6. Synchrotron radiation

6.1. Radiation of accelerated particles

Electrostatic potential Φ and vector potential $\vec{A}$ of a point charge:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\iiint_V \frac{\rho(\vec{r})}{|\vec{x}-\vec{r}|}d^3r \quad\rightarrow\quad \Phi(R) = \frac{e}{4\pi\varepsilon_0 R}$$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\iiint_V \frac{\vec{j}(\vec{r})}{|\vec{x}-\vec{r}|}d^3r \quad\rightarrow\quad \vec{A}(R) = \frac{\vec{\mathrm{v}}}{c^2}\Phi(R) = \frac{\mu_0}{4\pi}\cdot\frac{e\vec{\mathrm{v}}}{\mathrm{R}}$$

In case of a relativistic movement one obtains the so-called

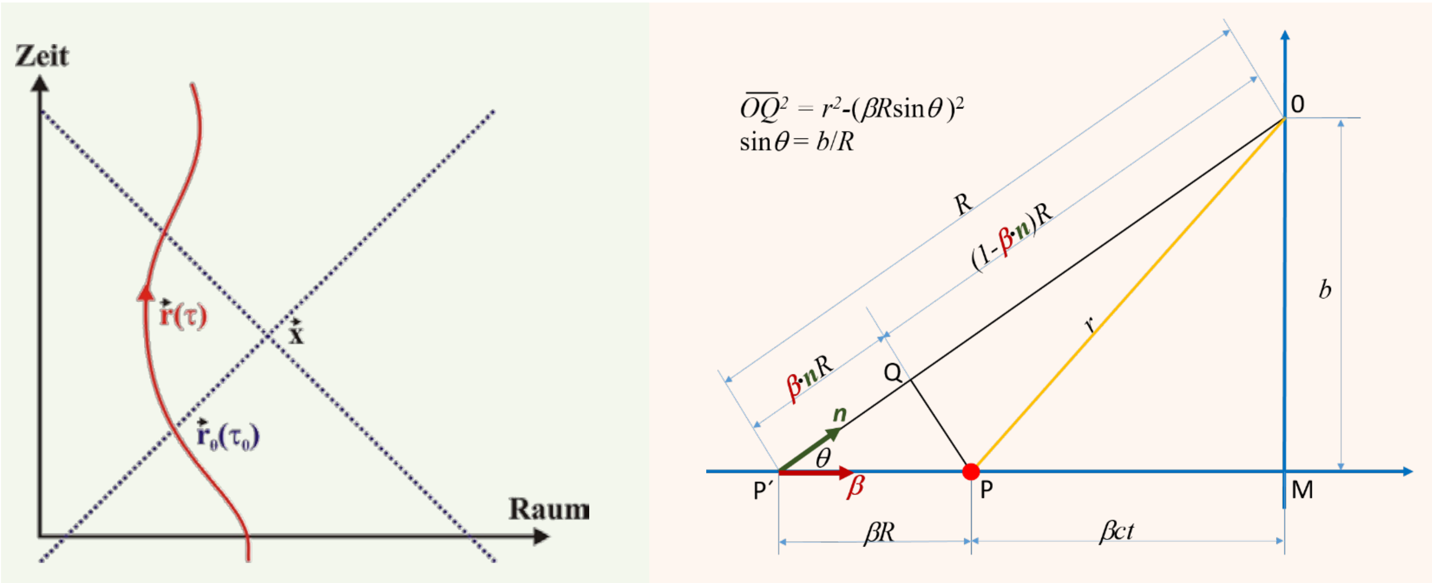
Liénard Wiechert potentials:

$$\Phi(\vec{x},t) = \left[\frac{1}{4\pi\varepsilon_0}\cdot\frac{e}{\left(1-\vec{\beta}\cdot\vec{n}\right)R}\right]_{ret}, \qquad \vec{A}(\vec{x},t) = \left[\frac{\mu_0 c}{4\pi}\cdot\frac{e\vec{\beta}}{\left(1-\vec{\beta}\cdot\vec{n}\right)R}\right]_{ret}$$

where the expressions in the brackets are to be evaluated at the retarded time

$$t_0{}'(\tau_0) = t_0 - R/c$$

and $\vec{n}$ is a unit vector in the direction of $\vec{x} - \vec{r}(\tau)$.



