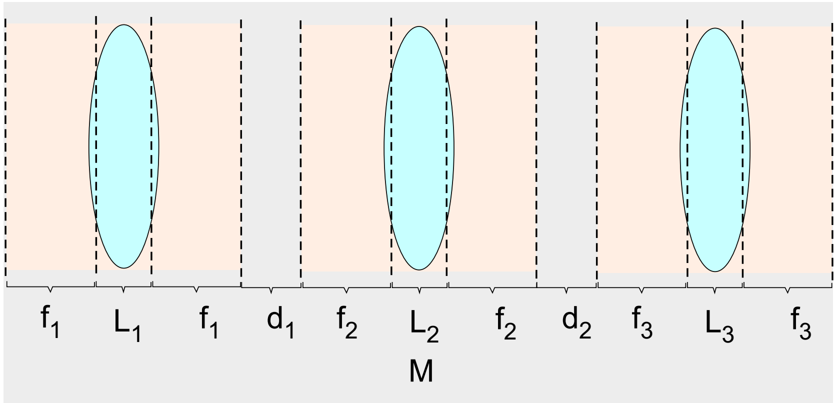
Principle planes:

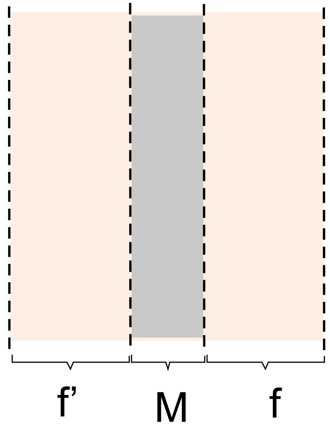
$$g = \frac{nK_2 \frac{d}{m}}{-K_1 - K_2 + K_1 K_2 \frac{d}{m}} \,, \quad b = \frac{n'K_1 \frac{d}{m}}{-K_1 - K_2 + K_1 K_2 \frac{d}{m}}$$

Focal planes:

$$f = \frac{\left(K_2 \frac{d}{m} - 1\right) n}{-K_1 - K_2 + K_1 K_2 \frac{d}{m}} \,, \quad f' = \frac{\left(K_1 \frac{d}{m} - 1\right) n'}{-K_1 - K_2 + K_1 K_2 \frac{d}{m}}$$

Example: zoom lens





$$M = \begin{pmatrix} 0 & -\frac{1}{f_3} \\ f_3 & 0 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ d_2 & 1 \end{pmatrix} \circ \begin{pmatrix} 0 & -\frac{1}{f_2} \\ f_2 & 0 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ d_1 & 1 \end{pmatrix} \circ \begin{pmatrix} 0 & -\frac{1}{f_1} \\ f_1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{d_2 f_1}{f_2 f_3} & \frac{f_2^{\,2} - d_1 d_2}{f_1 f_2 f_3} \\ -\frac{f_1 f_3}{f_2} & \frac{d_1 f_3}{f_1 f_2} \end{pmatrix}$$

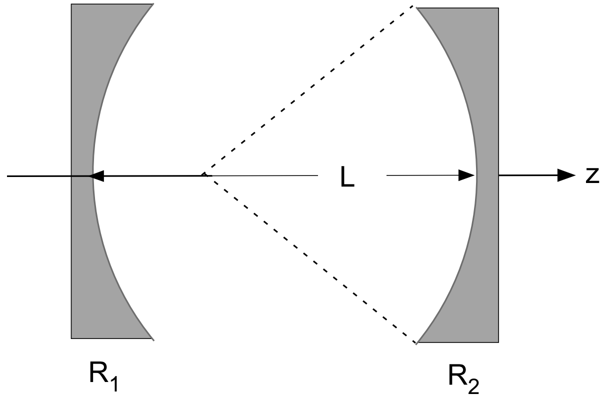
$$f = -\frac{A}{B} = \frac{-d_2 f_1^2}{f_2^2 - d_1 d_2}, \quad f' = -\frac{D}{B} = \frac{-d_1 f_3^2}{f_2^2 - d_1 d_2}$$

Special case: $f_1 = f_3 \equiv f_0, \; -d_2 = d_1 \equiv d \quad f + f' = 0, K = -B = \frac{d^2 - f_2^2}{f_2 f_0^2}, \; K_{max} = \frac{-f_2}{f_0^2}$

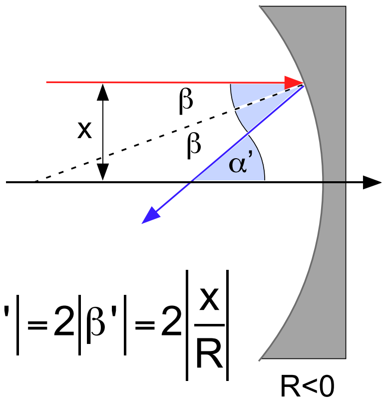
$$f_2 = -10 \text{ mm}, \; f_0 = 50 \text{ mm} \quad \Rightarrow \quad K_{max}^{-1} = 250 \text{ mm}$$

