5.3. Longitudinal Beam Dynamics

5.3.1. Equation of Motion in Phase Space

From the discussion of the momentum compaction (chapter 4.3.7.) we have obtained for the relative variation of the travel time $\Delta T/T_0$ and the angular revolution frequency $\Delta\omega/\omega_0$:

$$\frac{\Delta T}{T_0} = -\frac{\Delta\omega}{\omega_0} = -\left(\frac{1}{\gamma^2} - \alpha_c\right)\frac{\Delta p}{p_0} = -\eta\cdot\frac{\Delta p}{p_0}$$

The revolution frequency ω_0 is linked to the RF frequency ω_{RF} by the number *h* of circulating bunches, which is called the **harmonic number**. Using this relation we obtain for the phase shift $\Delta\varphi = \varphi - \varphi_0$ with respect to a reference particle (with reference phase φ_0):

$$\Delta\varphi = \omega_{RF}\cdot\Delta T = h\cdot\omega_0\cdot\Delta T$$

The phase shift per revolution can be linked to the relative momentum deviation by using the η-parameter:

$$\left(\Delta\varphi\right)_{rev} = -\eta\, h\, \omega_0\, T_0 \frac{\Delta p}{p_0} = -2\pi\, h\, \eta \frac{\Delta p}{p_0}$$

and may be expressed in terms of the relative energy deviation using

$$E^2 = m_0^2\, c^4 + p^2\, c^2 \;\Rightarrow\; 2E\cdot dE = 2\,p\,c^2\cdot dp \;\Rightarrow\; dE = \beta\, c\cdot dp \;\Rightarrow\; \frac{dE}{E} = \beta^2 \frac{dp}{p}$$

which gives:

$$\left(\Delta\varphi\right)_{rev} = -\frac{2\pi\, h\,\eta}{\beta^2}\frac{\Delta E}{E_0}$$

So far, we have expressed the phase shift $\left(\Delta\varphi\right)_{rev}$ per revolution in terms of $\Delta E = E - E_0$. In order to relate this to the energy gain per turn produced by acceleration, we first have to divide by the revolution time T_0 to get the change of the phase shift per unit time $\Delta\dot{\varphi}$:

$$\frac{d}{dt}\Delta\varphi = \frac{\left(\Delta\varphi\right)_{rev}}{T_0} = -\frac{2\pi\, h\,\eta}{\beta^2\, T_0\, E_0}\cdot\Delta E$$

We then have to built the second derivative to express this variation in terms of the energy gain $(\Delta E)_{rev}$ per turn

$$(\Delta E)_{rev} = eU(\varphi) - W(E) = eU_0 \sin\varphi - W(E)$$

where $W(E)$ represents the radiation losses per turn due to synchrotron radiation and $U(\varphi)$ is the acceleration voltage for a given phase φ. The energy gain per turn $(\Delta E)_{rev}$ is linked to the energy deviation ΔE with respect to the reference particle by

$$\frac{d}{dt}\Delta E = \frac{1}{T_0} \cdot (\Delta E)_{rev}$$

This gives

$$\frac{d^2 \Delta\varphi}{dt^2} + \frac{2\pi h \eta}{\beta^2 T_0 E_0} \cdot \frac{d\,\Delta E}{dt} = 0$$

and we finally obtain

$$\boxed{\frac{d^2 \Delta\varphi}{dt^2} + \frac{2\pi h \eta}{\beta^2 T_0^2 E_0} \cdot \left[eU_0 \sin\left(\varphi_0 + \Delta\varphi\right) - W(E) \right] = 0}$$

5.3.2. Small Oscillation Amplitudes

For small deviations $\Delta\varphi$ from the synchronous phase we can expand the acceleration voltage into a Taylor series and get

$$\frac{d}{dt}\Delta E = \frac{\Delta E}{T_0} \approx \frac{1}{T_0}\left\{ eU(\varphi_0) + e\frac{dU(\varphi_0)}{d\varphi}\cdot\Delta\varphi - W(E_0) - \frac{dW(E_0)}{dE}\cdot\Delta E \right\}$$