Only an intersection of the world line of a particle with the light cone contributes to the field at the "point" $(\vec{x}, t)$!

The electric and magnetic field are calculated by differentiation

$$\vec{E} = -\left[\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t'}\right]_{ret}, \qquad \vec{B} = \vec{\nabla}\times\vec{A},$$

and after some lengthy calculations one obtains:

$$\vec{E}(\vec{x},t) = \underbrace{\frac{e}{4\pi\varepsilon_0}\left[\frac{\vec{n}-\vec{\beta}}{\gamma^2\left(1-\vec{n}\cdot\vec{\beta}\right)^3 R^2}\right]_{ret}}_{\text{velocity fields}} + \underbrace{\frac{\mu_0 c}{4\pi}\left[\frac{\vec{n}\times\left\{\left(\vec{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right\}}{\left(1-\vec{n}\cdot\vec{\beta}\right)^3 R}\right]_{ret}}_{\text{acceleration fields}}, \qquad \vec{B} = \frac{1}{c}\cdot\left[\vec{n}\times\vec{E}\right]_{ret}$$

Reminder:

- **Velocity fields** essentially are static fields and drop by $1/R^2$: **Near fields!**
- **Acceleration fields** are radiation fields and drop by $1/R$: **Far fields!**

Now everything will be considered in a frame of reference in which the velocity of the charge is small compared to c ("*rest frame*"):

$$\vec{E}^* = \frac{\mu_0 c}{4\pi} \cdot \left[\frac{\vec{n} \times \left(\vec{n} \times \dot{\vec{\beta}}^* \right)}{R} \right]_{ret}$$

The energy flux is described by the Poynting vector:

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \stackrel{!}{=} \varepsilon_0 c \left| \vec{E}^* \right|^2 \vec{n}$$

The power radiated into the solid angle section Ω results therewith as

$$\frac{dP}{d\Omega} = \varepsilon_0 c \left| R \vec{E}^* \right|^2 = \frac{\mu_0 c}{(4\pi)^2} \cdot e^2 \cdot \left| \vec{n} \times \left(\vec{n} \times \dot{\vec{\beta}}^* \right) \right|^2$$

and can be expressed by the angle θ between $\dot{\vec{\beta}}^*$ and $\vec{n}$ as follows:

$$\frac{dP}{d\Omega} = \frac{\mu_0 c\, e^2}{(4\pi)^2} \cdot \left| \dot{\vec{\beta}}^* \right|^2 \cdot \sin^2 \theta$$

Integration over all angles yields for the total radiated power in non-relativistic approximation the famous **Larmor formula**:

$$P = \frac{e^2}{6\pi\varepsilon_0 c}\cdot\left|\dot{\vec{\beta}}^*\right|^2$$

Because of $\vec{p} = \vec{\beta}\gamma m_0 c$ it can be written as ($\beta \ll 1$!!!)

$$P = \frac{e^2}{6\pi\varepsilon_0 m_0^2 c^3}\cdot\left(\frac{d\,\vec{p}^*}{d\,t}\cdot\frac{d\,\vec{p}^*}{d\,t}\right)$$

With the proper time unit $d\tau = dt/\gamma$ and the four-momentum p_μ one gets the following Lorentz invariant form

$$P = -\frac{e^2}{6\pi\varepsilon_0 m_0^2 c^3}\cdot\left(\frac{d\,p_\mu}{d\,\tau}\frac{d\,\vec{p}^\mu}{d\,\tau}\right),$$

which for $\beta \to 0$ reduces to the Larmor formula.

