Lectures on classical optics

Part II

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Contents

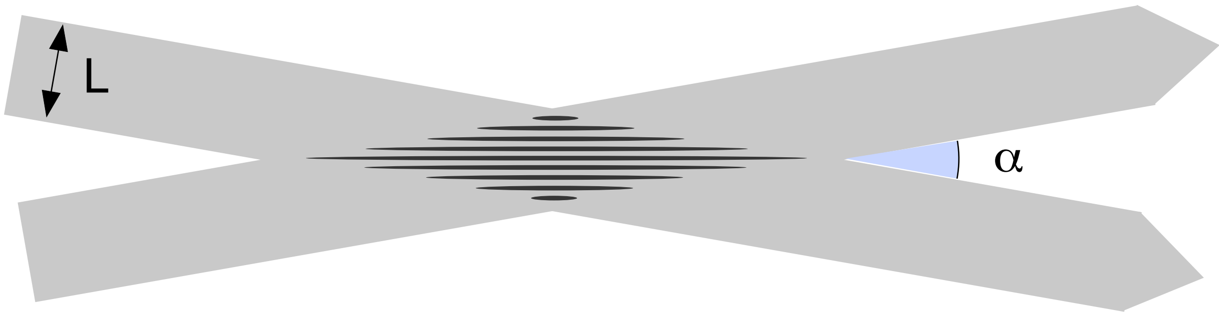
Light beams

Plane waves: parallel infinitely extending wave fronts

Laterally localized beams: must diverge

Assume laterally localized beam, which does not diverge

Choose α sufficiently small, such that the interference pattern in the intersection region exhibits a periodicity >> beam diameter L.
Hence, the intersection region may be chosen to be completely dark, which is incompatible with energy conservation $\nabla S = 0$.

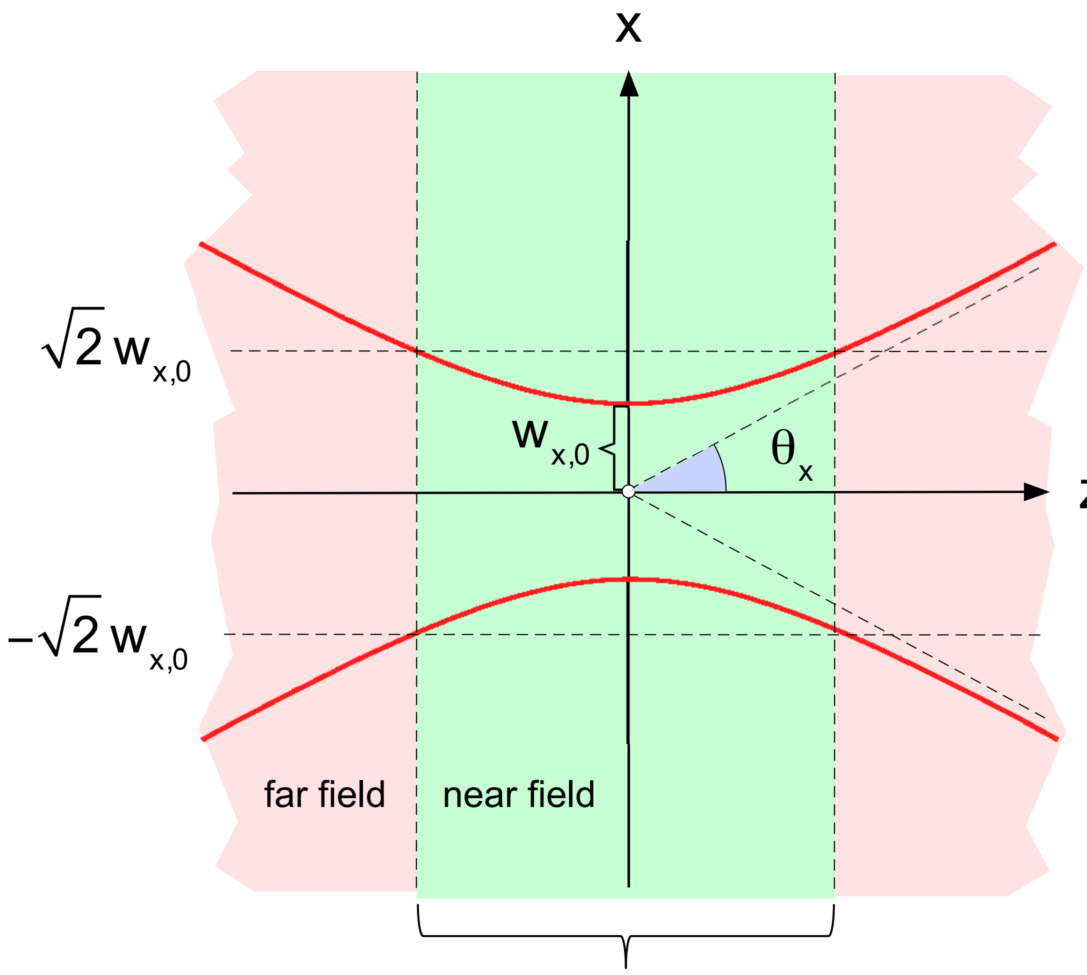


Distance between interference maxima $d = \frac{\lambda}{\alpha}$.

Energy conservation requires divergence angle θ such that $\forall \;\; L < d \;\; : \;\; \alpha = \frac{\lambda}{d} < \theta$

and hence $\lambda / L < \theta$

An important class of confined beams: Gaussian beams



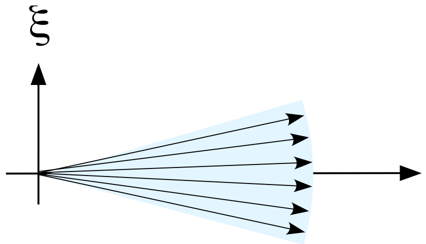
$$w_x(z) \equiv w_{x,0}\sqrt{1+\left(\frac{2z}{k\,w_{x,0}^2}\right)^2}$$

$$\tan(\theta_x) \equiv \lim_{z\to\infty}\frac{w_x(z)}{z} = \frac{2}{k\,w_{x,0}} = \frac{\lambda}{\pi\,w_{x,0}}$$

$b_x \equiv k\,w_{x,0}^2$ confocal parameter

Gaussian beam as a superposition of plane waves

$$\psi(x,y,z) \equiv \frac{1}{\pi\Delta\xi\Delta\eta}\int d\xi\, d\eta\; e^{-i(k_z(\xi,\eta)z+\xi x+\eta y)}\; e^{-\left(\frac{\xi}{\Delta\xi}\right)^2} e^{-\left(\frac{\eta}{\Delta\eta}\right)^2}$$



paraxial approximation: $k_z = \sqrt{k^2-\xi^2-\eta^2} \approx k - \frac{1}{2k}\left(\xi^2+\eta^2\right)$

With $e^{-\left(\frac{\xi}{\Delta\xi}\right)^2} e^{i\frac{\xi^2 z}{2k}} = e^{-\xi^2\left(\frac{1}{\Delta\xi^2}-i\frac{z}{2k}\right)^2}$ obtain

$$= \frac{e^{-ikz}}{\pi\Delta\xi\Delta\eta}\int d\xi\, d\eta\; e^{-i(\xi x+\eta y)}\; e^{-\left(\frac{\xi}{\Delta\tilde{\xi}}\right)^2} e^{-\left(\frac{\eta}{\Delta\tilde{\eta}}\right)^2}$$

with $\frac{1}{\Delta\tilde{\xi}^2} = \frac{1}{\Delta\xi^2} - i\frac{z}{2k}$, $\frac{1}{\Delta\tilde{\eta}^2} = \frac{1}{\Delta\eta^2} - i\frac{z}{2k}$

$$= \frac{\Delta\tilde{\xi}\,\Delta\tilde{\eta}}{\Delta\xi\,\Delta\eta}\, e^{-ikz}\, e^{-\left(\frac{x}{\Delta\tilde{x}}\right)^2} e^{-\left(\frac{y}{\Delta\tilde{y}}\right)^2}, \quad \Delta\tilde{x} \equiv \frac{2}{\Delta\tilde{\xi}}, \quad \Delta\tilde{y} \equiv \frac{2}{\Delta\tilde{\eta}}$$

use $$\int_{-\infty}^{\infty} d\xi\; e^{-i\xi x}\, e^{-(a\xi)^2} = \frac{\sqrt{\pi}}{a}\, e^{-\left(\frac{x}{2a}\right)^2}$$

Statement:

$$\Rightarrow \qquad \psi(x,y,z) \equiv \sqrt{\frac{w_{x,0}w_{y,0}}{w_x w_y}}\; e^{-i\left[kz-\frac{1}{2}\left(\Phi_x+\Phi_y\right)+\frac{k}{2}\left(\frac{x^2}{R_x}+\frac{y^2}{R_y}\right)\right]}\; e^{-\left(\frac{x^2}{w_x{}^2}+\frac{y^2}{w_y{}^2}\right)}$$

with abbreviations

$$w_x(z) \equiv w_{x,0}\sqrt{1+\left(\frac{2z}{k\,w_{x,0}{}^2}\right)^2}\;,\; w_y(z) \equiv w_{y,0}\sqrt{1+\left(\frac{2z}{k\,w_{y,0}{}^2}\right)^2}$$

beam diameters

$$w_{x,0} \equiv \frac{2}{\Delta\xi}\;,\quad w_{y,0} \equiv \frac{2}{\Delta\eta}$$

$$R_x(z) \equiv z\left[1+\left(\frac{k\,w_{x,0}{}^2}{2z}\right)^2\right],\; R_y(z) \equiv z\left[1+\left(\frac{k\,w_{y,0}{}^2}{2z}\right)^2\right]$$

curvature radii

$$\Phi_x(z) \equiv \arctan\left(\frac{2z}{kw_{x,0}^2}\right),\; \Phi_y(z) \equiv \arctan\left(\frac{2z}{kw_{y,0}^2}\right)$$

Guoy phases

Louis Georges Guoy
1854-1926

Proof

A. We show that $\frac{\Delta\tilde{\xi}}{\Delta\xi} = \sqrt{\frac{w_{x,0}}{w_x(z)}}\, e^{i\frac{1}{2}\Phi_x(z)}$

$$\frac{1}{\Delta\tilde{\xi}^2} = \frac{1}{\Delta\xi^2} - i\frac{z}{2k}\,,\quad w_{x,0} \equiv \frac{2}{\Delta\xi} \quad \Rightarrow$$

$$\frac{\Delta\xi^2}{\Delta\tilde{\xi}^2} = 1 - i\Delta\xi^2\frac{z}{2k} = 1 - i\frac{4}{w_{x,0}^{\;2}}\frac{z}{2k}$$

$$\frac{\Delta\tilde{\xi}^2}{\Delta\xi^2} = \frac{1 + i\frac{4}{w_{x,0}^{\;2}}\frac{z}{2k}}{1 + \left(\frac{4}{w_{x,0}^{\;2}}\frac{z}{2k}\right)^2} = \frac{e^{i\arctan\left(\frac{4}{w_{x,0}^{\;2}}\frac{z}{2k}\right)}}{\sqrt{1 + \left(\frac{4}{w_{x,0}^{\;2}}\frac{z}{2k}\right)^2}} = \frac{w_{x,0}}{w_x(z)}\, e^{i\Phi_x(z)}$$

use $c = \sqrt{\mathrm{Re}(c)^2 + \mathrm{Im}(c)^2}\, e^{i\arctan\left(\frac{\mathrm{Im}(c)}{\mathrm{Re}(c)}\right)}$

$$\frac{\Delta\tilde{\xi}}{\Delta\xi} = \sqrt{\frac{w_{x,0}}{w_x(z)}}\, e^{i\frac{1}{2}\Phi_x(z)}$$

B. We show that $e^{-\left(\frac{x}{\Delta\tilde{x}}\right)^2} = e^{-x^2\left[\frac{1}{w_x^2}+i\frac{k}{2R_x}\right]}$

$$(*)\ \Delta\tilde{\xi}^2 \underset{A.}{=} \Delta\xi^2 \frac{1+i\frac{4}{w_{x,0}{}^2}\frac{z}{2k}}{1+\left(\frac{4}{w_{x,0}{}^2}\frac{z}{2k}\right)^2} = \frac{4}{w_{x,0}{}^2\left(1+\left(\frac{2z}{k\,w_{x,0}{}^2}\right)^2\right)} + \frac{i2k}{z\left[1+\left(\frac{w_{x,0}{}^2}{2}\frac{k}{z}\right)^2\right]} = \frac{4}{w_x{}^2}+i\frac{2k}{R_x(z)}$$

$$\uparrow\quad \Delta\xi^2 = \frac{4}{w_{x,0}{}^2}$$

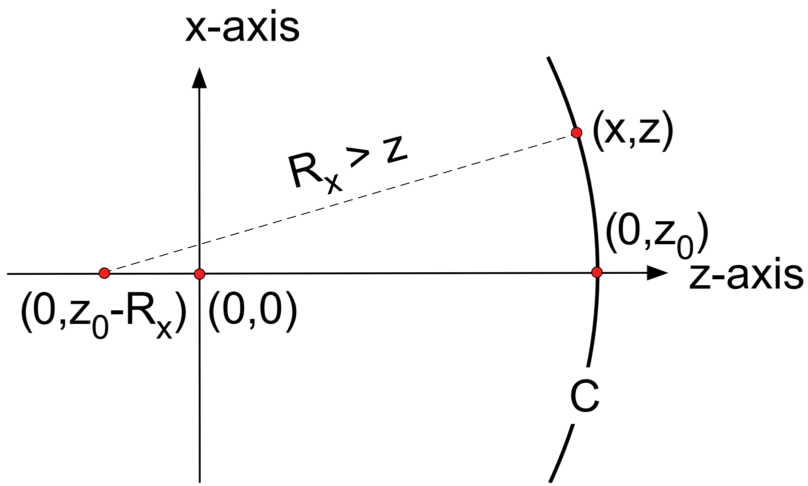
$$\Delta\tilde{x} = \frac{2}{\Delta\tilde{\xi}} \Rightarrow \quad e^{-\left(\frac{x}{\Delta\tilde{x}}\right)^2} = e^{-\left(\frac{\Delta\tilde{\xi}x}{2}\right)^2} \underset{(*)}{=} e^{-\left(\frac{x}{2}\right)^2\left[\frac{4}{w_x{}^2}+i\frac{2k}{R_x}\right]} = e^{-x^2\left[\frac{1}{w_x{}^2}+i\frac{k}{2R_x}\right]}$$

A. & B. lead to

$$\frac{\Delta\tilde{\xi}}{\Delta\xi}\, e^{-ikz}\, e^{-\left(\frac{x}{\Delta\tilde{x}}\right)^2} = \sqrt{\frac{w_{x,0}}{w_x(z)}}\; e^{i\frac{1}{2}\Phi_x(z)}\; e^{-ikz}\; e^{-x^2\left[\frac{1}{w_x{}^2}+i\frac{k}{2R_x}\right]}$$

Consider spherical hypersurfaces in paraxial approximation

$$C: \left\{ (x,z) \mid \left(z-(z_0-R_x)\right)^2 + x^2 = R_x{}^2 \right\}$$



$$\left(z-(z_0-R_x)\right)^2 + x^2 = R_x{}^2$$

$$\Leftrightarrow \quad \left(z-z_0\right)^2 + 2\left(z-z_0\right)R_x + x^2 = 0$$

$$\underbrace{\Leftrightarrow}_{z_0 \approx z} \; 2\left(z-z_0\right)R_x + x^2 = 0 \quad \Leftrightarrow \quad kz + \frac{kx^2}{2R_x} = kz_0$$

$$\Leftrightarrow \quad e^{-i\left[kz + \frac{kx^2}{2R_x}\right]} = e^{-ikz_0} = \text{constant}$$

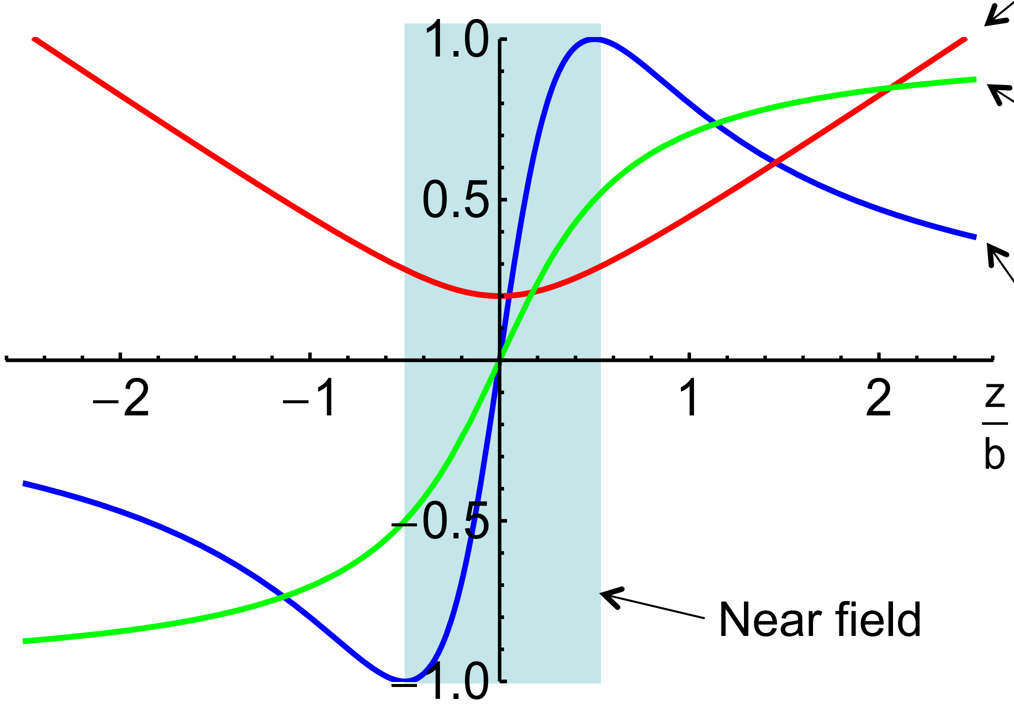
$$e^{-i\left[kz - \frac{1}{2}\left(\Phi_x + \Phi_y\right) + \frac{k}{2}\left(\frac{x^2}{R_x} + \frac{y^2}{R_y}\right)\right]}$$