At equilibrium we have $eU(\varphi_0) = W(E_0)$ and obtain the phase equation

$$\boxed{\frac{d^2\Delta\varphi}{dt^2} + 2\cdot\underbrace{\left(\frac{1}{2T_0}\cdot\frac{dW(E_0)}{dE}\right)}_{=\alpha_S}\cdot\frac{d\Delta\varphi}{dt} + \underbrace{\left(\frac{2\pi h\eta e}{\beta^2 T_0^2 E_0}\cdot U_0\cos\varphi_0\right)}_{=\Omega_S^2}\cdot\Delta\varphi = 0}$$

Particles orbiting in a circular accelerator therefore perform longitudinal oscillations with the angular frequency Ω_S, which are called **synchrotron oscillations**. These phase oscillations are damped or antidamped depending on the sign of the damping decrement α_S. For small oscillation amplitudes the movement can be described by a damped harmonic oscillator. In most cases we find the damping time much longer than the phase oscillation period

$$\tau_S = \frac{1}{\alpha_S} \ll \frac{2\pi}{\Omega_S} = \frac{1}{Q_S}$$

and the synchrotron tune Q_S, defined by the number of longitudinal oscillations per turn, much smaller than the transverse tunes Q_X, Q_Z.

The oscillations are stable for a real angular frequency Ω_S and therefore for a positive product $\eta \cdot \cos\varphi_0$. From $\eta = 1/\gamma^2 - 1/\gamma_{tr}^2$ and the equilibrium condition $eU_0 \sin\varphi_0 = W(E_0) > 0$ we derive the condition for stable phase focusing:

$$0 < \varphi_0 < \frac{\pi}{2} \qquad \text{for} \qquad \gamma < \gamma_{tr}$$

$$\frac{\pi}{2} < \varphi_0 < \pi \qquad \text{for} \qquad \gamma > \gamma_{tr}$$

Neglecting the small damping term the equation of motion reads

$$\frac{d^2\Delta\varphi}{dt^2} + \Omega_S^2 \cdot \Delta\varphi = 0$$

and is solved by a harmonic oscillation

$$\Delta\varphi = \widehat{\Delta\varphi} \cdot \cos\left(\Omega_S t + \phi\right)$$

Building the first derivative and relating $\Delta\dot{\varphi}$ to the relative energy deviation $\Delta E / E_0$,

we obtain for the amplitude $\widehat{\Delta\varphi}$ of the oscillation

$$\Delta\dot{\varphi} = -\Omega_S \cdot \widehat{\Delta\varphi} \cdot \sin\left(\Omega_S t + \phi\right) = -\frac{2\pi h \eta}{\beta^2 T_0} \cdot \frac{\Delta E}{E_0} = \eta\,\omega_{RF} \cdot \frac{\Delta p}{p_0}$$

$$\Rightarrow \qquad \widehat{\Delta\varphi} = \frac{\eta\,\omega_{RF}}{\beta^2 \Omega_S} \cdot \left(\frac{\Delta E}{E_0}\right)_{\max} = \frac{\eta\,\omega_{RF}}{\Omega_S} \cdot \left(\frac{\Delta p}{p_0}\right)_{\max}$$

All particles of a beam perform incoherent phase oscillations about a common reference point and generate thereby the appearance of a steady longitudinal distribution of particles which we call a particle bunch.

The total bunch length l_b can be determined from the maximum longitudinal excursion of particles from the bunch center and is twice the amplitude of the phase variation.:

$$\frac{l_b}{2} = \frac{\lambda_{RF}}{2\pi} \cdot \widehat{\Delta\varphi} = \frac{c}{h\omega_0} \cdot \widehat{\Delta\varphi}$$

Using the equation derived above, this gives

$$\boxed{l_b = 2 \cdot \frac{c\sqrt{2\pi}}{\beta\omega_0} \cdot \sqrt{\frac{\eta E_0}{heU_0 \cos\varphi_0}} \cdot \left(\frac{\Delta E}{E_0}\right)}$$