(Remark: It's interesting to see how this is e.g. "derived" in Jackson's classical textbook "Classical Electrodynamics"!)

Expressing it by the energy $E = \gamma m_0 c^2$ and the momentum $\vec{p} = \vec{\beta}\gamma m_0 c$ ensues

$$-\frac{d\,p_\mu}{d\,\tau}\frac{d\,\vec{p}^\mu}{d\,\tau} = \left(\frac{d\,\vec{p}}{d\,\tau}\right)^2 - \frac{1}{c^2}\left(\frac{d\,E}{d\,\tau}\right)^2 = \left(\gamma\, m_0\, c\right)^2 \cdot \left[\left(\frac{d\,\gamma\vec{\beta}}{d\,t}\right)^2 - \left(\frac{d\,\gamma}{d\,t}\right)^2\right]$$

and using the relations $\dot{\beta}^2 \big/ \gamma^2 + \left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2 = \dot{\beta}^2 - \left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2$ and $\dot{\gamma} = \left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)\gamma^3$ one finally gets:

$$P = \frac{e^2\,\gamma^6}{6\pi\,\varepsilon_0\,c}\cdot\left[\dot{\beta}^2 - \left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2\right]$$

In accelerators either of the two cases of linear acceleration $\dot{\vec{\beta}}_{\parallel}$ when passing through accelerating sections and centripetal acceleration $\dot{\vec{\beta}}_{\perp}$ during deflection in magnetic fields occurs separately. The resulting radiated powers are:

$$P_{\parallel} = \frac{e^2 \gamma^6}{6\pi \varepsilon_0 c} \cdot \dot{\beta}_{\parallel}^{\,2} = \frac{e^2}{6\pi \varepsilon_0 m_0^{\,2} c^3} \cdot \left(\frac{d \vec{p}_{\parallel}}{d t} \right)^2 = \frac{e^2}{6\pi \varepsilon_0 m_0^{\,2} c^3} \cdot \left(\frac{d E}{d x} \right)^2$$

$$P_{\perp} = \frac{e^2 \gamma^4}{6\pi \varepsilon_0 c} \cdot \dot{\beta}_{\perp}^{\,2} = \frac{e^2 \gamma^2}{6\pi \varepsilon_0 m_0^{\,2} c^3} \cdot \left(\frac{d \vec{p}_{\perp}}{d t} \right)^2 = \frac{e^2 c \beta^4}{6\pi \varepsilon_0 \left(m_0 c^2 \right)^4} \cdot \frac{E^4}{R^2}$$

Implications:

- Linear accelerator: $\frac{d E}{d x} \overset{\text{typ}}{\leq} 15\ \text{MeV/m}$, thus $\frac{P}{d E / d x} \approx 5 \cdot 10^{-14}$!!!
- Electron- / proton accelerator: $\frac{P_e}{P_p} = \left(\frac{m_p}{m_e} \right)^4 \approx 10^{13}$!!!

The synchrotron radiation $P_{\perp}$ has been theoretically predicted by Liénard by the end of the penultimate century. The experimental verification only succeeded at the end of the forties at the 70 MeV synchrotron of General Electric in the USA.

6.2. Circumference voltage

The energy loss per revolution $\Delta E = P \cdot T$ results from the radiation loss P and the staying time $T = \frac{2\pi R}{\beta c}$ per revolution in the deflecting magnets

$$\Delta E = \frac{e^2 \beta^3}{3\varepsilon_0 \left(m_0 c^2\right)^4} \cdot \frac{E^4}{R}$$

and is called circumference voltage. Approximately the following applies:

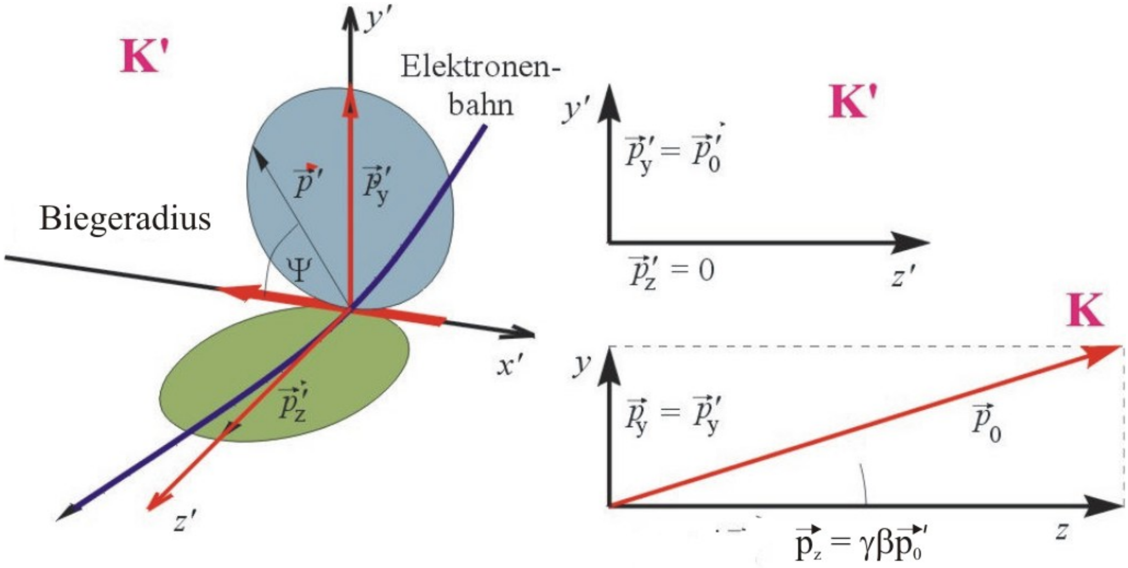
$$\Delta E[\mathrm{keV}] \approx 88{,}5 \cdot \frac{E^4[\mathrm{GeV}]}{R[\mathrm{m}]}$$

For illustration purposes we give here some numerical values:

Accelerator	L / m	E / GeV	R / m	B / T	ΔE / MeV
BESSY I (Berlin)	62.4	0.8	1.78	1.50	0.02
DELTA (Dortmund)	115	1.5	3.34	1.50	0.134
ELSA (Bonn)	164.2	3.5	11.0	1.08	1.22
DORIS II (Hamburg)	288	5.0	12,2	1.37	4.53
PETRA (Hamburg)	2304	23.5	195	0.40	138
LEP (Geneva)	27000	104.6	3000	0.116	3450

6.3. Time structure and angular distribution

In order to provide a clarifying illustration, the situation of a circulating electron will be considered here in an approximation. In a frame of reference moving uniformly with the electron, the latter, being located in the origin at the time t=0, only experiences an acceleration along the x´-axis:



The radiation characteristic corresponds to the Hertzian dipole. A photon being emitted in y´-direction holds the momentum:

$$p_y' = p_0' = \frac{\hbar\omega'}{c}\hat{e}_y'$$

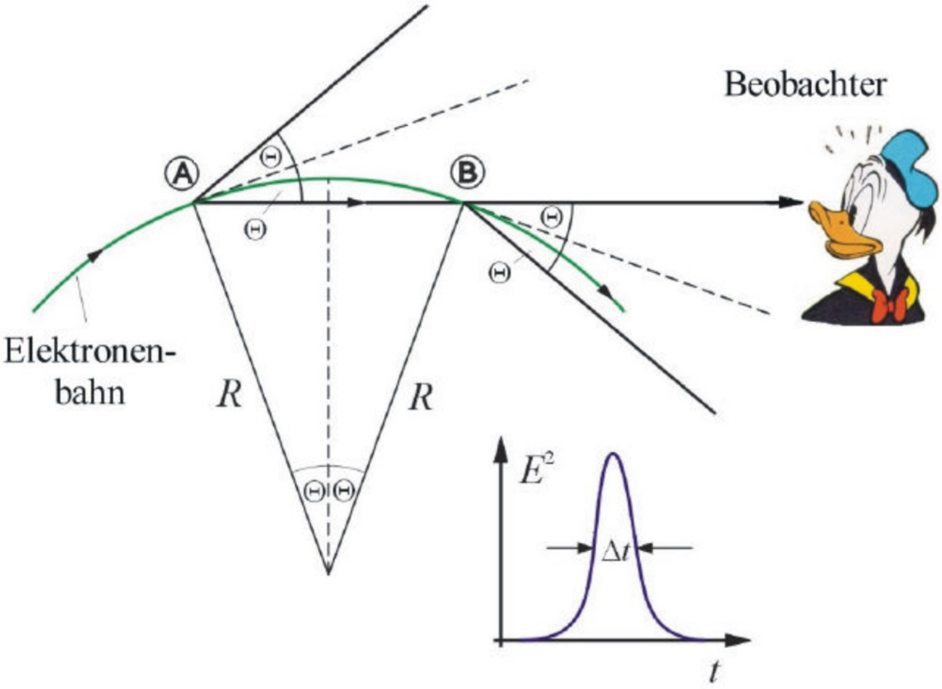
By Lorentz transformation, one gets the momentum in the laboratory frame:

$$P_\mu = \begin{bmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{bmatrix} \bullet \begin{pmatrix} \hbar\omega'/c \\ 0 \\ p_0' \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma\hbar\omega'/c \\ 0 \\ p_0' \\ \beta\gamma\hbar\omega'/c \end{pmatrix}$$

Herefrom ensues for the angle Θ under which the photon is emitted in the laboratory frame:

$$\tan\Theta = \frac{p_y}{p_z} \approx \frac{1}{\gamma}$$

The result is a focusing of the light beam with the angle of aperture $\Theta \approx 1/\gamma$ in the shape of a narrow club which, like the beam of a headlamp, is emitted tangentially to the electron path and sweeps over the observer:



Only during the flight from point A to point B the emitted light reaches the observer. He sees a pulse whose length corresponds to Δt, the delay between electron and photon from A to B ($\beta\gamma \approx \gamma - 1/2\gamma$):

$$\Delta t = \frac{2R\Theta}{\beta c} - \frac{2R\sin\Theta}{c} \approx \frac{2R}{c}\left(\frac{\Theta}{\beta} - \Theta + \frac{\Theta^3}{3!}\right) \approx \frac{2R}{c}\left(\frac{1}{\beta\gamma} - \frac{1}{\gamma} + \frac{1}{6\gamma^3}\right) \approx \frac{4R}{3c\gamma^3}$$

The following frequency corresponds to this pulse length:

$$\omega_{typ} = \frac{2\pi}{\Delta t} = \pi\frac{3c\gamma^3}{2R} \equiv \pi\cdot\omega_c$$

The thereby defined frequency ω_c is called the critical frequency (see below).

6.4. Spectrum of the synchrotron radiation

For simplification matters we do set in the following $\vec{A}(t) = \sqrt{\varepsilon_0 c}\left[R\vec{E}\right]_{ret}$ and

thereby get: $$\frac{dP}{d\Omega} = \left|\vec{A}(t)\right|^2$$

The total irradiated energy W then is

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} \frac{dP}{d\Omega} dt = \int_{-\infty}^{\infty} \left|\vec{A}(t)\right|^2 dt$$

This can be expressed through the Fourier transform of $\vec{A}$:

$$\frac{dW}{d\Omega} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \, \vec{A}^*(\omega') \cdot \vec{A}(\omega) \cdot e^{i(\omega-\omega')\cdot t} = \int_{-\infty}^{\infty} \left|\vec{A}(\omega)\right|^2 d\omega$$