$\Leftrightarrow$ **C is an area of constant phase**

Relative minimum of $R_x(z) = z + \frac{1}{z}\left(\frac{b_x}{2}\right)^2 \,, \; b_x \equiv k\, w_{x,0}{}^2$

$$0 = \frac{\partial}{\partial z} R_x(z) \quad \Rightarrow \quad z_{min} = \frac{b_x}{2} \,, \; R_{x,min} = R_x(z_{min}) = b_x$$

Gaussian beam parameters



beam radius

$$\frac{w(z)}{5\,w_0} = \sqrt{1+\left(\frac{2z}{b}\right)^2}$$

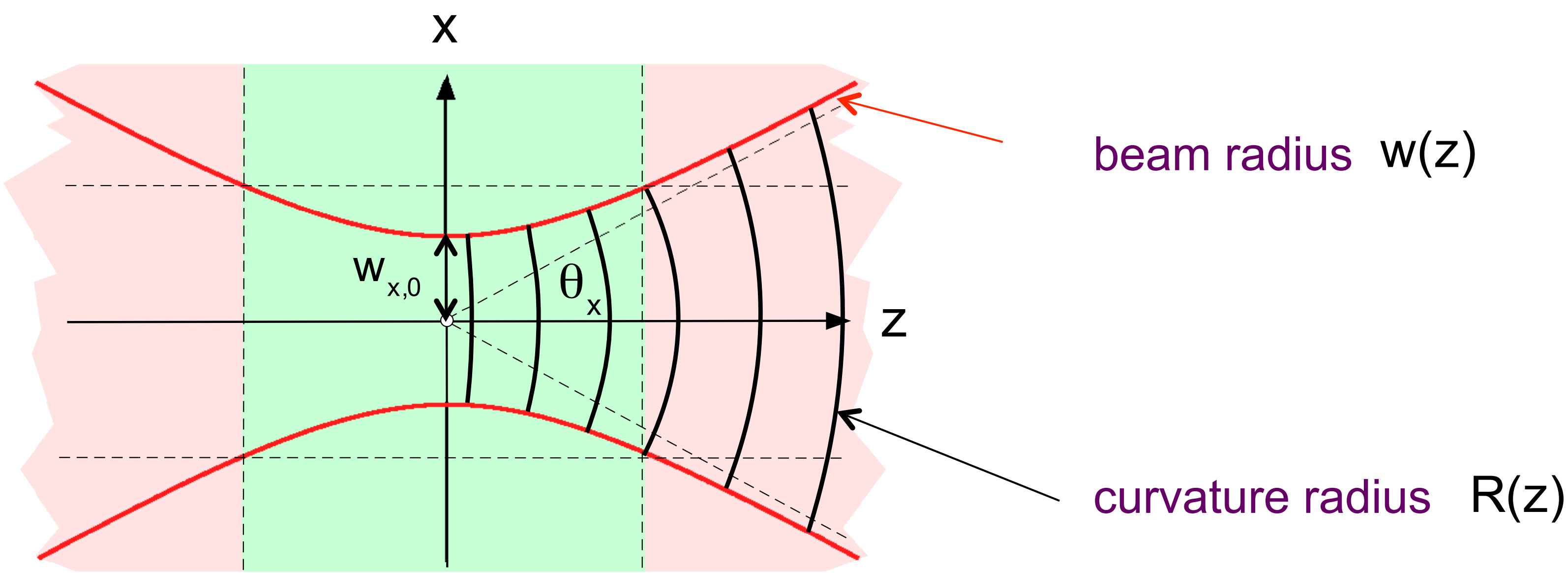
Guoy phase

$$\frac{2}{\pi}\Phi(z) \equiv \frac{2}{\pi}\arctan\left(\frac{2z}{b}\right)$$

Curvature

$$\frac{R_{min}}{R(z)} = \frac{4\,z\,b}{4z^2 + b^2}$$

Gaussian beam: beam radius and curvature radius



Near field

$$\tan(\theta_x) \equiv \lim_{z\to\infty} \frac{w_x(z)}{z} = \frac{2}{k\,w_{x,0}} = \frac{\lambda}{\pi\,w_{x,0}}$$

Paraxial wave equation

Helmholtz equation $\left(\Delta + k^2\right)\psi(x,y,z) = 0\,,\ \ \psi(x,y,z) \equiv u(x,y,z)\,e^{-ikz}$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}-2ik\frac{\partial u}{\partial z}=0$$

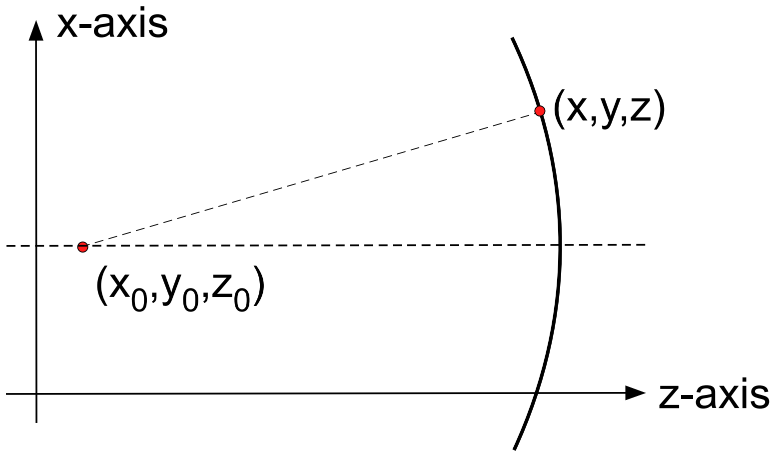
paraxial approximation: $\left|\frac{\partial^2 u}{\partial z^2}\right| \ll \left|k\frac{\partial u}{\partial z}\right|,\ \left|\frac{\partial^2 u}{\partial x^2}\right|,\ \left|\frac{\partial^2 u}{\partial y^2}\right|$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-2ik\frac{\partial u}{\partial z}=0$$ paraxial wave equation

Analogous equation in quantum mechanics: Schrödinger equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-i\hbar\frac{\partial u}{\partial t}=0$$

Paraxial spherical waves



Spherical wave $\psi \propto \dfrac{e^{-ik\left|\vec{r}-\vec{r}_0\right|}}{\left|\vec{r}-\vec{r}_0\right|}$

is an exact solution to wave equation

expand $\left|\vec{r}-\vec{r}_0\right| = z-z_0 + \dfrac{\left(x-x_0\right)^2 + \left(y-y_0\right)^2}{2\left(z-z_0\right)} + \cdots$

$$\psi \propto e^{-ik\left(z-z_0\right)}\, u(x,y,z)\,,\;\; u(x,y,z) = \frac{1}{z-z_0} e^{-ik\frac{\left(x-x_0\right)^2+\left(y-y_0\right)^2}{2\left(z-z_0\right)}}$$

is an exact solution of the paraxial wave equation (for arbitrary complex source points x_0, y_0, z_0)

Generalization for different curvature radii in x- and y-directions

$$u(x,y,z) = \frac{1}{\sqrt{z-z_x}\sqrt{z-z_y}}\, e^{-ik\frac{\left(x-x_0\right)^2}{2\left(z-z_x\right)}}\; e^{-ik\frac{\left(y-y_0\right)^2}{2\left(z-z_y\right)}}$$

Set $x_0 = y_0 = 0$ and choose complex source ccordinates

$$z_x = z_{x,0} - i\frac{k w_{x,0}^2}{2}, \quad z_y = z_{y,0} - i\frac{k w_{y,0}^2}{2}$$

define q-parameters $q_\nu \equiv z - z_\nu = z - z_{\nu,0} + i\frac{k w_{\nu,0}^2}{2}, \; \nu \in \{x,y\}$

$$\Rightarrow \quad u(x,y,z) = \frac{1}{\sqrt{q_x q_y}} e^{-ik\frac{x^2}{2q_x}} e^{-ik\frac{y^2}{2q_y}}$$

Astigmatic Gauss mode

use $$\frac{1}{q_\nu} = \frac{1}{z - z_\nu} = \frac{1}{R_\nu(z - z_{\nu,0})} - i\frac{2}{k w_\nu(z - z_{\nu,0})^2} = \frac{-2i}{k w_{\nu,0}{}^2} \frac{w_{\nu,0}}{w_\nu(z - z_{\nu,0})} e^{i\Phi_\nu(z - z_{\nu,0})} \quad (*)$$

cf. next page

to show that

$$\frac{1}{\sqrt{q_\nu}} e^{-ik\frac{x^2}{2q_\nu}} = \sqrt{\frac{-2i}{k w_{\nu,0}{}^2}} \sqrt{\frac{w_{\nu,0}}{w_\nu(z - z_{\nu,0})}} e^{i\frac{1}{2}\Phi_\nu(z - z_{\nu,0})} e^{-\frac{x^2}{w_\nu(z - z_{\nu,0})^2}} e^{-ik\frac{x^2}{2R_\nu(z - z_{\nu,0})}}$$

Résumé: Gauss beams arise as paraxial spherical waves with complex source coordinates

Proof of (*)

$$\frac{1}{q_\nu} = \frac{1}{z - z_{\nu,0} + i\,\frac{k\,w_{\nu,0}^{\;2}}{2}} = \frac{z - z_{\nu,0} - i\,\frac{k\,w_{\nu,0}^{\;2}}{2}}{\left(z - z_{\nu,0}\right)^2 + \left(\frac{k\,w_{\nu,0}^{\;2}}{2}\right)^2} = -i\,\frac{2}{k\,w_{\nu,0}^{\;2}}\,\frac{1 + i\frac{2\left(z - z_{\nu,0}\right)}{kw_{x,0}^{\;2}}}{1 + \left(\frac{2\left(z - z_{\nu,0}\right)}{kw_{x,0}^{\;2}}\right)^2}$$

$$= -i\,\frac{2}{k\,w_{\nu,0}^{\;2}}\,\frac{e^{i\arctan\left(\frac{2\left(z - z_{\nu,0}\right)}{kw_{x,0}^{\;2}}\right)}}{\sqrt{1 + \left(\frac{2\left(z - z_{\nu,0}\right)}{kw_{x,0}^{\;2}}\right)^2}} = -i\,\frac{2}{k\,w_{\nu,0}}\,\frac{e^{i\Phi\left(z - z_{\nu,0}\right)}}{w_\nu\left(z - z_{\nu,0}\right)}$$

$\uparrow$

use $c = \sqrt{\mathrm{Re}(c)^2 + \mathrm{Im}(c)^2}\; e^{i\arctan\left(\frac{\mathrm{Im}(c)}{\mathrm{Re}(c)}\right)}$

Basis for paraxial wave equation (Cartesian coordinates)

In the following we provide a basis system of orthonormal Gauss Hermite modes such that each solution of the paraxial wave equation arises as a superposition of basis members

$$u(x,y,z) \equiv f(x,z)\, g(y,z) \;\text{ and }\; f, g \;\text{ satisfy }\; \frac{\partial^2 f}{\partial x^2} - 2ik\,\frac{\partial f}{\partial z} = 0$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 2ik\,\frac{\partial u}{\partial z} = 0$$

Ansatz $\qquad f(x,z) \equiv A(q(z)) \cdot H\left(\frac{x}{p(z)}\right) \cdot e^{-i\frac{kx^2}{2q(z)}}$

$$\Rightarrow \quad 0 = \left(\frac{\partial^2 f}{\partial x^2} - 2ik\,\frac{\partial f}{\partial z}\right) = \frac{f(x,z)}{p(z)^2 H\left(\frac{x}{p(z)}\right)} \bullet$$

$$\left[H''\left(\frac{x}{p(z)}\right) - 2ikx\left(\frac{p(z)}{q(z)} - p'(z)\right)H'\left(\frac{x}{p(z)}\right) - ik\frac{p(z)^2}{q(z)}H\left(\frac{x}{p(z)}\right)\left(1 + 2q(z)q'(z)\,\frac{A'(q(z))}{A(q(z))} + i\frac{kx^2}{q(z)}\left(q'(z) - 1\right)\right)\right] \qquad (1)$$

Proposition: because of

$$0 = H''\left(\frac{x}{p(z)}\right) - 2ikx\left(\frac{p(z)}{q(z)} - p'(z)\right)H'\left(\frac{x}{p(z)}\right) - ik\frac{p(z)^2}{q(z)}H\left(\frac{x}{p(z)}\right)\left(1 + 2q(z)q'(z)\frac{A'(q(z))}{A(q(z))} + i\frac{kx^2}{q(z)}(q'(z)-1)\right) \qquad (1)$$

H satifies the Hermitian differential equation

$$H_n''\left(\frac{x}{p}\right) - \frac{2x}{p}H_n'\left(\frac{x}{p}\right) + 2nH_n\left(\frac{x}{p}\right) = 0 \qquad (2a)$$

$$\Leftrightarrow \quad p'(z) = \frac{p(z)}{q(z)} + \frac{i}{kp(z)} \qquad (2b)$$

$$A'(q(z))q'(z) = A(q(z))\left(\frac{in}{kp(z)^2} - \frac{1}{2q(z)} - i\frac{kx^2}{2q(z)^2}(q'(z)-1)\right) \qquad (2c)$$

Proof: equations (1) and (2a)

$$\Leftrightarrow \quad -\frac{2x}{p} = -2ikx\left(\frac{p(z)}{q(z)} - p'(z)\right)$$

$$-ik\frac{p(z)^2}{q(z)}\left(1 + 2q(z)q'(z)\frac{A'(q(z))}{A(q(z))} + i\frac{kx^2}{q(z)}(q'(z)-1)\right) = 2n$$

$\Leftrightarrow$ (2b) and (2c)

Define $q(z), p(z), R(z), w(z)$ by the relations

$$q(z) = z + q_0,\ q_0 = i\frac{kw_0^2}{2} = i\frac{b}{2},\ \frac{1}{q(z)} \equiv \frac{1}{R(z)} - i\frac{2}{kw(z)^2},\ p(z) \equiv \frac{1}{\sqrt{2}}w(z)$$

It follows: $q^*q = z^2 + \tfrac{1}{4}b^2,\ q^* - q = -ib$ (3a)

and hence: $p^2(z) = \tfrac{1}{2}w^2 = \tfrac{1}{2}w_0{}^2\left(1+\left(\tfrac{2z}{b}\right)^2\right) = \tfrac{2}{bk}\left(1+\left(\tfrac{2z}{b}\right)^2\right) = \tfrac{2}{ik}\dfrac{z^2+\frac{b^2}{4}}{-ib} = \tfrac{2}{ik}\dfrac{q^*q}{q^*-q}$ (3b)

It follows

$$q'(z) = 1 \quad \text{(4a)}$$

$$p'(z) = \frac{p(z)}{q(z)} + \frac{i}{kp(z)} \quad \text{(4b)}$$

Proof of (4b) : with (3b) obtain $2pp' = \dfrac{4z}{kb}$

furthermore

$$2p\left(\frac{p}{q} + \frac{i}{kp}\right) = \frac{2p^2}{q} + \frac{2i}{k} = \frac{4}{kb}\frac{z^2+\frac{1}{4}b^2}{z+i\frac{1}{2}b} + \frac{2i}{k}$$

$$= \frac{\frac{4}{kb}\left(z^2+\frac{1}{4}b^2\right)\left(z-i\frac{1}{2}b\right)+\frac{2i}{k}\left(z^2+\frac{1}{4}b^2\right)}{z^2+\frac{1}{4}b^2} = \frac{4z}{kb}$$

It follows with (2c), (3b) that

$$\frac{A'(q(z))}{A(q(z))} \overset{(2c)}{=} \frac{in}{k\,p(z)^2} - \frac{1}{2q(z)} \overset{(3b)}{=} \frac{n}{2q^*(z)} - \frac{n+1}{2q(z)}$$

and hence

$$d\ln(A(q(z))) = \frac{n}{2q^*(z)}dz - \frac{n+1}{2q(z)}dz = \frac{n}{2q^*(z)}dq^* - \frac{n+1}{2q(z)}dq$$

$$= d\left[\frac{n}{2}\ln(q^*(z)) - \frac{n+1}{2}\ln(q(z))\right] = d\left[\ln\left(\frac{q^{*\,n/2}}{q^{(n+1)/2}}\right)\right] \quad \Rightarrow \quad A = A_0\,\frac{q^{*n/2}}{q^{(n+1)/2}}$$