5.3.3. Large Amplitude Oscillations

We will ignore the small damping term for the following discussions. This allows us to rewrite the equation of motion (without any further approximation) to

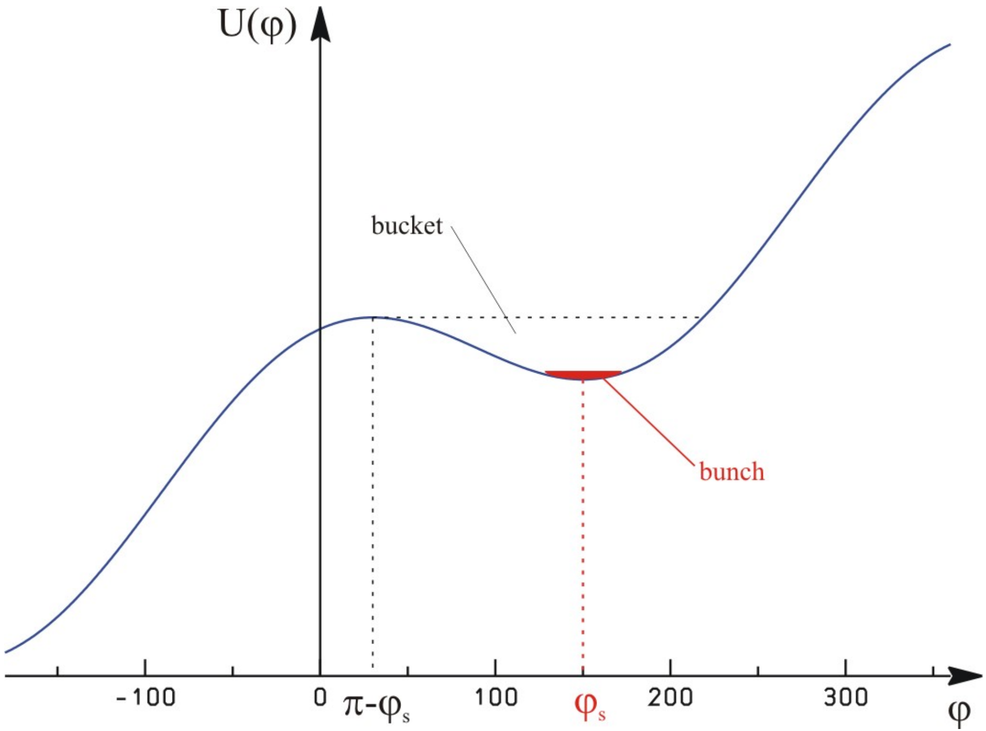
$$\boxed{\ddot{\varphi} + \frac{\Omega_S^2}{\cos\varphi_0}\left[\sin\varphi - \sin\varphi_0\right] = 0}$$

with the synchrotron frequency Ω_S defined above and $\varphi = \varphi_0 + \Delta\varphi$.