Paraxial resonators

Resonators can be described by
a periodic array of lenses



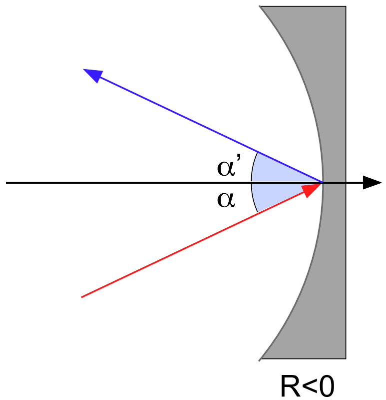
Single spherical mirror:



$$|\alpha'| = 2|\beta'| = 2\left|\frac{x}{R}\right|$$

$$\begin{pmatrix}\frac{2}{R}\\1\end{pmatrix}x = \begin{pmatrix}\alpha'\\x'\end{pmatrix} = \begin{pmatrix}A & B\\C & D\end{pmatrix}\begin{pmatrix}0\\x\end{pmatrix} = \begin{pmatrix}B\\D\end{pmatrix}x$$

$$\Rightarrow\ B = \frac{2}{R}, D = 1$$



$$\begin{pmatrix}\alpha\\0\end{pmatrix} = \begin{pmatrix}\alpha'\\0\end{pmatrix} = \begin{pmatrix}A & B\\C & D\end{pmatrix}\begin{pmatrix}\alpha\\0\end{pmatrix} = \begin{pmatrix}A\\C\end{pmatrix}\alpha$$

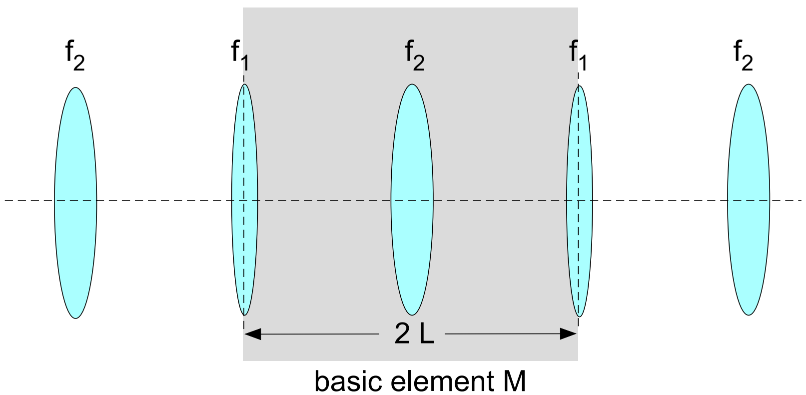
$$\Rightarrow\ A = 1, C = 0$$

hence obtain: $$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{f} \\ 0 & 1 \end{pmatrix}, \quad f = -\frac{R}{2}$$

Spherical mirror acts as thin lens with focal ength $f = -\frac{R}{2}$

Resonator may be replaced by periodic array of mirrors

$$M = \begin{pmatrix} 1 & -\frac{1}{R_1} \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ L & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & -\frac{2}{R_2} \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ L & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & -\frac{1}{R_1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2g_1g_2 - 1 & \frac{2}{L} g_1 (g_1 g_2 - 1) \\ 2Lg_2 & 2g_1g_2 - 1 \end{pmatrix}$$



$$g_\nu \equiv 1 - \frac{L}{R_\nu}, \quad \det(M) = 1$$

Stability of paraxial resonators

Consider eigenvalue problem: $M \cdot \xi = \lambda \xi$ eigenvalues: $\lambda_\pm = \frac{m}{2} \pm \sqrt{\frac{m^2}{4} - 1}$

$$0 = \det(M - \lambda 1) = 1 - \lambda \underbrace{\mathrm{trace}(M)}_{m} + \lambda^2$$

eigenvectors: $\xi_\pm$

Generally $\lambda_\pm$ are complex and, if $\xi_\pm$ linearly independent, every vector may be composed as $\xi = c_+ \xi_+ + c_- \xi_-$

case 1: discriminante negative or zero $\left|\frac{m}{2}\right| \le 1,\ \lambda_\pm = \frac{m}{2} \pm i\sqrt{1 - \frac{m^2}{4}}$

$$\Rightarrow |\lambda_\pm| = 1,\ \lambda_\pm = e^{\pm i\theta}, \cos(\theta) = \frac{m}{2}$$

For arbitrary light rays $\xi = c_+ \xi_+ + c_- \xi_-$ after n roundtrips: $M^n \xi = c_+ e^{in\theta} \xi_+ + c_- e^{-in\theta} \xi_-$

and hence $\left|M^n \xi\right| = \left|c_+ e^{in\theta} \xi_+ + c_- e^{-in\theta} \xi_-\right| \le \left|c_+ \xi_+\right| + \left|c_- \xi_-\right|$ bounded independent of n

$\Rightarrow$ Resonator is stable

case 2: discriminante positive

$\Rightarrow \left|\frac{m}{2}\right| > 1 \Rightarrow |\lambda_+| > 1$ or $|\lambda_-| > 1$ resonator is instable

Stability of resonator with two mirrors

$$|2g_1g_2 - 1| = \left|\frac{m}{2}\right| \le 1 \quad \Rightarrow \quad 0 \le g_1 g_2 \le 1$$

1. Concentric resonator $R_1 = R_2 = \frac{L}{2}$

2. Confocal resonator $R_1 = R_2 = L$

3. Plane resonator $R_1 = R_2 = \infty$