Together

$$f_n(x,z) \equiv \left(\frac{2}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^n n!\,w_{x,0}}} \left(\frac{q_{x,0}}{q_x(z)}\right)^{1/2} \left(\frac{q_{x,0}\,q_x^{\,*}(z)}{q_{x,0}^{\ \ *}\;q_x(z)}\right)^{n/2} H_n\left(\frac{\sqrt{2}\,x}{w_x(z)}\right) \cdot e^{-i\frac{kx^2}{2q_x(z)}}$$

$$\psi_{nm}(x,y,z) = f_n(x,z)\,f_m(y,z)\,e^{-ikz}$$

Normalization

$$\int dx\,dz\,f_n(x,z)\,f_m^*(x,z) = \delta_{nm}$$

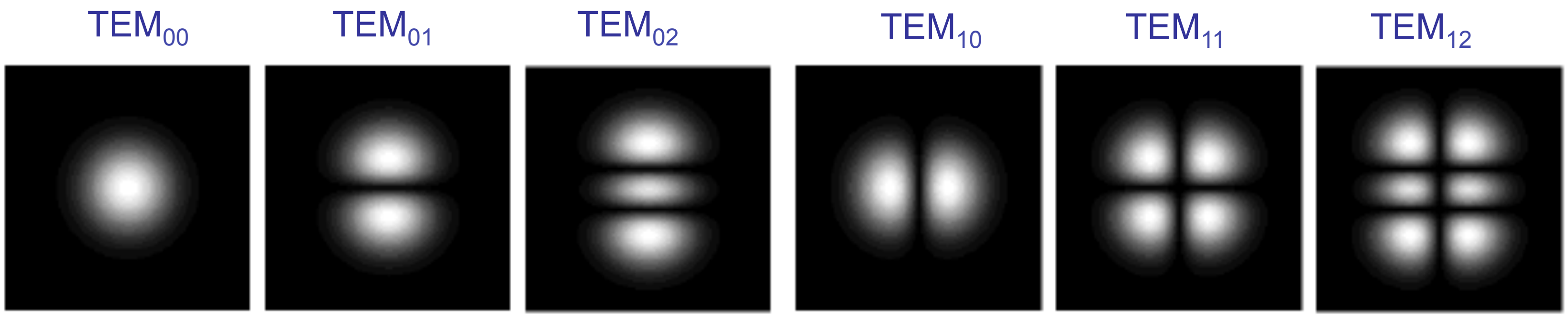
use $\left(\frac{q_{x,0}}{q_x(z)}\right)^{1/2}\left(\frac{q_{x,0}\,q_x^*(z)}{q_{x,0}^*\,q_x(z)}\right)^{n/2} = \sqrt{\frac{w_{\nu,0}}{w_\nu(z-z_{\nu,0})}}\;e^{i(n+1/2)\Phi_\nu(z-z_{\nu,0})}$ to obtain

$$f_n(x,z) = \left(\frac{2}{\pi}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}\,\frac{1}{\sqrt{w_\nu(z-z_{\nu,0})}}\,e^{i(n+1/2)\Phi_\nu(z-z_{\nu,0})}\,H_n\!\left(\frac{\sqrt{2}\,x}{w_x(z-z_{\nu,0})}\right)\cdot e^{-i\frac{kx^2}{2q_x(z-z_{\nu,0})}}$$

$z_{x,0} \neq z_{y,0} \Rightarrow$ astigmatic beam

$w_{x,0} \neq w_{y,0} \Rightarrow$ elliptic beam

Various Gauss-modes



Diffraction

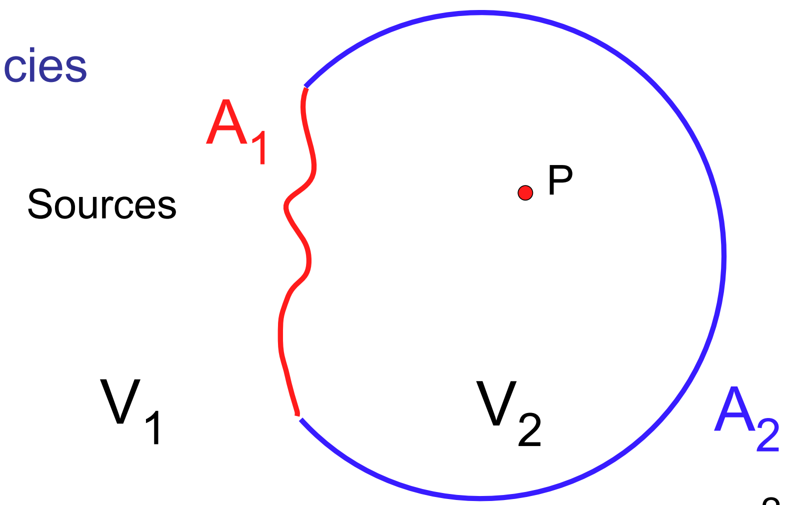
Outline of problem:

A volume V_2 is bounded by the surfaces A_1 and A_2. A_1 may, for example, represent a screen with some apertures. The electric field, originating from sources inside the volume V_1, is known on A_1 and on A_2.

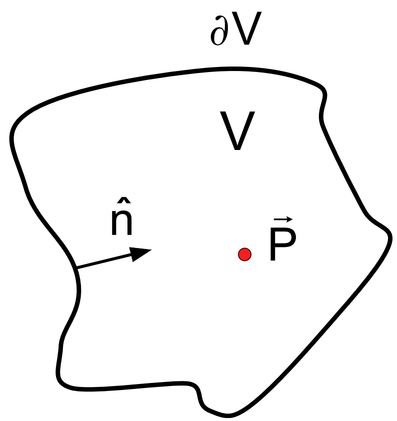
Determine field at point P

We apply a scalar theory: only the electric field interacts with the environment in a polarization-independent way. The magnetic field follows the electric field.

- good approximation for optical frequencies
- not true in the microwave domain



Derivation of Kirchhoff 's Integral



$$\left(\Delta + k^2\right)\psi(\vec{x}) = 0$$

$$\left(\Delta + k^2\right)G(\vec{x},\vec{p}) = \delta(\vec{x}-\vec{p})$$

$$\Rightarrow \quad \psi(\vec{p}) = \int_{\partial V}\left(\psi\frac{\partial}{\partial n}G - G\frac{\partial}{\partial n}\psi\right)dA \qquad (1)$$

Proof $\quad \psi(\vec{p}) \overset{(a)}{=} \int_V\left(\psi\Delta G - G\Delta\psi\right)dV \overset{(b)}{=} \int_{\partial V}\left(\psi\frac{\partial}{\partial n}G - G\frac{\partial}{\partial n}\psi\right)dA$

(a) follows with $\left(\Delta + k^2\right)\psi(\vec{x}) = 0,\ \left(\Delta + k^2\right)G(\vec{x},\vec{p}) = \delta(\vec{x}-\vec{p})$

$$\int_V\left(\psi\Delta G - G\Delta\psi\right)dV = \int_V\left(-\psi\, k^2 G + \psi\,\delta(\vec{x}-\vec{p}) + G\,k^2\psi\right)dV = \int_V \psi\,\delta(\vec{x}-\vec{p})\,dV = \psi(\vec{p})$$

(b) use Gauß-theorem for vector field $\psi\vec{\nabla}G - G\vec{\nabla}\psi$

$$\int_V\left(\psi\Delta G - G\Delta\psi\right)dV = \int_V \vec{\nabla}\left(\psi\vec{\nabla}G - G\vec{\nabla}\psi\right)dV = \int_{\partial V}\left(\psi\vec{\nabla}G - G\vec{\nabla}\psi\right)d\vec{A} = \int_{\partial V}\left(\psi\frac{\partial}{\partial n}G - G\frac{\partial}{\partial n}\psi\right)dA$$

The Greens-function for empty space: $G(\vec{x},\vec{p}) \equiv -\dfrac{e^{ik|\vec{x}-\vec{p}|}}{4\pi|\vec{x}-\vec{p}|}$ (2)

satifies $\left(\Delta + k^2\right) G(\vec{x},\vec{p}) = \delta(\vec{x}-\vec{p})$

Proof $\vec{\nabla}|\vec{x}-\vec{p}| = \dfrac{\vec{x}-\vec{p}}{|\vec{x}-\vec{p}|}, \quad \vec{\nabla}\dfrac{(\vec{x}-\vec{p})}{|\vec{x}-\vec{p}|^2} = \dfrac{1}{|\vec{x}-\vec{p}|^2}, \quad \vec{\nabla}\dfrac{1}{|\vec{x}-\vec{p}|} = \dfrac{-\vec{\nabla}|\vec{x}-\vec{p}|}{|\vec{x}-\vec{p}|^2}, \quad \vec{\nabla}e^{ik|\vec{x}-\vec{p}|} = ik\,e^{ik|\vec{x}-\vec{p}|}\,\vec{\nabla}|\vec{x}-\vec{p}|$

$$\Rightarrow \quad \vec{\nabla}G = -ik\frac{e^{ik|\vec{x}-\vec{p}|}}{4\pi|\vec{x}-\vec{p}|}\vec{\nabla}|\vec{x}-\vec{p}| - \frac{e^{ik|\vec{x}-\vec{p}|}}{4\pi}\vec{\nabla}\frac{1}{|\vec{x}-\vec{p}|} = -ik\frac{e^{ik|\vec{x}-\vec{p}|}}{4\pi}\frac{(\vec{x}-\vec{p})}{|\vec{x}-\vec{p}|^2} - \frac{e^{ik|\vec{x}-\vec{p}|}}{4\pi}\vec{\nabla}\frac{1}{|\vec{x}-\vec{p}|} \quad (*)$$

$$\Delta G = \frac{k^2 e^{ik|\vec{x}-\vec{p}|}}{4\pi}\left(\vec{\nabla}|\vec{x}-\vec{p}|\right)\frac{(\vec{x}-\vec{p})}{|\vec{x}-\vec{p}|^2} - ik\frac{e^{ik|\vec{x}-\vec{p}|}}{4\pi}\vec{\nabla}\frac{(\vec{x}-\vec{p})}{|\vec{x}-\vec{p}|^2} - ik\frac{e^{ik|\vec{x}-\vec{p}|}}{4\pi}\left(\vec{\nabla}|\vec{x}-\vec{p}|\right)\vec{\nabla}\frac{1}{|\vec{x}-\vec{p}|} - \frac{e^{ik|\vec{x}-\vec{p}|}}{4\pi}\Delta\frac{1}{|\vec{x}-\vec{p}|}$$

$$= \frac{k^2 e^{ik|\vec{x}-\vec{p}|}}{4\pi}\frac{1}{|\vec{x}-\vec{p}|} - ik\frac{e^{ik|\vec{x}-\vec{p}|}}{4\pi}\frac{1}{|\vec{x}-\vec{p}|^2} + ik\frac{e^{ik|\vec{x}-\vec{p}|}}{4\pi}\frac{1}{|\vec{x}-\vec{p}|^2} - \frac{e^{ik|\vec{x}-\vec{p}|}}{4\pi}\Delta\frac{1}{|\vec{x}-\vec{p}|}$$

with $\Delta\dfrac{1}{|\vec{x}-\vec{p}|} = -4\pi\,\delta\left(\vec{x}-\vec{p}\right)$, $f(\vec{x})\delta\left(\vec{x}\right) = f(0)\delta\left(\vec{x}\right)$ obtain $\Delta G = -k^2 G(\vec{x},\vec{p}) + \delta\left(\vec{x}-\vec{p}\right)$

Kirchhoff 's integral

$$\psi(\vec{p}) = -\frac{1}{4\pi}\int_{\partial V} dA\,\hat{n}\,\frac{e^{ikR}}{R}\left(\hat{R}\left(ik-\frac{1}{R}\right)\psi - \vec{\nabla}\psi\right) \qquad (3)$$

$$\vec{R} \equiv \vec{x}-\vec{p}\,,\quad R \equiv \left|\vec{x}-\vec{p}\right|\,,\quad \hat{R} \equiv \vec{R}/R$$

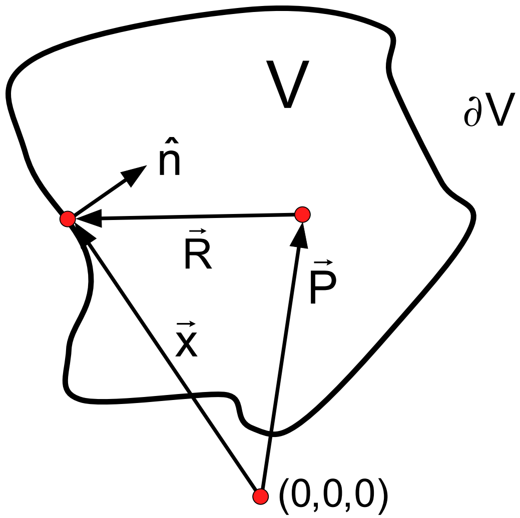
Proof

use (1) $\psi(\vec{p}) = \int_{\partial V}\left(\psi\frac{\partial}{\partial n}G - G\frac{\partial}{\partial n}\psi\right)dA$ and (2) $G(\vec{x},\vec{p}) \equiv -\frac{e^{ik\left|\vec{x}-\vec{p}\right|}}{4\pi\left|\vec{x}-\vec{p}\right|}$

$$\vec{\nabla}G \overset{(*)}{=} -ik\frac{e^{ikR}}{4\pi R}\hat{R} - \frac{e^{ikR}}{4\pi}\vec{\nabla}\frac{1}{R} = -ik\frac{e^{ikR}}{4\pi R}\hat{R} + \frac{e^{ikR}}{4\pi R}\frac{1}{R}\hat{R} = -\frac{e^{ikR}}{4\pi R}\hat{R}\left(ik-\frac{1}{R}\right)$$

$$\hat{n}\vec{\nabla}G = -\frac{1}{4\pi}\hat{n}\frac{e^{ik\left|\vec{x}-\vec{p}\right|}}{\left|\vec{x}-\vec{p}\right|}\frac{\vec{x}-\vec{p}}{\left|\vec{x}-\vec{p}\right|}\left(ik-\frac{1}{\left|\vec{x}-\vec{p}\right|}\right) = -\frac{1}{4\pi}\hat{n}\frac{e^{ikR}}{R}\hat{R}\left(ik-\frac{1}{R}\right)$$

Kirchhoff 's integral

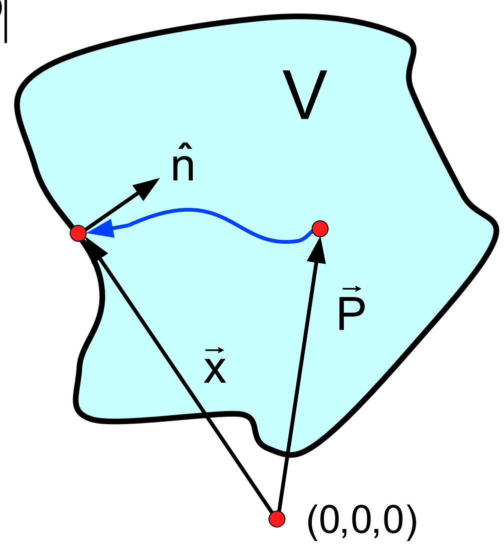


$$\psi(\vec{p}) = -\frac{1}{4\pi}\int_{\partial V} dA\,\hat{n}\,\frac{e^{ikR}}{R}\left(\left(ik\hat{R} - \frac{\hat{R}}{R}\right)\psi - \vec{\nabla}\psi\right)$$

$$\vec{R} \equiv \vec{x} - \vec{p}\,,\;\; R \equiv \left|\vec{x} - \vec{p}\right|,\;\; \hat{R} \equiv \vec{R}/R$$

Assume inhomogeneous medium inside the volume V

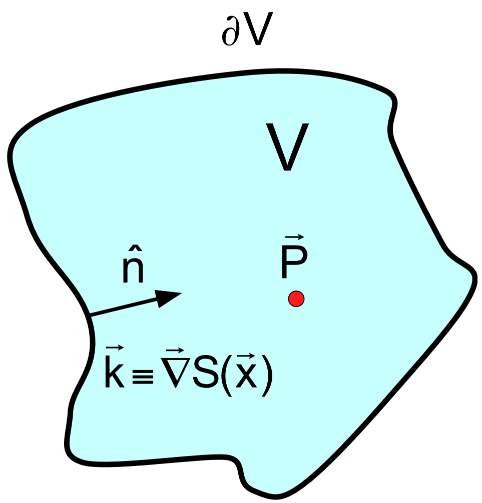
Approximate $S(\vec{x}) - S(\vec{p}) \cong k\left|\vec{x} - \vec{p}\right| = kR,\; \vec{k} \equiv \vec{\nabla}S(\vec{x}) = k\,\frac{\vec{x} - \vec{p}}{\left|\vec{x} - \vec{p}\right|} = k\hat{R}$



And rewrite Kirchhoff integral as

$$\psi(\vec{p}) = -\frac{1}{4\pi}\int_{\partial V} dA\,\hat{n}\,\frac{e^{i\left(S(\vec{x}) - S(\vec{p})\right)}}{R}\left(\left(i\vec{\nabla}S(\vec{x}) - \frac{\hat{R}}{R}\right)\psi - \vec{\nabla}\psi\right)$$