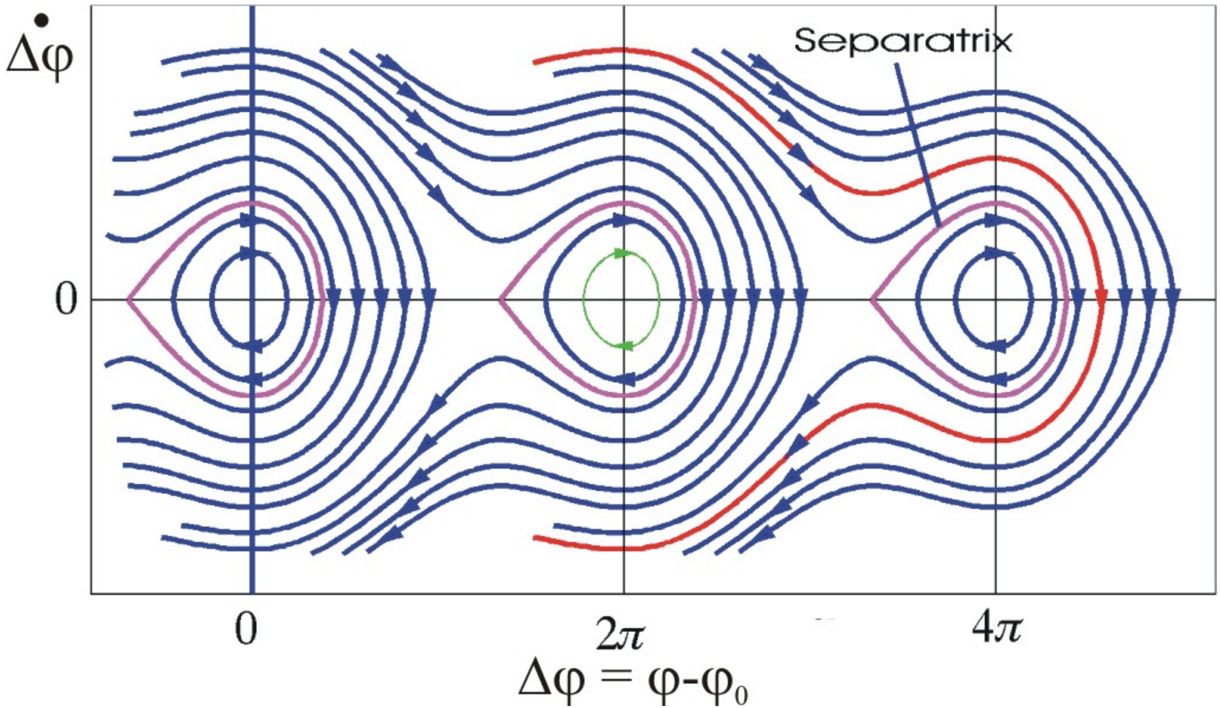
This can easily be integrated to the potential equation

$$\underbrace{\frac{\dot{\varphi}^2}{2}}_{\text{kinetic energy}} + \underbrace{\left\{-\frac{\Omega_S^2}{\cos\varphi_0}\left[\cos\varphi + \varphi\sin\varphi_0\right]\right\}}_{\text{potential energy}} = \text{const.}$$

The potential energy function corresponds to the sum of a linear function and a sinusoidal one. An oscillation can only take place if the particle is trapped in the potential well:



$\varphi_1^{\max} = \pi - \varphi_0$ is an extreme elongation corresponding to a stable motion. The corresponding curve in phase space is called separatrix and the area delimited by this curve is called the RF bucket. Part of this area is filled with particles, forming the bunch.