If one restrains the integration to positive frequency values, one gets for the radiated energy per solid angle unit and frequency interval:

$$\frac{d^2 I}{d\omega d\Omega} = \left|\vec{A}(\omega)\right|^2 + \left|\vec{A}(-\omega)\right|^2 \overset{A\text{ real}}{=} 2\left|\vec{A}(\omega)\right|^2$$

After calculating the Fourier transform taking into account the retardation condition

$$t' + R(t')/c = t$$

and the approximation for large distances (allowed by $R(t') \approx x - \vec{n} \cdot \vec{r}(t')$) as well as the decomposition into orthogonal polarizations one gets for the radiated power on a circular orbit (see e.g. Jackson):

$$\frac{d^2 I}{d\omega d\Omega} = \frac{3e^2}{16\pi^3 \varepsilon_0 c} \cdot \gamma^2 \left(\frac{\omega}{\omega_c}\right)^2 \left\{1 + \gamma^2\theta^2\right\}^2 \cdot \left[\underbrace{K_{2/3}^2(\xi)}_{\text{radiation polarized in the orbit plane}} + \underbrace{\frac{\gamma^2\theta^2}{1+\gamma^2\theta^2} \cdot K_{1/3}^2(\xi)}_{\text{radiation polarized perpendicularly to the orbit plane}} \right]$$

where:

- θ = polar angle to the orbit plane,
- $\omega_c = \dfrac{3c\gamma^3}{2R}$ "**critical frequency**" (R = radius of curvature of the orbit),
- $\xi = \dfrac{\omega}{2\omega_c}\left(1 + \gamma^2\theta^2\right)^{3/2}$ dimensionless parameter as the argument.

In the orbit plane, the light therefore is completely linearly polarized, outside of it weakly elliptically polarized!

In the low and high frequency range, the following approximations apply for $\theta \approx 0$:

$$\left.\frac{d^2 I}{d\omega\, d\Omega}\right|_{\theta=0} \approx \begin{cases} \dfrac{e^2}{4\pi\varepsilon_0 c}\left[\dfrac{\Gamma(2/3)}{\pi}\right]^2 \left(\dfrac{3}{4}\right)^{1/3}\left(\dfrac{\omega R}{c}\right)^{2/3}, & \text{if } \omega \ll \omega_c \\ \dfrac{3e^2}{4\pi^2\varepsilon_0 c}\gamma^2 \dfrac{\omega}{\omega_c} e^{-\omega/\omega_c}, & \text{if } \omega \gg \omega_c \end{cases}$$

For the "critical angles" (for a more precise definition see Jackson) we have:

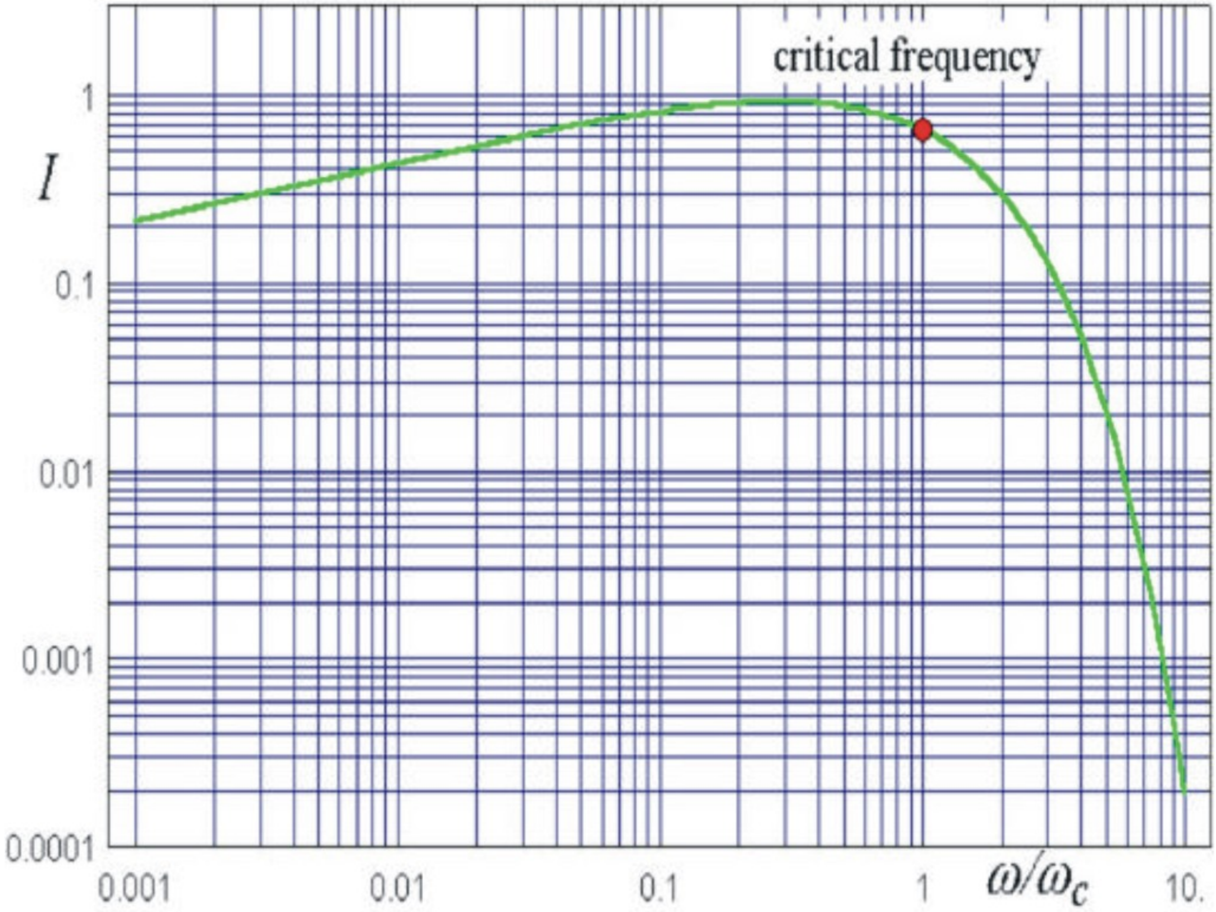
$$\theta_c \approx \begin{cases} \dfrac{1}{\gamma}\left(\dfrac{\omega_c}{2\omega}\right)^{1/3}, & \text{if } \omega \ll \omega_c \qquad [\xi(\theta_c) = \xi(0) + 1] \\ \dfrac{1}{\gamma}\left(\dfrac{\omega_c}{6\omega}\right)^{1/2}, & \text{if } \omega \gg \omega_c \qquad [I(\theta_c) = I(0)/e] \end{cases}$$

The low frequency components are emitted under a much higher, the high frequency components under a much smaller angle than the average radiation!

Integration over the polar angle θ (allowed because of the small aperture angle) yields the spectral energy distribution:

$$\frac{dI}{d\omega} = \frac{\sqrt{3}e^2}{4\pi\varepsilon_0 c}\gamma\frac{\omega}{\omega_c}\int_{\omega/\omega_c}^{\infty} K_{5/3}(x)\,dx \approx \begin{cases} \dfrac{e^2}{4\pi\varepsilon_0 c}\sqrt[3]{\dfrac{\omega R}{c}}, & \text{if } \omega \ll \omega_c \\ \dfrac{\sqrt{6\pi}e^2}{4\pi\varepsilon_0 c}\gamma\sqrt{\dfrac{\omega}{\omega_c}}e^{-\omega/\omega_c}, & \text{if } \omega \gg \omega_c \end{cases}$$

The critical energy $\hbar\omega_c$ divides the spectrum of the synchrotron radiation into two equal parts.


critical frequency
I
1
0.1
0.01
0.001
0.0001
0.001
0.01
0.1
1
ω/ω_c
10.