Generalized Kirchhoff Integral



Assume inhomogeneous medium inside the volume V

$$\left(\Delta + k^2\right)\psi(\vec{x}) = 0$$

$$\left(\Delta + k^2\right)G(\vec{x},\vec{p}) = \delta(\vec{x}-\vec{p})$$

$$\vec{k} \equiv \vec{\nabla}S(\vec{x}),\quad k \equiv \left|\vec{k}\right|,\quad 2\vec{\nabla}\frac{1}{\left|\vec{x}-\vec{p}\right|}\,\vec{\nabla}S(\vec{x}) + \frac{1}{\left|\vec{x}-\vec{p}\right|}\Delta S(\vec{x}) \cong 0 \qquad (*)$$

Note that (*) holds if $S(\vec{x}) - S(\vec{p}) \cong k\left|\vec{x}-\vec{p}\right| = kR,\quad \vec{k} = \vec{\nabla}S(\vec{x}) = k\,\dfrac{\vec{x}-\vec{p}}{\left|\vec{x}-\vec{p}\right|} = k\hat{R}$

Use Green-function for empty space:

$$G(\vec{x},\vec{p}) \equiv -\frac{e^{i\left(S(\vec{x})-S(\vec{p})\right)}}{4\pi\left|\vec{x}-\vec{p}\right|} \qquad (4)$$

to obtain generalized Kirchhoff integral

$$\psi(\vec{p}) = -\frac{1}{4\pi}\int_{\partial V} dA\,\hat{n}\,\frac{e^{i\left(S(\vec{x})-S(\vec{p})\right)}}{R}\left(\left(i\vec{\nabla}S(\vec{x}) - \frac{\hat{R}}{R}\right)\psi - \vec{\nabla}\psi\right) \qquad (5)$$

Proof

A. Showing that : $\left(\Delta+\left|\vec{\nabla}S\right|^{2}\right)\,G(\vec{x},\vec{p})=\delta(\vec{x}-\vec{p})$

$$\vec{\nabla}G(\vec{x},\vec{p})=-\vec{\nabla}\frac{e^{i\left(S(\vec{x})-S(\vec{p})\right)}}{4\pi\left|\vec{x}-\vec{p}\right|}=-\frac{e^{i\left(S(\vec{x})-S(\vec{p})\right)}}{4\pi\left|\vec{x}-\vec{p}\right|}i\vec{\nabla}S(\vec{x})-\frac{e^{i\left(S(\vec{x})-S(\vec{p})\right)}}{4\pi}\vec{\nabla}\frac{1}{\left|\vec{x}-\vec{p}\right|}=-\frac{e^{i\left(S(\vec{x})-S(\vec{p})\right)}}{4\pi}\left(\vec{\nabla}\frac{1}{\left|\vec{x}-\vec{p}\right|}+\frac{i\vec{\nabla}S(\vec{x})}{\left|\vec{x}-\vec{p}\right|}\right)\qquad (*)$$

$$\Delta G(\vec{x},\vec{p})=-\vec{\nabla}\frac{e^{i\left(S(\vec{x})-S(\vec{p})\right)}}{4\pi}\left(\vec{\nabla}\frac{1}{\left|\vec{x}-\vec{p}\right|}+\frac{i\vec{\nabla}S(\vec{x})}{\left|\vec{x}-\vec{p}\right|}\right)-\frac{e^{i\left(S(\vec{x})-S(\vec{p})\right)}}{4\pi}\vec{\nabla}\left(\vec{\nabla}\frac{1}{\left|\vec{x}-\vec{p}\right|}+\frac{i\vec{\nabla}S(\vec{x})}{\left|\vec{x}-\vec{p}\right|}\right)$$

$$=-\frac{e^{i\left(S(\vec{x})-S(\vec{p})\right)}}{4\pi}i\vec{\nabla}S(\vec{x}-\vec{p})\left(\vec{\nabla}\frac{1}{\left|\vec{x}-\vec{p}\right|}+\frac{i\vec{\nabla}S(\vec{x})}{\left|\vec{x}-\vec{p}\right|}\right)-\frac{e^{i\left(S(\vec{x})-S(\vec{p})\right)}}{4\pi}\left(\Delta\frac{1}{\left|\vec{x}-\vec{p}\right|}+i\vec{\nabla}\frac{\vec{\nabla}S(\vec{x})}{\left|\vec{x}-\vec{p}\right|}\right)$$

$$=-\frac{e^{i\left(S(\vec{x})-S(\vec{p})\right)}}{4\pi}i\vec{\nabla}S(\vec{x}-\vec{p})\left(\vec{\nabla}\frac{1}{\left|\vec{x}-\vec{p}\right|}+\frac{i\vec{\nabla}S(\vec{x})}{\left|\vec{x}-\vec{p}\right|}\right)-\frac{e^{i\left(S(\vec{x})-S(\vec{p})\right)}}{4\pi}\left(\Delta\frac{1}{\left|\vec{x}-\vec{p}\right|}+i\frac{\Delta S(\vec{x})}{\left|\vec{x}-\vec{p}\right|}+i\vec{\nabla}S(\vec{x})\vec{\nabla}\frac{1}{\left|\vec{x}-\vec{p}\right|}\right)$$

$$=-\frac{e^{i\left(S(\vec{x})-S(\vec{p})\right)}}{4\pi}\left[\cancel{i2\vec{\nabla}S(\vec{x})\vec{\nabla}\frac{1}{\left|\vec{x}-\vec{p}\right|}}+\cancel{i\frac{\Delta S(\vec{x})}{\left|\vec{x}-\vec{p}\right|}}+\Delta\frac{1}{\left|\vec{x}-\vec{p}\right|}-\frac{\vec{\nabla}S(\vec{x})\vec{\nabla}S(\vec{x})}{\left|\vec{x}-\vec{p}\right|}\right]$$

use $2\vec{\nabla}\frac{1}{R}\vec{\nabla}S(\vec{x})+\frac{1}{R}\Delta S(\vec{x})\cong 0$ and $\Delta\frac{1}{R}=-4\pi\,\delta(R)$

$$=-\frac{e^{i\left(S(\vec{x})-S(\vec{p})\right)}}{4\pi}\left[\Delta\frac{1}{\left|\vec{x}-\vec{p}\right|}-\frac{\vec{\nabla}S(\vec{x})\vec{\nabla}S(\vec{x})}{\left|\vec{x}-\vec{p}\right|}\right]=-\frac{e^{i\left(S(\vec{x})-S(\vec{p})\right)}}{4\pi}\left[-4\pi\delta(\vec{x}-\vec{p})-\frac{\vec{\nabla}S(\vec{x})\vec{\nabla}S(\vec{x})}{\left|\vec{x}-\vec{p}\right|}\right]=\left[\delta(\vec{x}-\vec{p})-G(\vec{x},\vec{p})\left|\vec{\nabla}S(\vec{x})\right|^{2}\right]$$

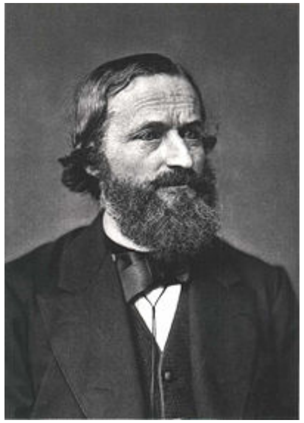
B. use (1), (4) , A.

$$\psi(\vec{p}) = \int_{\partial V} \left(\psi \frac{\partial}{\partial n} G - G \frac{\partial}{\partial n} \psi \right) dA \qquad (1)$$

$$G(\vec{x},\vec{p}) = -\frac{e^{i\left(S(\vec{x})-S(\vec{p})\right)}}{4\pi\left|\vec{x}-\vec{p}\right|} \qquad (4)$$

$$\vec{\nabla}G(\vec{x},\vec{p}) = -\frac{e^{i\left(S(\vec{x})-S(\vec{p})\right)}}{4\pi\left|\vec{x}-\vec{p}\right|} \left(i\vec{\nabla}S(\vec{x}) - \frac{\vec{x}-\vec{p}}{\left|\vec{x}-\vec{p}\right|} \right) \qquad \text{A. } (*)$$

to obtain $$\psi(\vec{p}) = -\frac{1}{4\pi} \int_{\partial V} dA\, \hat{n}\, \frac{e^{i\left(S(\vec{x})-S(\vec{p})\right)}}{R} \left(\left(i\vec{\nabla}S(\vec{x}) - \frac{\hat{R}}{R} \right) \psi - \vec{\nabla}\psi \right)$$



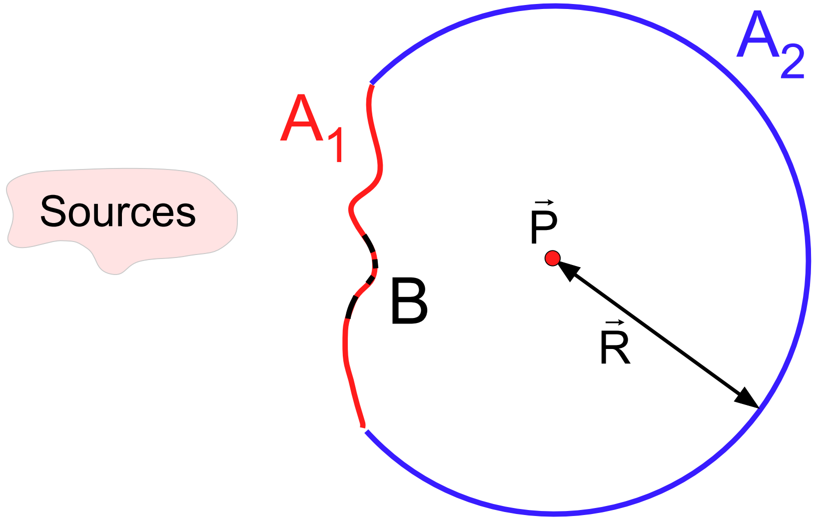
Gustav Robert Kirchhoff
1824-1887

Assume that

A_1 is a screen with apertures B.

A_2 is pushed to infinity $\vec{R} \to \infty$

In accordance with Sommerfeld 's radiation condition no waves come in from infinity, hence $\psi|_{A_2} = \frac{\partial}{\partial n}\psi|_{A_2} = 0$



Kirchhoff 's boundary conditions

K1 $\psi|_A = \frac{\partial}{\partial n}\psi|_A = 0$

K2 $\psi|_B, \frac{\partial}{\partial n}\psi|_B$ determined by sources

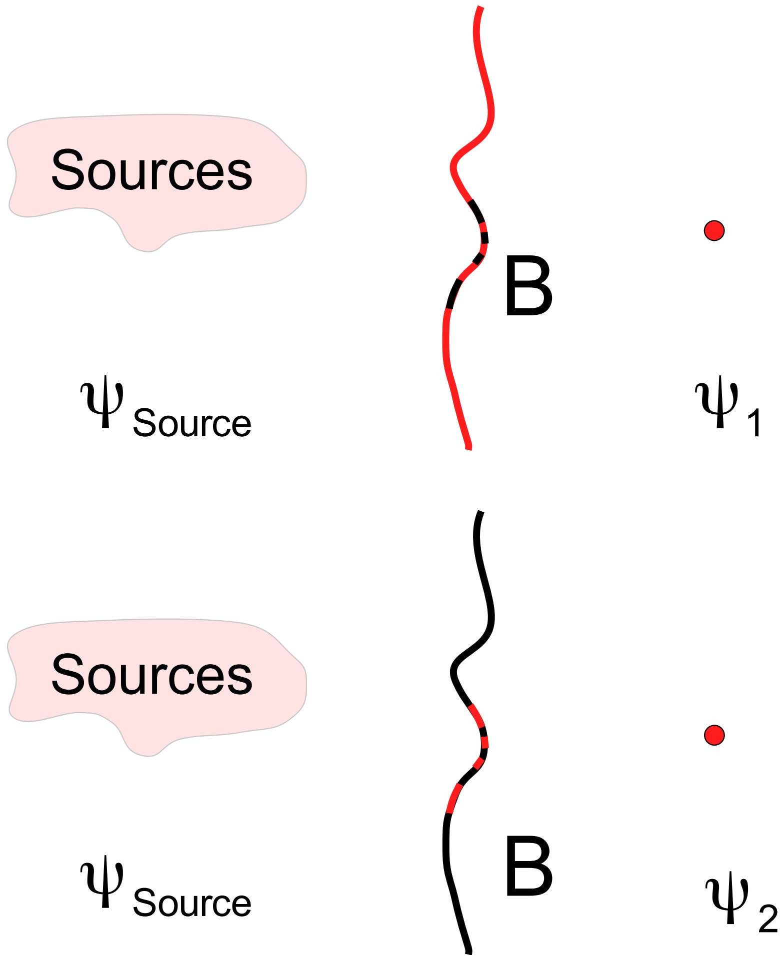
Diffraction integral $$\psi(\vec{p}) = -\frac{1}{4\pi}\int_B dA\,\hat{n}\,\frac{e^{ikR}}{R}\left(\hat{R}\left(ik - \frac{1}{R}\right)\psi - \vec{\nabla}\psi\right)$$

Note: K1, K2 are not compatible with Helmholtz equation. Improved solution: choose adequate Green functions for Dirichlet or von Neumann boundary conditions

Babinet`s principle

$$\psi_1(\vec{p}) + \psi_2(\vec{p}) = \psi_{Source}(\vec{p})$$

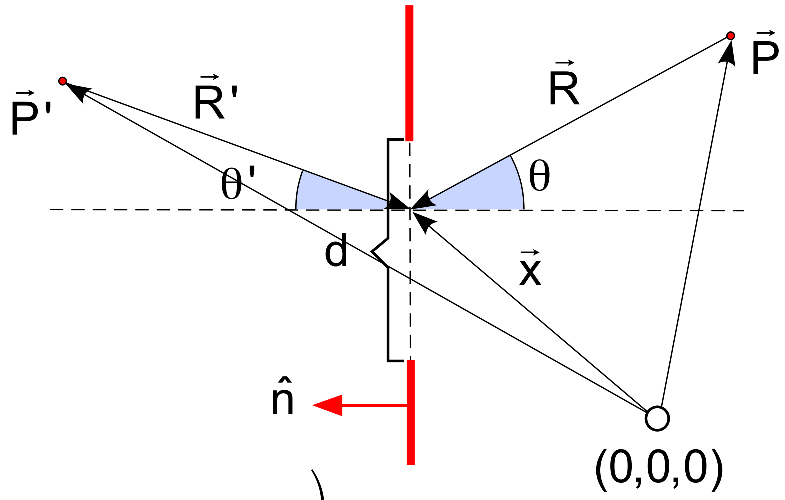
Jacques Babinet
1794-1872



Special case: point source & flat screen

$\vec{R} \equiv \vec{x} - \vec{P}$

$\vec{R}' \equiv \vec{x} - \vec{P}'$



$$\psi_{source}(\vec{x}) = \frac{e^{ikR'}}{R'}\,,\; \vec{\nabla}\psi_{source}(\vec{x}) = \frac{e^{ikR'}}{R'}\left(ik - \frac{1}{R'}\right)\hat{R}'$$

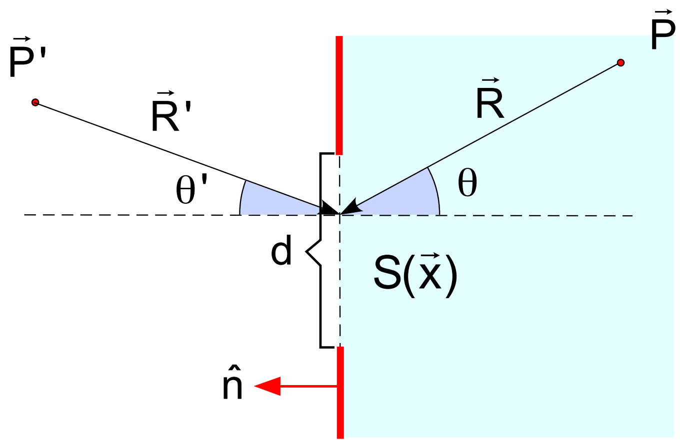
$$\psi(\vec{P}) = -\frac{1}{4\pi}\int_{apertures} dA\, \hat{n}\, \frac{e^{ikR}}{R}\left(\frac{\vec{R}}{R}\left(ik - \frac{1}{R}\right)\psi_{source}(\vec{x}) - \vec{\nabla}\psi_{source}(\vec{x})\right)$$

$$= -\frac{1}{4\pi}\int_{apertures} dA\, \frac{e^{ikR}e^{ikR'}}{R\,R'}\left(\left(ik - \cancel{\frac{1}{R}}\right)\hat{n}\hat{R} - \left(ik - \cancel{\frac{1}{R'}}\right)\hat{n}\hat{R}'\right) \qquad (*)$$

$$\cong \frac{1}{i\lambda}\frac{Q}{R\,R'}\int_{apertures} dA\; e^{ik(R+R')}\,, \quad Q \equiv \frac{1}{2}\hat{n}\left(\hat{R} - \hat{R}'\right) = \frac{1}{2}\left(\cos(\theta) + \cos(\theta')\right) \qquad (K1)$$