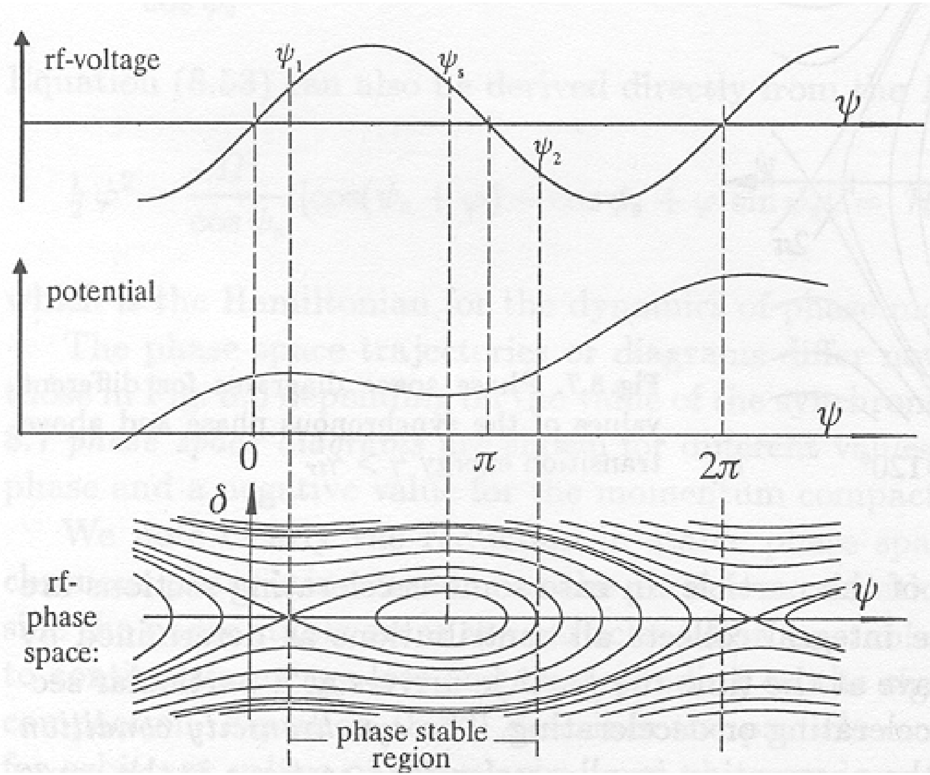


The equation of the separatrix is

$$\boxed{\frac{\dot{\varphi}^2}{2} - \frac{\Omega_S^2}{\cos\varphi_0}\left[\cos\varphi + \varphi\cdot\sin\varphi_0\right] = -\frac{\Omega_S^2}{\cos\varphi_0}\left[\cos\left(\pi-\varphi_0\right) + \left(\pi-\varphi_0\right)\cdot\sin\varphi_0\right]}$$

The other extreme elongation $\varphi_2^{\max}$ (second value for which $\dot{\varphi}=0$), is such that

$$\cos\varphi_2^{\max} + \varphi_2^{\max}\cdot\sin\varphi_0 = \cos\left(\pi-\varphi_0\right) + \left(\pi-\varphi_0\right)\cdot\sin\varphi_0$$


rf-voltage
ψ_1
ψ_s
ψ
ψ_2
potential
ψ
0
π
2π
δ
rf-
phase
space:
ψ
phase stable
region

From the equation of motion it is also seen that $\dot{\varphi}$ reaches a maximum when $\ddot{\varphi} = 0$ corresponding to $\varphi = \varphi_0$. This gives the maximum stable values of $\dot{\varphi}$ and the maximum energy spread $\Delta E_{\max}$, which is called the **RF acceptance**:

$$\boxed{\begin{aligned} \dot{\varphi}_{\max}^2 &= 2\Omega_S^2 \left[2 - \left(\pi - 2\varphi_0 \right) \cdot \tan \varphi_0 \right] \\ \left(\frac{\Delta E}{E_0} \right)_{\max} &= \pm \beta \sqrt{ \frac{eU_0}{\pi h \eta E_0} \cdot \left[2\cos\varphi_0 - \left(\pi - 2\varphi_0 \right) \cdot \sin\varphi_0 \right] } \end{aligned}}$$

In accelerator physics one usually defines an **over voltage factor** q by

$$\boxed{q = \frac{\text{maximum RF voltage}}{\text{desired energy gain}} = \frac{eU_0}{eU_0 \sin\varphi_0} = \frac{1}{\sin\varphi_0}}$$

Using this factor, we can rewrite the RF acceptance to

$$\left(\frac{\Delta E}{E_0} \right)_{\max} = \beta \sqrt{ \frac{2eU_0 \sin\varphi_0}{\pi h \eta E_0} \cdot \left(\sqrt{q^2 - 1} - \arccos\frac{1}{q} \right) } \le \beta \sqrt{ \frac{2eU_0}{\pi h \eta E_0} }$$

Using $\eta = \left(\gamma^{-2} - \alpha_C\right)$, $\alpha_C \approx 1/Q_x^{\,2}$ and $\omega_{RF} = h \cdot \omega_0$ we finally note the important scaling:

$$\boxed{\left(\frac{\Delta E}{E_0}\right)_{\max} \sim \frac{1}{\sqrt{\omega_{RF}}}}, \qquad \boxed{\left(\frac{\Delta E}{E_0}\right)_{\max} \overset{\gamma \gg 1}{\sim} Q_S}, \qquad \boxed{\left(\frac{\Delta E}{E_0}\right)_{\max} \sim \sqrt{\frac{eU_0 \sin\varphi_0}{E_0}}}$$