$R, R' > d\,, \quad R, R' \gg \lambda \quad (*)$

Generalized case: point source & flat screen around $\vec{x}=0$



$$\vec{R} \equiv \vec{x} - \vec{P}$$

$$\vec{R}' \equiv \vec{x} - \vec{P}'$$

$$\psi_{source}(\vec{x}) = \frac{e^{ikR'}}{R'} \;,\; \vec{\nabla}\psi_{source}(\vec{x}) = \psi_{source}(\vec{x})\left(ik - \frac{1}{R'}\right)\hat{R}'$$

$$\psi(\vec{P}) = -\frac{1}{4\pi}\int_{apertures} dA\,\hat{n}\,\frac{e^{i\left(S(\vec{x})-S(\vec{p})\right)}}{R}\left(\left(i\vec{\nabla}S(\vec{x}) - \frac{\hat{R}}{R}\right)\psi_{source} - \vec{\nabla}\psi_{source}\right)$$

$$= -\frac{1}{4\pi}\int_{apertures} dA\;\psi_{source}(\vec{x})\,\frac{e^{i\left(S(\vec{x})-S(\vec{p})\right)}}{R}\left(\hat{n}\left(i\vec{\nabla}S(\vec{x}) - \cancel{\frac{\hat{R}}{R}}\right) - \left(ik - \cancel{\frac{1}{R'}}\right)\hat{n}\hat{R}'\right) \qquad (*)$$

$$\cong \boxed{\frac{1}{i\lambda}\frac{Q}{R}\int_{apertures} dA\,\psi_{source}(\vec{x})\,e^{i\left(S(\vec{x})-S(\vec{p})\right)}} \;,\; Q \equiv \frac{1}{2}\hat{n}\left(\frac{1}{k}\vec{\nabla}S(\vec{x}) - \hat{R}'\right) \cong \frac{1}{2}\hat{n}\left(\hat{R}-\hat{R}'\right) \cong \frac{1}{2}\left(\cos(\theta)+\cos(\theta')\right)$$

(K2)

$R, R' > d \;,\quad R, R' >> \lambda \quad (*)$

$\vec{\nabla}S(\vec{x}) \cong k\hat{R}$

Expansion of R and R ‘

$R, R' >> d\,, \quad R, R' >> \lambda$

$$\frac{R}{P} = \frac{|\vec{x} - \vec{P}|}{P} = 1 - \frac{\hat{P}}{|\vec{P}|}\vec{x} + \frac{1}{2}\frac{1}{P^2}\left(\vec{x}^2 - \left(\hat{P}\vec{x}\right)^2\right) + \frac{1}{2}\frac{\hat{P}\vec{x}}{P^3}\left(\vec{x}^2 - \left(\hat{P}\vec{x}\right)^2\right) + \cdots$$

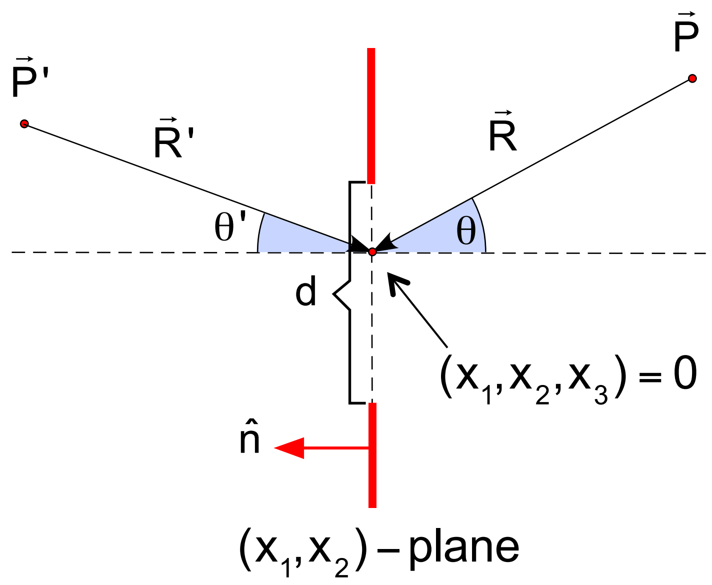
$$\uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow$$

$$\propto \frac{d}{P} \qquad\qquad \propto \left(\frac{d}{P}\right)^2 \qquad\qquad \propto \left(\frac{d}{P}\right)^3$$

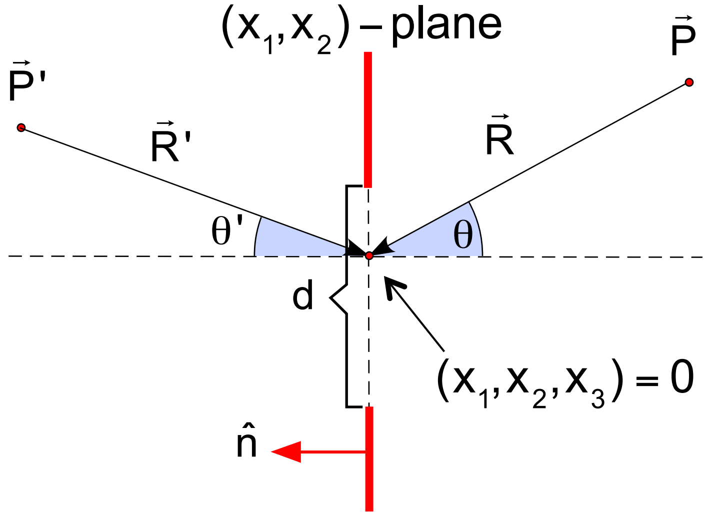
Fraunhofer limit $\quad \frac{R}{P} \approx 1 - \frac{\hat{P}}{|\vec{P}|}\vec{x}$

Fresnel limit $\quad \frac{R}{P} \approx 1 + \frac{1}{2P^2}\left(\vec{x}^2 - \left(\hat{P}\vec{x}\right)^2\right)$

$\hat{P}\vec{x} \approx 0$



Fraunhofer diffraction



$R, R' >> d\ ,\quad R, R' >> \lambda$

$$\frac{R}{P} \approx 1 - \frac{\hat{P}}{|\vec{P}|}\vec{x}$$

$$e^{ik(R+R')} \approx e^{ik(P+P')}\ e^{-ik(\hat{P}+\hat{P}')\ \vec{x}}$$

$$\psi(\vec{P}) \overset{(K1)}{=} \frac{k}{2\pi i}\frac{Q}{R\,R'} \int_{\text{apertures}} dA\ e^{ik(R+R')}\ ,\quad Q \equiv \frac{1}{2}\hat{n}\left(\hat{R}-\hat{R}'\right)$$

$$= \frac{k}{2\pi i}\frac{Q}{R\,R'} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dA\ \ \tau(x_1,x_2)\ e^{ik(R+R')} \approx \frac{k\ Q}{2\pi i}\frac{e^{ik(P+P')}}{P\ P'} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dA\ \ \tau(x_1,x_2)\ e^{-ik(\hat{P}+\hat{P}')\vec{x}}$$

$$= \frac{k\ Q}{2\pi i}\frac{e^{ik(P+P')}}{P\ P'} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx_1 dx_2\ \tau(x_1,x_2)\ e^{-ik\left[\left(\frac{P_1}{P}+\frac{P_1'}{P'}\right)x_1 + \left(\frac{P_2}{P}+\frac{P_2'}{P'}\right)x_2\right]}$$

Joseph von Fraunhofer
1787-1826

$\tau(\vec{x})$ transmission function describes the apertures

Fraunhofer diffraction

$R, R' \gg d \,, \quad R, R' \gg \lambda$

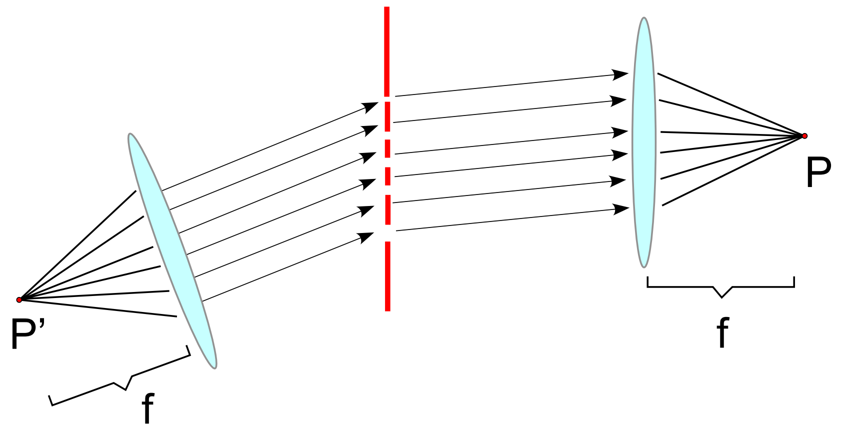
$$\psi(\vec{P}) = \frac{k\hat{n}\left(\hat{R}-\hat{R}'\right)}{4\pi\, i} \, \frac{e^{ik(P+P')}}{P\,P'} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx_1\, dx_2\; \tau(x_1, x_2)\; e^{-ik\left[\left(\frac{P_1}{P}+\frac{P_1'}{P'}\right)x_1 \;+\; \left(\frac{P_2}{P}+\frac{P_2'}{P'}\right)x_2\right]}$$

$$\frac{P_\nu}{P} = \sin(\theta_\nu)\,, \quad \frac{P_\nu'}{P'} = \sin(\theta_\nu')\,, \;\; \nu = 1,2$$

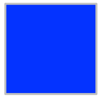
$$= \frac{k\hat{n}\left(\hat{R}-\hat{R}'\right)}{4\pi\, i} \, \frac{e^{ik(P+P')}}{P\,P'} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx_1\, dx_2\; \tau(\vec{x})\; e^{-ik\left[\left(\sin(\theta_1)+\sin(\theta_1')\right)x_1 \;+\; \left(\sin(\theta_2)+\sin(\theta_2')\right)x_2\right]} \qquad \text{(F1)}$$

$\Rightarrow \quad \psi(\vec{P}) \propto$ Fourier – transform of $\tau(\vec{x})$

Experimental realisation:



Fraunhofer diffraction for rectangular aperture



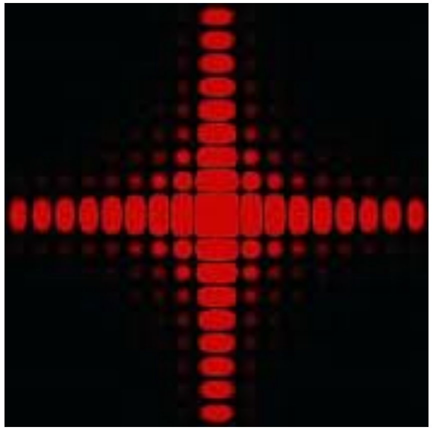
$$\psi(\vec{P}) \overset{(F1)}{=} \frac{k\hat{n}\left(\hat{R}-\hat{R}'\right)}{4\pi\, i} \frac{e^{ik\left(P+P'\right)}}{P\,P'} \int_{-a_1}^{a_1}\int_{-a_2}^{a_2} dx_1 dx_2\; \tau(\vec{x})\; e^{-ik\left[\left(\sin(\theta_1)+\sin(\theta_1')\right)x_1 \,+\, \left(\sin(\theta_2)+\sin(\theta_2')\right)x_2\right]}$$

$$= \frac{\hat{n}\left(\hat{R}-\hat{R}'\right)e^{ik\left(P+P'\right)}}{\pi\, i\, P\, P'} \frac{\sin\left[k\left(\sin(\theta_1)+\sin(\theta_1')\right)a_1\right]}{k\left(\sin(\theta_1)+\sin(\theta_1')\right)} \frac{\sin\left[k\left(\sin(\theta_2)+\sin(\theta_2')\right)a_2\right]}{k\left(\sin(\theta_2)+\sin(\theta_2')\right)}$$

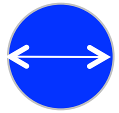
Intensity:

$$\theta_1' = \theta_2' = 0 \quad \Rightarrow \quad I \;=\; I_0 \frac{\sin^2\left[k\sin(\theta_1)\, a_1\right]}{k^2\sin^2(\theta_1)} \frac{\sin^2\left[k\sin(\theta_2)\, a_2\right]}{k^2\sin^2(\theta_2)}$$

$$\text{minima} \quad \Leftrightarrow \quad \sin(\theta_1)\, a_1 = n\frac{\lambda}{2} \;\text{ or }\; \sin(\theta_2)\, a_2 = n\frac{\lambda}{2}$$

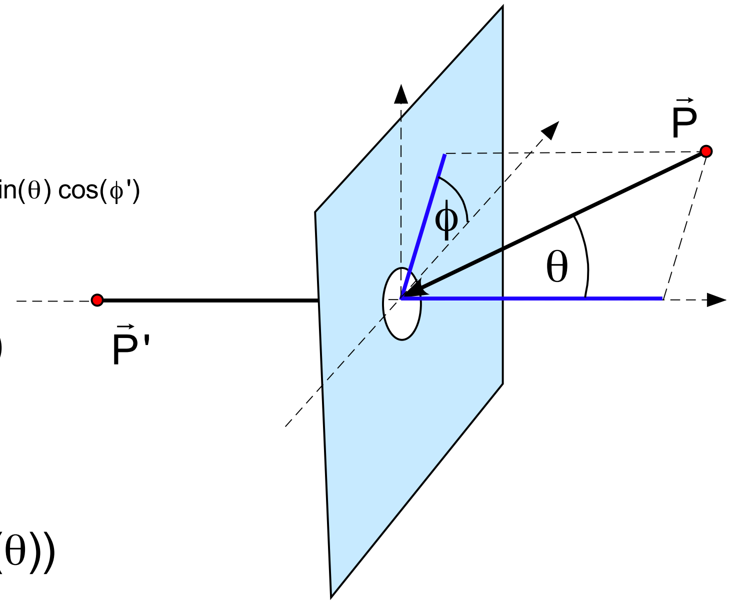


Fraunhofer diffraction for circular aperture



$$P_1 = P\sin(\theta)\cos(\phi), P_1 = P\sin(\theta)\sin(\phi),$$

$$P'_1 = P_2' = 0,\ x_1 = r\cos(\phi'),\ x_2 = r\sin(\phi'),\ dx_1 dx_2 = r dr$$



$$\psi(\vec{P}) \propto \int_0^a r\,dr \int_0^{2\pi} d\phi'\, e^{-ikr\sin(\theta)\cos(\phi-\phi')} = \int_0^a r\,dr \int_0^{2\pi} d\phi'\, e^{-ikr\sin(\theta)\cos(\phi')}$$

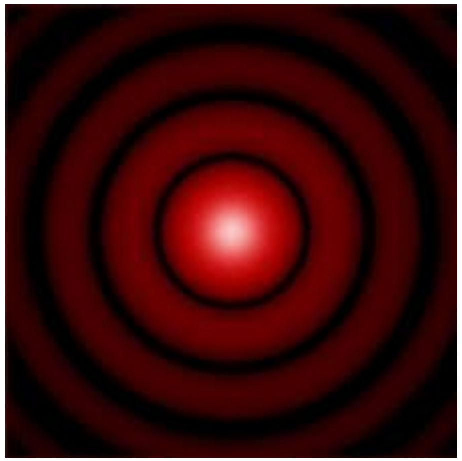
$$= 2\pi \int_0^a r\,dr\, J_0(-kr\sin(\theta)) = \frac{2\pi}{k^2\sin^2(\theta)} \int_0^{ka\sin(\theta)} dx\ x\ J_0(x)$$

$x = -kr\sin(\theta)$

$$= \frac{2\pi}{k^2\sin^2(\theta)} \int_0^{ka\sin(\theta)} dx\, \frac{d}{dx}\left(x\ J_1(x)\right) = \frac{2\pi a}{k\sin(\theta)}\ J_1(ka\sin(\theta))$$

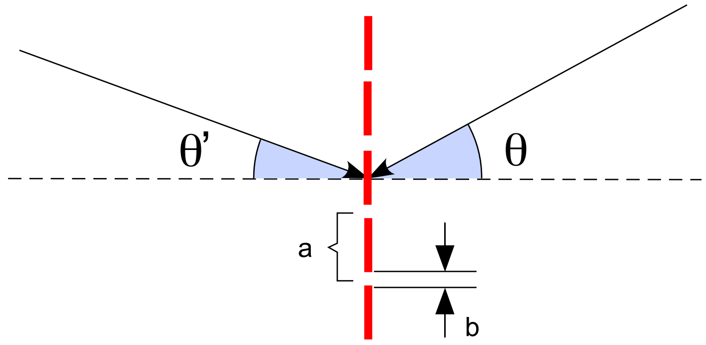
use $\frac{d}{dx}\left(x\ J_1(x)\right) = x\ J_0(x)\ ,\ \ J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} d\alpha\ e^{ix\cos(\alpha)}$

Intensity: $I = I_0 \left(\frac{2J_1(ka\sin(\theta))}{ka\sin(\theta)}\right)^2$



Diffraction at a transmission grating

$$\psi \propto \int_{-\infty}^{\infty} dx\ \tau(x)\ e^{-ik\left(\sin(\theta)+\sin(\theta')\right)x} \qquad (**)$$



$$\psi \propto FT[\tau](k\eta)\ ,\quad \tau = f * g\ ,\quad \eta \equiv (\sin(\theta)+\sin(\theta'))$$

$$f(x) \equiv \begin{cases} 1 \ \text{if}\ \ x \in [-\frac{b}{2},\frac{b}{2}] \\ 0 \ \ \text{otherwise} \end{cases} \quad ,\quad g(x) \equiv \sum_{n=0}^{N-1} \delta(x-na)$$

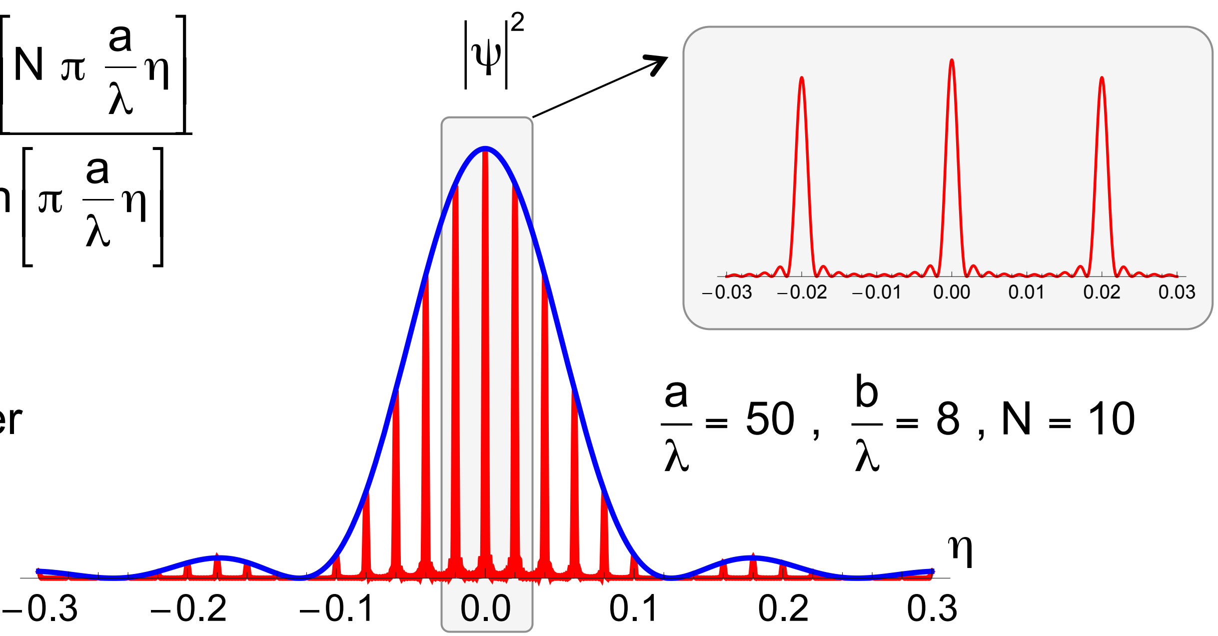
$$\psi \propto FT(f * g) = FT(f)\cdot FT(g) \ \propto\ \text{sinc}\left[\pi\,\frac{b}{\lambda}\eta\right] \sum_{n=0}^{N-1} e^{i\,n\,2\pi\frac{a}{\lambda}\eta}$$

$$\sum_{n=0}^{N-1} e^{i2nz} = e^{i2(N-1)z}\,\frac{\sin(Nz)}{\sin(z)}$$

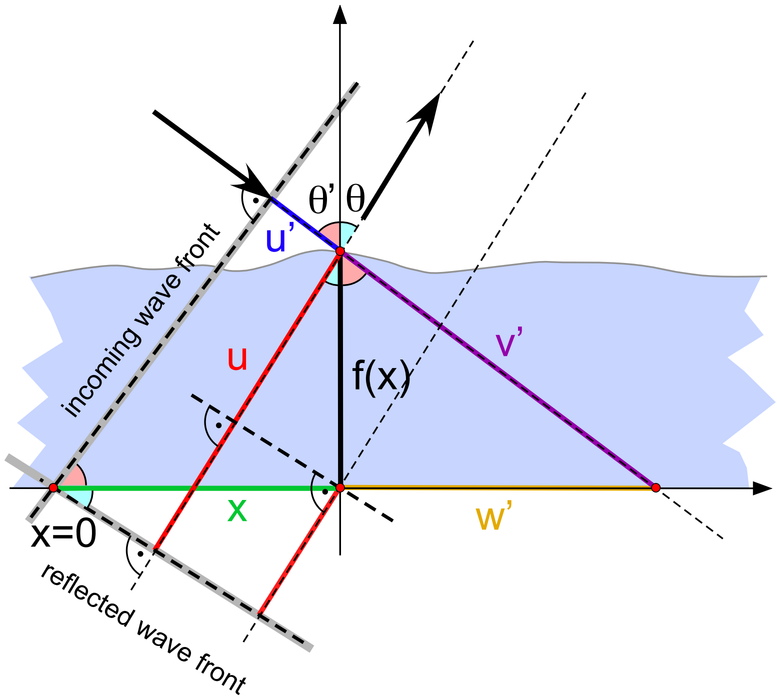
$$\propto\ \text{sinc}\left[\pi\,\frac{b}{\lambda}\eta\right]\ \frac{\sin\left[N\,\pi\,\frac{a}{\lambda}\eta\right]}{\sin\left[\pi\,\frac{a}{\lambda}\eta\right]}$$

envelope

Maxima: $\frac{a}{\lambda}\eta$ integer



Diffraction at a reflective surface



(1) $u' + v' = \left(x + w'\right)\sin(\theta')$

(2) $u = x\sin(\theta) + f(x)\cos(\theta)$

(3) $v' = \dfrac{f(x)}{\cos(\theta')}\,, \quad w' = f(x)\tan(\theta')$

(1),(3)

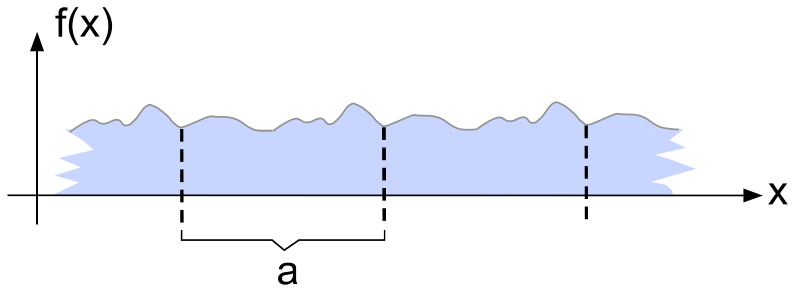
$$\Rightarrow \quad u' = x\sin(\theta') + f(x)\left(\tan(\theta')\sin(\theta') - \frac{1}{\cos(\theta')}\right) = x\sin(\theta') - f(x)\cos(\theta') \qquad (4)$$

(2),(4)

$$\Rightarrow \quad e^{ik\left(u'-u\right)} = e^{ik\left[x\,\alpha - f(x)\,\beta\right]}\,, \quad \alpha \equiv \sin(\theta') - \sin(\theta)\,, \quad \beta \equiv \cos(\theta') + \cos(\theta) \qquad (5)$$

$$\Rightarrow \quad \psi \propto \int dx\, e^{ik\left[x\,\alpha - f(x)\,\beta\right]} \propto FT\left[e^{-i\,k\,f(x)\,\beta}\right]\Big|_{k\alpha}$$

Diffraction at a reflective grating



Assume periodic function $f(x+na)=f(x)$ (*)

$$\psi \propto \int_0^{Na} dx\, e^{ik[x\alpha - f(x)\beta]} = \sum_{n=0}^{N-1} \int_{na}^{(n+1)a} dx\, e^{ik[x\alpha - f(x)\beta]}$$

substitute $x = x' + na$

$$= \sum_{n=0}^{N-1} \int_0^a dx'\, e^{ik[(x'+na)\alpha - f(x'+na)\beta]} \overset{(*)}{=} \left(\sum_{n=0}^{N-1} e^{ikna\alpha}\right) \cdot \int_0^a dx'\, e^{ik[x'\alpha - f(x')\beta]}$$

$$= e^{i(N-1)\pi\frac{a}{\lambda}\alpha} \frac{\sin\left[N\pi\frac{a}{\lambda}\alpha\right]}{\sin\left[\pi\frac{a}{\lambda}\alpha\right]} \cdot \int_0^a dx\, e^{ik[x\alpha - f(x)\beta]} \qquad \text{(DG)}$$

envelope

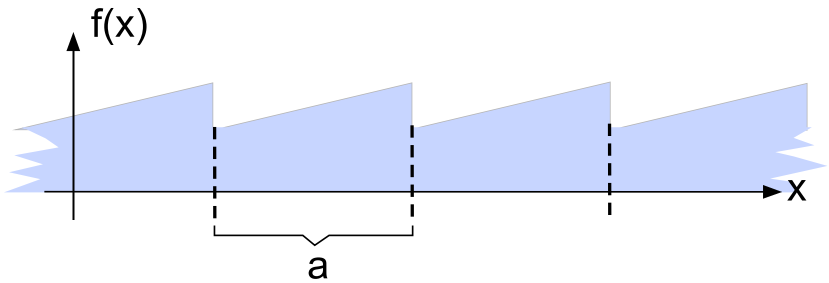
Trivial case: flat surface

$$f(x)=f_0 \Rightarrow \int_0^a dx\, e^{ik[x\alpha - f(x)\beta]} = e^{-ikf_0\beta} e^{i\pi\frac{a}{\lambda}\alpha} \frac{\sin(\pi\frac{a}{\lambda}\alpha)}{\frac{\pi}{\lambda}\alpha} \Rightarrow \psi \propto e^{iN\pi\frac{a}{\lambda}\alpha} e^{-ikf_0\beta} \frac{\sin\left[N\pi\frac{a}{\lambda}\alpha\right]}{\frac{\pi}{\lambda}\alpha}$$

$\Rightarrow$ maximum for $\alpha = 0$ $\Rightarrow$ specular reflection $\theta' = \theta$

1st minimum for $\alpha = \lambda/(Na) \xrightarrow[N\to 0]{} 0$

Special case: blazed grating



$$f(x)\Big|_{[0,a]} = \tan(\phi)x$$

$$\int_0^a dx\; e^{ik[x\alpha - f(x)\,\beta]} = \int_0^a dx\; e^{ik(\alpha - \tan(\phi)\beta)x} = e^{i\frac{\pi}{\lambda}(\alpha - \tan(\phi)\beta)a}\,\frac{\sin\left(\frac{\pi}{\lambda}(\alpha - \tan(\phi)\beta)a\right)}{\frac{\pi}{\lambda}(\alpha - \tan(\phi)\beta)}$$

$$\psi^{(DG)} \propto e^{i(N-1)\pi\frac{a}{\lambda}\alpha}\,\frac{\sin\left[N\pi\frac{a}{\lambda}\alpha\right]}{\sin\left[\pi\frac{a}{\lambda}\alpha\right]} \cdot \int_0^a dx\; e^{ik[x\alpha - f(x)\,\beta]}$$

$$= e^{i\pi\frac{a}{\lambda}(\alpha N - \tan(\phi)\beta)}\,\frac{\sin\left[N\pi\frac{a}{\lambda}\alpha\right]}{\sin\left[\pi\frac{a}{\lambda}\alpha\right]} \cdot \frac{\sin\left(\pi\frac{a}{\lambda}(\alpha - \tan(\phi)\beta)\right)}{\frac{\pi}{\lambda}(\alpha - \tan(\phi)\beta)}$$

← envelope

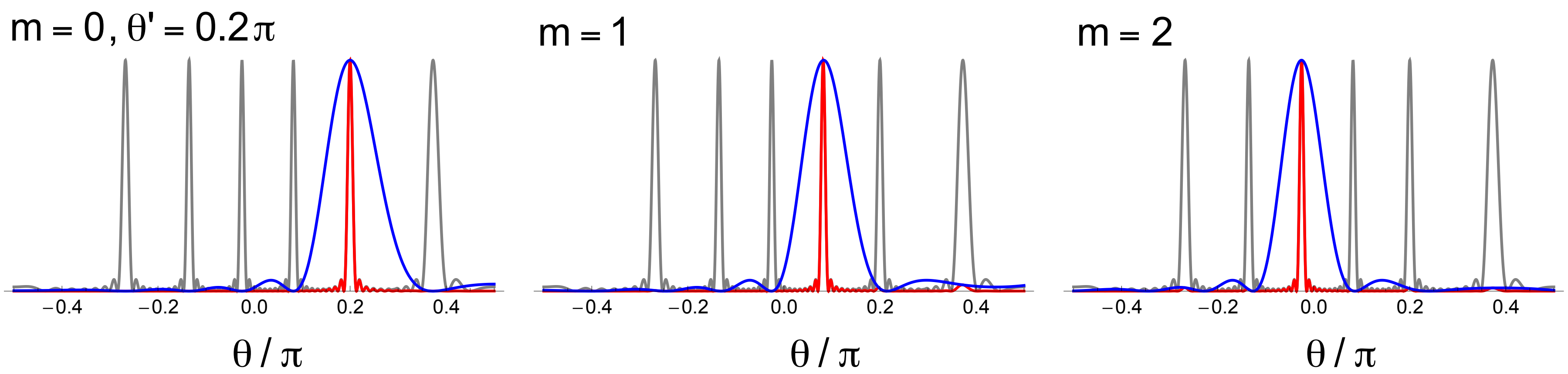
Very useful example: blazed grating

$$\psi \propto e^{i\pi\frac{a}{\lambda}(\alpha N - \tan(\phi)\beta)} \frac{\sin\left[N\pi\frac{a}{\lambda}\alpha\right]}{\sin\left[\pi\frac{a}{\lambda}\alpha\right]} \cdot \frac{\sin\left(\pi\frac{a}{\lambda}\left(\alpha - \tan(\phi)\beta\right)\right)}{\frac{\pi}{\lambda}\left(\alpha - \tan(\phi)\beta\right)}$$

envelope

Maxima: $m = \alpha\frac{a}{\lambda}$ integer $\Rightarrow \theta = \theta_m(\theta')$ = m-th diffraction maximum

By chosing the appropriate blaze angle ϕ one may position the maximum of the envelope to a higher order maximum $m \neq 0$:

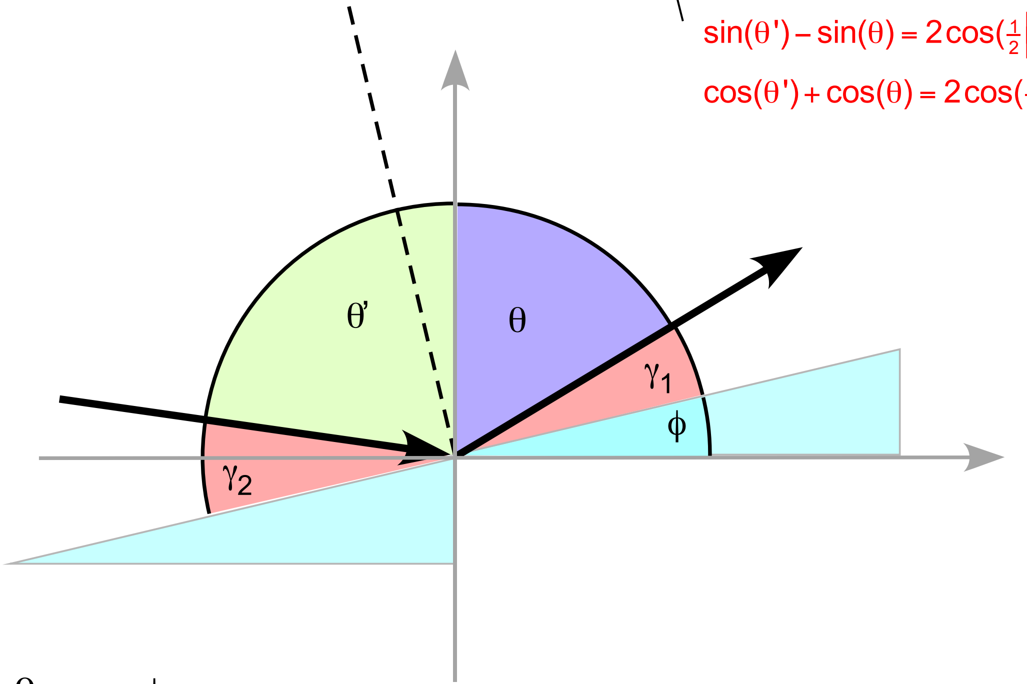


$$\psi \propto \frac{\sin\left[N\,\pi\,m\right]}{\sin\left[\pi\,m\right]} \cdot \frac{a\,\sin\left(\pi m\left(1-\tan(\phi)\frac{\beta}{\alpha}\right)\right)}{\pi m\left(1-\tan(\phi)\frac{\beta}{\alpha}\right)} \Rightarrow \phi = \arctan\left(\frac{\alpha}{\beta}\right)$$

$$\phi = \arctan\left(\frac{\alpha}{\beta}\right) = \arctan\left(\frac{\sin(\theta')-\sin(\theta)}{\cos(\theta')+\cos(\theta)}\right) = \frac{1}{2}\left(\theta'-\theta\right) \text{ with } \theta=\theta_m \qquad (*)$$

$$\sin(\theta')-\sin(\theta) = 2\cos(\tfrac{1}{2}\left[\theta'+\theta\right])\sin(\tfrac{1}{2}\left[\theta'-\theta\right])$$

$$\cos(\theta')+\cos(\theta) = 2\cos(\tfrac{1}{2}\left[\theta'+\theta\right])\cos(\tfrac{1}{2}\left[\theta'-\theta\right])$$



$$\frac{\pi}{2} = \theta + \gamma_1 + \phi$$
$$\frac{\pi}{2} = \theta' + \gamma_2 - \phi$$
$$\Rightarrow \theta - \theta' + \gamma_1 - \gamma_2 + 2\phi = 0 \quad \underset{(*)}{\Rightarrow} \quad \gamma_1 = \gamma_2$$

Fresnel Diffraction

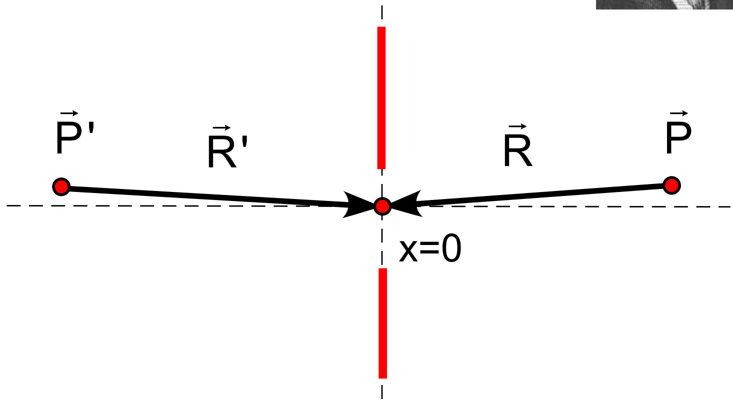
Augustin Jean Fresnel
1788 - 1827

Fresnel approximations

$$\hat{P}\vec{x} \approx 0, \; \hat{P}'\vec{x} \approx 0$$

$$\frac{R}{P} \approx 1 + \frac{1}{2P^2}\left(\vec{x}^2 - \left(\hat{P}\vec{x}\right)^2\right) \approx 1 + \frac{\vec{x}^2}{2P^2}$$

$$Q \equiv \frac{1}{2}\hat{n}\left(\hat{R} - \hat{R}'\right) \cong 1, \quad e^{ik(R+R')} \cong e^{ik(P+P')} e^{i\frac{\pi}{\lambda}\left(\frac{1}{P}+\frac{1}{P'}\right)\vec{x}^2}$$

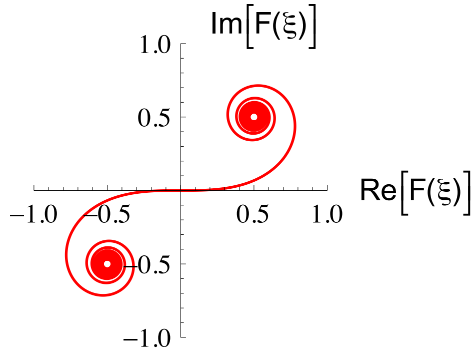


$$\psi(\vec{P}) \overset{(K1)}{\cong} \frac{k}{2\pi i} \frac{Q}{R R'} \int_{\text{apertures}} dA \; e^{ik(R+R')} \cong \frac{1}{i\lambda} \frac{e^{ik(P+P')}}{P P'} \int_{\text{apertures}} dx_1 dx_2 \; e^{i\frac{\pi}{\lambda}\left(\frac{1}{P}+\frac{1}{P'}\right)(x_1^2+x_2^2)}$$

$$\xi_\nu \equiv C(P,P')\, x_\nu, \; C(P,P') \equiv \sqrt{\frac{2}{\lambda}\left(\frac{1}{P}+\frac{1}{P'}\right)} \Rightarrow \psi(\vec{P}) \cong \frac{1}{2i} \frac{e^{ik(P+P')}}{P+P'} \int_{c(P,P')\times \text{apertures}} d\xi_1 d\xi_2 \; e^{i\frac{\pi}{2}(\xi_1^2+\xi_2^2)}$$

Fresnel integral: (Cornu spiral)

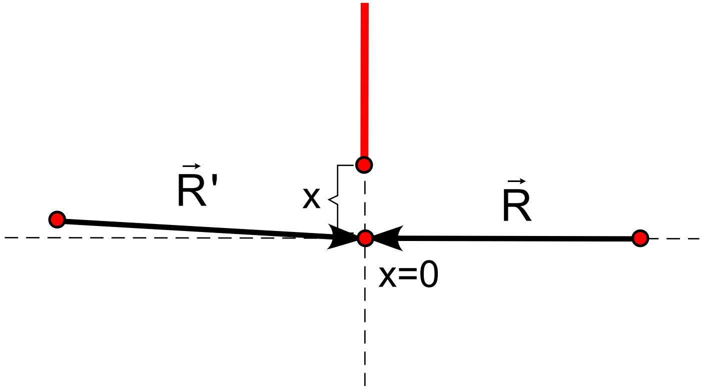
$$F(\xi) \equiv \int_0^\xi dz \; e^{i\frac{\pi}{2}z^2}$$



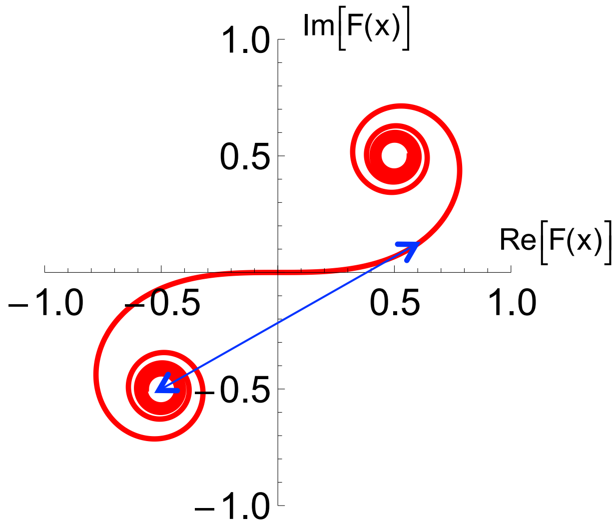
Diffraction at an edge

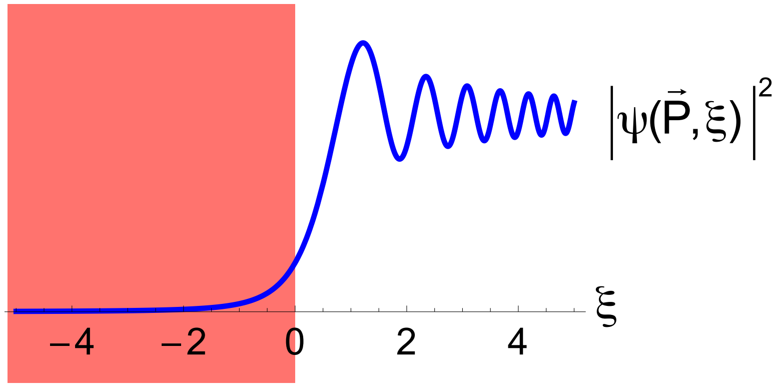
$$\psi(\vec{P},\xi) \cong \frac{1}{2i}\frac{e^{ik(P+P')}}{P+P'}\left(F(\xi)-F(-\infty)\right)\left(F(\infty)-F(-\infty)\right)$$

$$= \frac{1+i}{2i}\frac{e^{ik(P+P')}}{P+P'}\left(F(\xi)-\frac{1+i}{2}\right),\ \xi=\sqrt{\frac{2}{\lambda}\left(\frac{1}{P}+\frac{1}{P'}\right)}\,x$$



$$\left|\psi(\vec{P},\xi)\right| = \frac{1}{\sqrt{2}\left(P+P'\right)}\left|\ F(\xi)-F(-\infty)\right|,\ \xi=\sqrt{\frac{2}{\lambda}\left(\frac{1}{P}+\frac{1}{P'}\right)}\,x$$

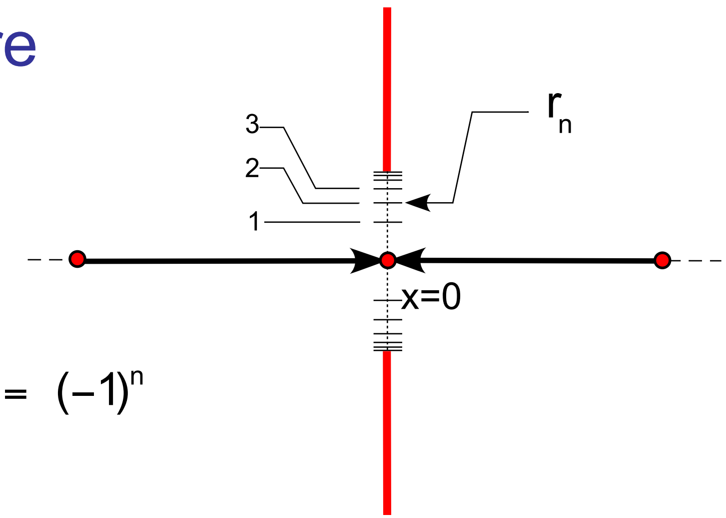




Fresnel diffraction at circular apperture

$$r_n \equiv \sqrt{\frac{n\lambda}{\frac{1}{P}+\frac{1}{P'}}}\ ,\quad n = 0,1,2,\ldots \quad \Rightarrow$$

$$\text{Fresnel-integrand}\quad e^{i\frac{\pi}{\lambda}\left(\frac{1}{P}+\frac{1}{P'}\right)r_n^2} = (-1)^n$$



The region between r_n and r_{n-1} is called the n-th Fresnel zone. Each zone encloses the same area $A = \pi\left(r_n^{\,2} - r_{n-1}^{\,2}\right) = \dfrac{\pi\lambda}{\frac{1}{P}+\frac{1}{P'}}$

$$\psi \cong \frac{1}{i\lambda}\frac{e^{ik(P+P')}}{P\,P'}\int_{\text{apertures}} dx_1 dx_2\; e^{i\frac{\pi}{\lambda}\left(\frac{1}{P}+\frac{1}{P'}\right)\left(x_1^2+x_2^2\right)} = \frac{2\pi}{i\lambda}\frac{e^{ik(P+P')}}{P\,P'}\int_0^\infty dr\; r\;\tau(r)\; e^{i\frac{\pi}{\lambda}\left(\frac{1}{P}+\frac{1}{P'}\right)r^2}$$

Fresnel zone plate

Covering the even or odd zones yields Fresnel ‘s zone plate: on the optical axis locations arise with constructive or destructive interference.

Consider Fresnel zone plate with $r_n = \sqrt{n\lambda f}$

$$\psi \propto \int_0^{r_{max}} dr\, r\, e^{i\frac{\pi}{\lambda}\left(\frac{1}{P}+\frac{1}{P'}\right) r^2} = \sum_{n=0}^{n_{max}} \int_{r_n}^{r_{n+1}} dr\; r\; e^{i\frac{\pi}{\lambda}\left(\frac{1}{P}+\frac{1}{P'}\right) r^2} = \frac{1}{2} \sum_{n=0}^{n_{max}} \int_{r_n^2}^{r_{n+1}^2} dz\; e^{i\frac{\pi}{\lambda}\left(\frac{1}{P}+\frac{1}{P'}\right) z}$$

$$\propto \sum_{n=0}^{n_{max}} \left(e^{i\frac{\pi}{\lambda}\left(\frac{1}{P}+\frac{1}{P'}\right) r_{n+1}^2} - e^{i\frac{\pi}{\lambda}\left(\frac{1}{P}+\frac{1}{P'}\right) r_n^2} \right) = \sum_{n=0}^{n_{max}} \left(e^{i\pi\left(\frac{1}{P}+\frac{1}{P'}\right)(n+1)f} - e^{i\pi\left(\frac{1}{P}+\frac{1}{P'}\right) n f} \right)$$

$$= \left(e^{i\pi\left(\frac{1}{P}+\frac{1}{P'}\right) f} - 1 \right) \sum_{n=0}^{n_{max}} e^{i\pi\left(\frac{1}{P}+\frac{1}{P'}\right) n f}$$

If $\left(\frac{1}{P}+\frac{1}{P'}\right) f = 2m+1$ is odd integer $\Rightarrow \psi \propto -2 \sum_{n=0}^{n_{max}} (-1)^n$

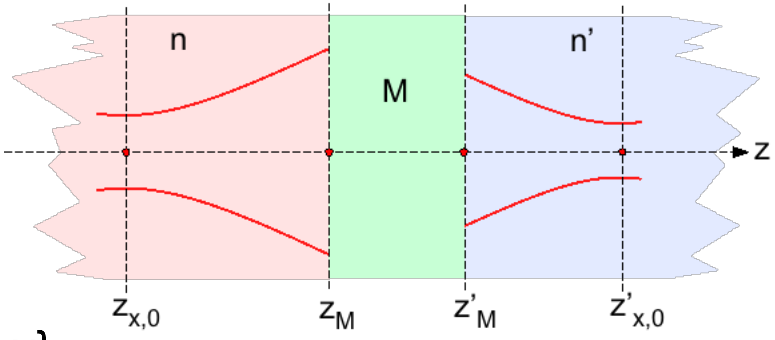
Hence, constructive inerference arises if odd or even zones are covered.

The first focus n=1 arises, if $\frac{1}{P}+\frac{1}{P'} = \frac{1}{f}$

Paraxial optics with Gaussian beams (ABCD law)

Assume that a Gaussian beam

(cf. Page 14)



$$u(x,y,z) = \frac{1}{\sqrt{q_x(z)q_y(z)}}\, e^{-ik\frac{x^2}{2q_x(z)}}\, e^{-ik\frac{y^2}{2q_y(z)}}$$

$$q_\nu(z) \equiv z - z_{\nu,0} + i\frac{k w_{\nu,0}^2}{2},\quad q_{\nu,M} \equiv q_\nu(z_M),\ \nu \in \{x,y\}$$

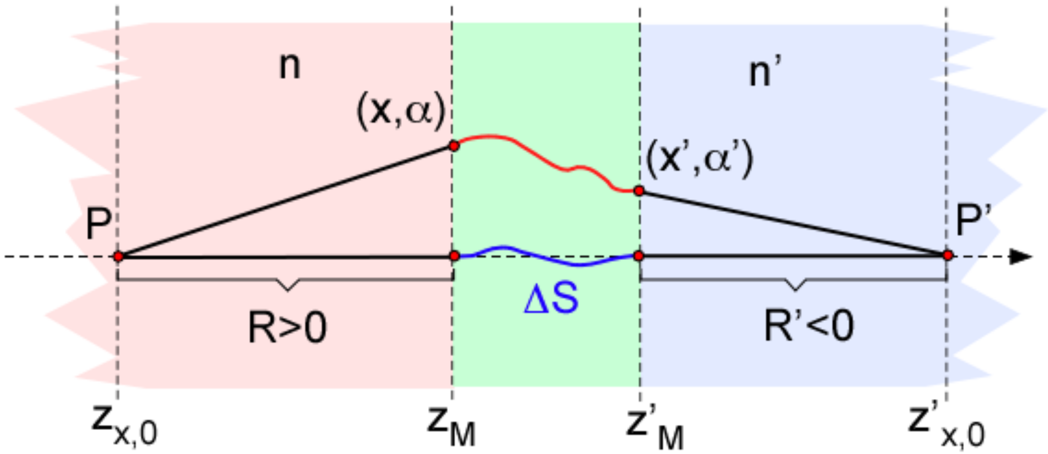
is incident upon an optical system described by a transfer matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

Statement: the transmitted beam is a Gaussian beam with

$$q'_\nu(z) \equiv z - z'_{\nu,0} + i\frac{k w'^2_{\nu,0}}{2},\ q'_{\nu,M} \equiv q'_\nu(z'_M),\ \nu \in \{x,y\}$$

$$\frac{q'_{\nu,M'}}{n'} = \frac{C + \frac{q_{\nu,M}}{n} D}{A + \frac{q_{\nu,M}}{n} B}$$

Proof



Consider conjugate points P, P ‘. According to Fermat ‘s principle, each path connecting P and P ‘ has the same optical path length:

$$S_1 = k\,n\,R + \Delta S - k\,n'R'$$

$$S_2 = k\,n\left(R + \frac{x^2}{2R}\right) + S(x',z_M') - S(x,z_M) - k\,n'\left(R' + \frac{x'^2}{2R'}\right)$$

$$S_1 = S_2 \quad \Rightarrow \quad S(x',z_M') - S(x,z_M) = \Delta S + k\,n'\frac{x'^2}{2R'} - k\,n\frac{x^2}{2R}$$

Determination of $S(x',z_M') - S(x,z_M)$

$$x' = Cn\alpha + Dx \qquad (1)$$

$$n'\alpha' = An\alpha + Bx \qquad (2)$$

$$AD - BC = 1 \qquad (3)$$

$$\begin{pmatrix} n'\alpha' \\ x' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} n\alpha \\ x \end{pmatrix}$$

$$R = \frac{x}{\alpha} = \frac{xCn}{\alpha Cn} \overset{(1)}{=} \frac{xCn}{x' - Dx} \qquad (4)$$

$$R' = \frac{x'}{\alpha'} = \frac{x'n'}{\alpha' n'} \overset{(2)}{=} \frac{x'n'}{An\alpha + Bx} \overset{(3)}{=} \frac{x'n'C}{An\alpha C + (AD-1)x} \overset{(1)}{=} \frac{x'n'C}{Ax' - x} \qquad (5)$$

$$S(x',z_M') - S(x,z_M) = \Delta S + kn'\frac{x'^2}{2R'} - kn\frac{x^2}{2R}$$

$$\underset{(4,5)}{=} \Delta S + \frac{k}{2C}x'(Ax' - x) - \frac{k}{2C}x(x' - Dx) = \Delta S + \frac{k}{2C}\left(Ax'^2 + Dx^2 - 2xx'\right) \qquad (6)$$

Assume that at z_M the ν-distribution of the incident field is

$$u(x,z_M) = u_0\, e^{-ikn\frac{x^2}{2q}}\ ,\quad q = q_{\nu,M} = q_\nu(z_M)\ ,\quad L \equiv \left(z'_M - z_M\right)\ ,\quad u_0 \equiv \left(\frac{1}{2\pi}\right)^{1/4}\left(\frac{kw_0}{q}\right)^{1/2}$$

Using Kirchhoff 's generalized integral (Page 32 with R = L, Q = 1)

$$u'(x',z_M') \cong \left(\frac{i}{\lambda L}\right)^{1/2}\int_{-\infty}^{\infty} dx\, u(x,z_M)e^{-i\left(S(z_M')-S(z_M)\right)} \overset{(6)}{=} \left(\frac{i}{\lambda L}\right)^{1/2}\int_{-\infty}^{\infty} dx\, u(x,z_M)e^{-i\left[\Delta S+\frac{k}{2C}\left(Ax'^2+Dx^2-2xx'\right)\right]}$$

$$= \left(\frac{i}{\lambda L}\right)^{1/2} u_0\ e^{-i\Delta S}\, e^{-i\frac{k}{2C}Ax'^2}\int_{-\infty}^{\infty} dx\, e^{-\left[ik\left(\frac{n}{2q}+\frac{D}{2C}\right)\right]x^2}\, e^{-2\left(i\frac{-kx'}{2C}\right)x}$$

$$= \left(\frac{i}{\lambda L}\right)^{1/2} u_0\ e^{-i\Delta S}\sqrt{\frac{2\pi C}{ik\left(\frac{n}{q}C+D\right)}}\ e^{-i\frac{n'kx'^2}{2}\left[\frac{1}{n'C}\left(A-\left(\frac{n}{q}C+D\right)^{-1}\right)\right]}$$

use $$\int_{-\infty}^{\infty} dx\, e^{-ax^2}e^{-bx} = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{a}}$$

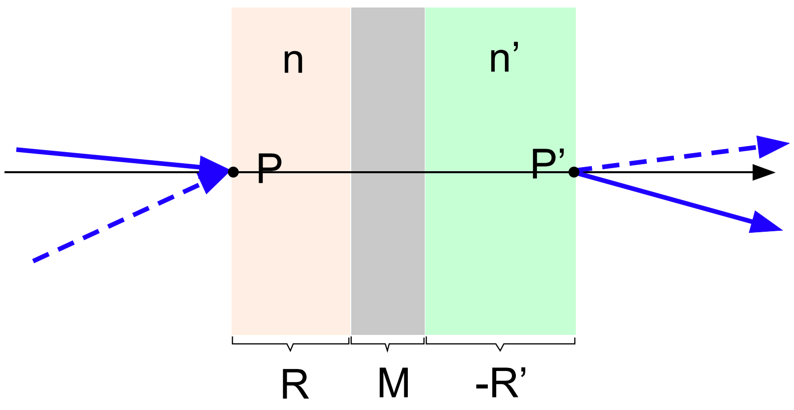
$$= \left(\frac{i}{\lambda L}\right)^{1/2} u_0 \; e^{-i\Delta S} \sqrt{\frac{2\pi C}{ik\left(\frac{n}{q}C + D\right)}} \; e^{-i\frac{n'kx'^2}{2}\left[\frac{1}{n'C}\left(A - \left(\frac{n}{q}C+D\right)^{-1}\right)\right]}$$

$$= \; e^{-i\Delta S} u_0 \sqrt{\frac{C}{\left(\frac{n}{q}C+D\right)L}} \; e^{-ikn'\frac{x'^2}{2q'}}, \quad \frac{1}{q'} = \frac{1}{n'C}\left(A - \left(\frac{n}{q}C+D\right)^{-1}\right)$$

$$\frac{n'}{q'} = \frac{1}{C}\left(A - \frac{1}{\frac{n}{q}C+D}\right) = \frac{1}{C}\left(\frac{\frac{n}{q}AC + AD - 1}{\frac{n}{q}C+D}\right) = \left(\frac{\frac{n}{q}A+B}{\frac{n}{q}C+D}\right)$$

$$\Rightarrow \quad \frac{q'}{n'} = \frac{\frac{n}{q}C+D}{\frac{n}{q}A+B} = \frac{C+\frac{q}{n}D}{A+\frac{q}{n}B}$$

Recall geometrical optics expression for conjugate points



$$\tilde{M} = \begin{pmatrix} A + \frac{R}{n}B & B \\ C + \frac{R}{n}D - \frac{R'}{n'}\left(A + \frac{R}{n}B\right) & D - \frac{R'}{n'}B \end{pmatrix}$$

with $M \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

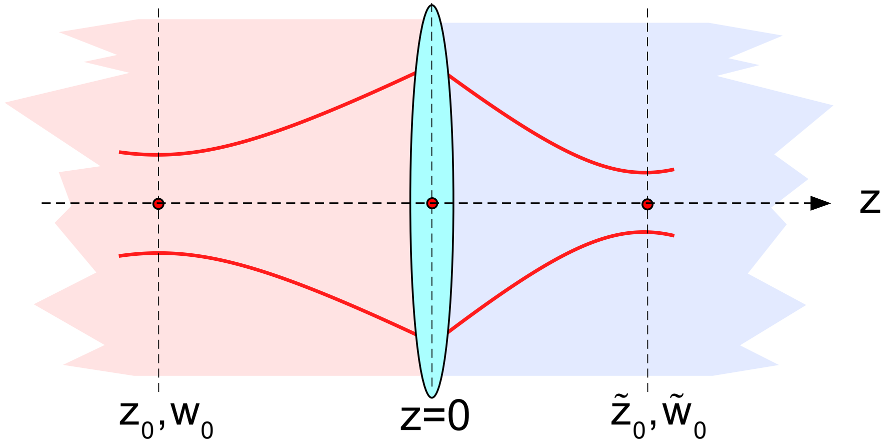
$$0 = C + \frac{R}{n}D - \frac{R'}{n'}\left(A + \frac{R}{n}B\right) \Rightarrow \frac{R'}{n'} = \frac{C + \frac{R}{n}D}{A + \frac{R}{n}B}$$

Paraxial wave optics $\quad q \leftrightarrow R \quad$ Paraxial geom. optics

Theorem: $q_1 = f_{M_1}(q_0)\,,\quad q_2 = f_{M_2}(q_1) \;\Rightarrow\; q_2 = f_{M_2 \circ M_1}(q_0)$

Example 1: lens with focal length f

$$M \equiv \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}, \quad B = -\frac{1}{f}$$



$$q \equiv q(0) = -z_0 + q_0, \quad q_0 = i\frac{\pi w_0^2}{\lambda}$$

$$\tilde{q} \equiv \tilde{q}(0) = -\tilde{z}_0 + \tilde{q}_0, \quad \tilde{q}_0 = i\frac{\pi \tilde{w}_0^2}{\lambda}$$

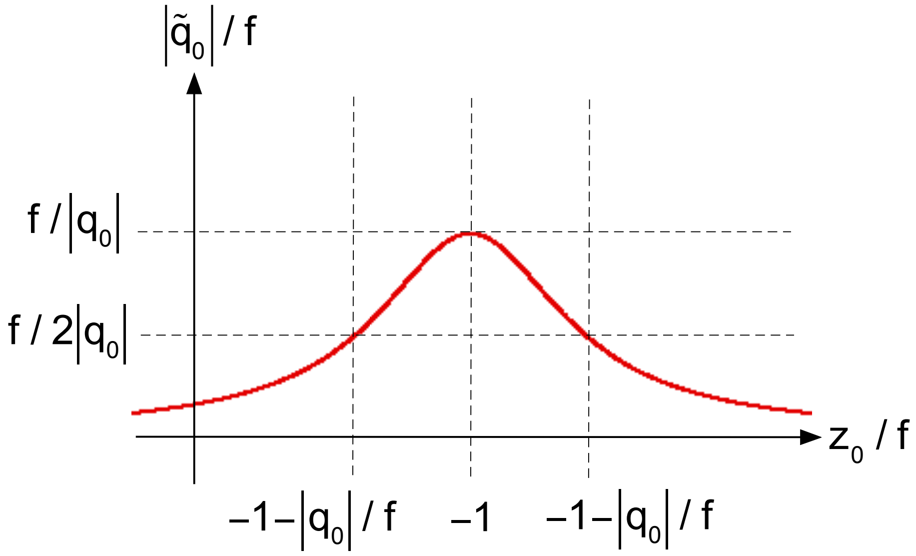
$\Rightarrow$

$$\mathrm{Re}[\tilde{q}] = \frac{\mathrm{Re}[q] + |q|^2 B}{1 + 2\mathrm{Re}[q]\,B + |q|^2 B^2}$$

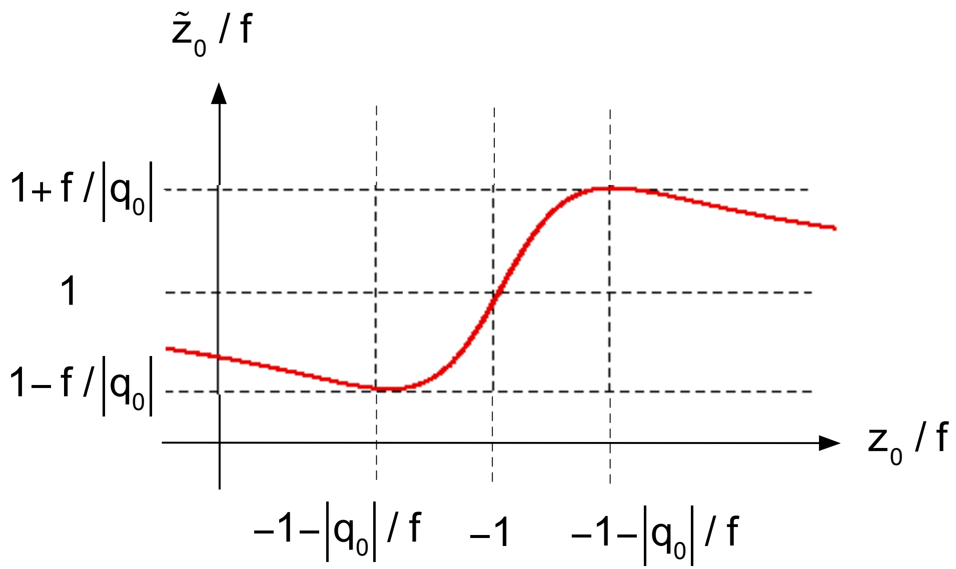
$$\mathrm{Im}[\tilde{q}] = \frac{\mathrm{Im}[q]}{1 + 2\mathrm{Re}[q]\,B + |q|^2 B^2}$$

Example 1: lens with focal length f

$$\Rightarrow \qquad \frac{|\tilde{q}_0|}{f} = \frac{\dfrac{|q_0|}{f}}{\left(1+\dfrac{z_0}{f}\right)^2 + \dfrac{|q_0|^2}{f^2}}$$



$$\frac{\tilde{z}_0}{f} = 1 - \frac{1+\dfrac{z_0}{f}}{\left(1+\dfrac{z_0}{f}\right)^2 + \dfrac{|q_0|^2}{f^2}}$$



Special case:

$$z_0 = -f \quad \Rightarrow \quad |\tilde{q}_0||q_0| = f^2 \,, \quad \tilde{z}_0 = f$$

Gaussian beams in optical resonators

consider a resonator with unit cell $M \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $\tilde{q} = \frac{C + qD}{A + qB}$

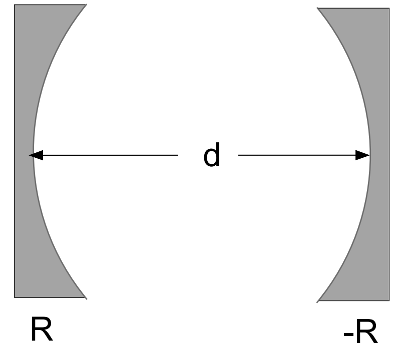
$$\tilde{q} = f_M(q) = q \quad \Leftrightarrow \quad q^2 + q\left(\frac{A - D}{B}\right) - \frac{C}{B} = 0$$

$$\Leftrightarrow \quad q = \frac{D - A}{2B} \pm \sqrt{\frac{(A + D)^2 - 4}{4B^2}} = \frac{D - A}{2B} \pm i\sqrt{\frac{4 - (A + D)^2}{4B^2}}$$

criterion for stability: $\frac{1}{2}\mathrm{Trace}(M) = \frac{1}{2}|A + D| < 1$

same condition as in geometrical optics:

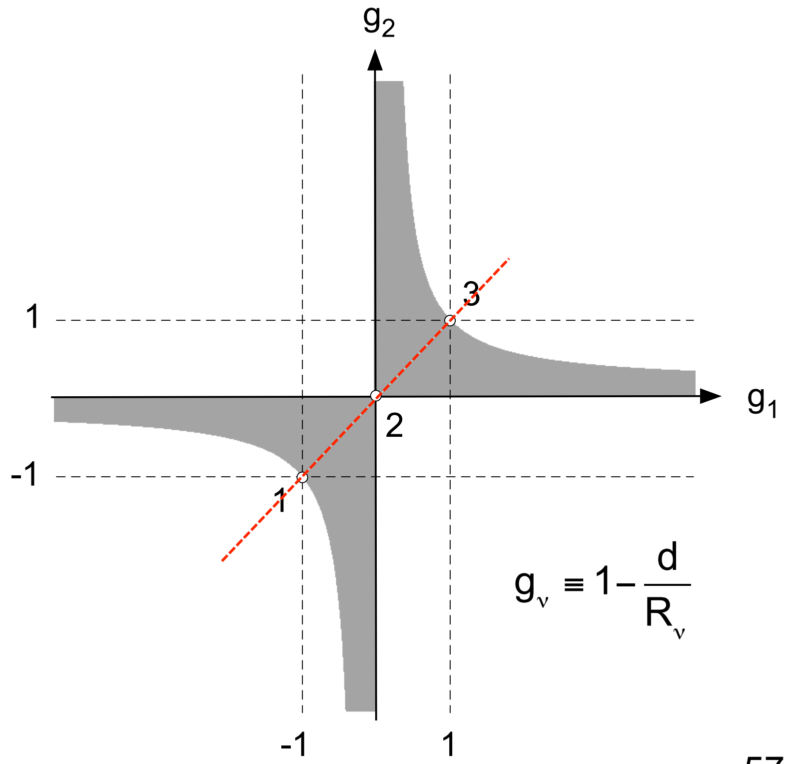
Example: resonator with two identical mirrors



unit cell

$$M = \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{2}{R} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{2}{R} \\ d & 1-\frac{2d}{R} \end{pmatrix}$$

$$\Rightarrow \qquad q = \frac{d}{2} \pm i \, \frac{1}{2}\sqrt{(2R-d)\,d}$$



1. concentric resonator

$$\text{waist position} = \frac{d}{2}, \ \text{confocal parameter} = 0$$

2. confocal resonator

$$\text{waist position} = \frac{d}{2}, \ \text{confocal parameter} = d$$

3. plane resonator

$$\text{waist position} = \frac{d}{2}, \ \text{confocal parameter} = \infty$$

Resonance condition $w_0 \equiv w_{x,0} = w_{y,0}$

$$kd - (n+m+1)\left(\Phi(\tfrac{d}{2}) - \Phi(-\tfrac{d}{2})\right) = \nu\,\pi \ , \nu \ \text{integer}$$

$$\Phi(z) \equiv \arctan\left(\frac{2z}{b}\right), \quad b = \frac{2\pi w_0^2}{\lambda} = \sqrt{(2R-d)\,d}$$

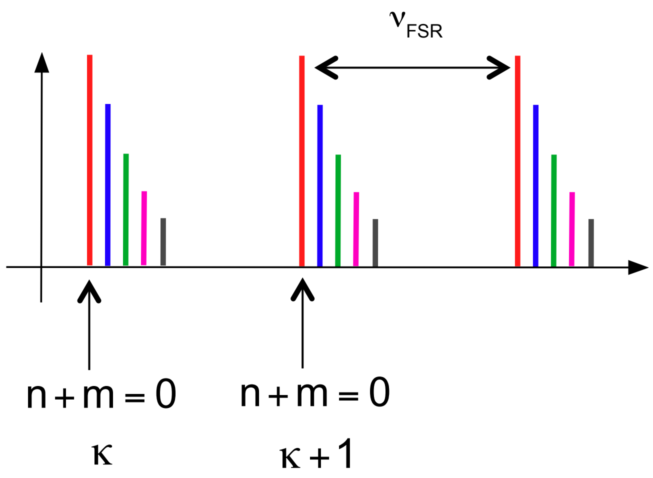
$$\Rightarrow \quad kd - (n+m+1)\ 2\ \text{arctg}\left(\frac{d}{\sqrt{(2R-d)\,d}}\right) = \kappa\,\pi \ , \kappa \ \text{integer}$$

use $2\ \text{arctg}(x) = \text{arctg}\left(\frac{2x}{1-x^2}\right), \quad \text{arctg}(y) = \arccos\left(\frac{1}{\sqrt{1+y^2}}\right)$

$$\Rightarrow \quad kd - (n+m+1)\arccos\left(1-\frac{d}{R}\right) = \kappa\,\pi \ , \kappa \ \text{integer}$$

$$\Rightarrow \quad \frac{\nu}{\nu_{FSR}} = \kappa - \frac{1}{\pi}(n+m+1)\ \arccos\left(1-\frac{d}{R}\right), \kappa \ \text{integer}$$

nearly plane resonator $R \gg d$



confocal resonator $R = d$ $\Rightarrow$ $\frac{\nu}{\nu_{FSR}} = \kappa - \frac{1}{2}(n+m+1)$, κ integer