ADVANCED QUANTUM FIELD THEORY

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Contents

1. Introduction

Quantum field theory, in the special form of the so-called Standard Model, describes elementary particle physics very well and provides a basis from which, at least in principle, other branches of physics, in particular nuclear and atomic physics, but also condensed matter physics and chemistry can be derived. The only visible part of physics which is not included is gravitational physics. The main reason for the difficulty in incorporating gravity into quantum field theory is the different role spacetime plays in both theories, quantum field theory and General Relativity. In quantum field theory, spacetime is a background on which fields live; actually, the interpretation of field theory in terms of particles heavily relies on an

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analysis of the spacetime structure of scattering events, as most clearly illustrated by the multiple traces recorded in modern particle detectors. It is crucial, that the events themselves have to be analyzed by statistical means. General Relativity, on the other side, considers spacetime as dynamical and strongly influenced by the distribution of matter. But if matter is governed by the laws of quantum physics, also geometry had to be quantized, so the interpretation of scattering events would be difficult.

In addition to the conceptual differences, both theories are rather complex, a fact which slows down all approaches towards a solution. Moreover, in spite of the impressive experimental confirmations of both theories, no effect has been identified which hints towards a possible theory of quantum gravity.

There are courageous attempts to overcome these difficulties as e.g. string theory and loop quantum gravity. But up to now, it is not known whether any of them will lead to a valid theory of quantum gravity. A good criterion such a theory should fulfil would be that it describes in a certain limit quantum field theory on a generic curved spacetime. Actually, most of the presently observable physics should be covered by this limit. It is the aim of these lectures to discuss how quantum field theory can be defined on generic curved spacetime.

Interest on quantum field theory on curved spacetime started essentially in the 1970's and culminated in Hawking's prediction of black hole evaporation. What became gradually clear in these years was that various concepts on which quantum field theory on Minkowski space is based cannot easily be transferred to generic curved spacetime.

The first of these concepts one has to give up is the concept of the vacuum. Originally, the vacuum was understood as empty space where all particles are removed. But this idea does not take into account the unavoidable presence of fluctuations in quantum physics, the simplest example being the null point fluctuations of the harmonic oscillator. As a matter of fact, even in free quantum field theory there is no state where the quantum fields do not fluctuate, as a consequence of the canonical commutation relations between the basic fields. Consider e.g. the free scalar field at time zero and its time derivative. They satisfy the commutation relation

$$[\varphi(\mathbf{x}), \dot{\varphi}(\mathbf{y})] = i\hbar\delta(\mathbf{x}-\mathbf{y}) \ .$$

Now consider the smeared field $\varphi(f) := \int d\mathbf{x} f(\mathbf{x})\varphi(\mathbf{x})$. Then we have the uncertainty relation

$$\Delta(\varphi(f))\Delta(\dot{\varphi}(f)) \geq \frac{\hbar}{2}||f||^2$$

with the squared L^2-norm $||f||^2 = \int d\mathbf{x}|f(\mathbf{x})|^2$. A similar relation holds for the product of the uncertainties of the magnetic and the electric field.

These fluctuations are an unavoidable consequence of the laws of quantum physics. They are also visible in several physical effects, as e.g. the Casimir effect. They imply that the vacuum, defined as a ground state with respect to the total energy, is not a ground state for the energy density in a given point of spacetime. But on a spacetime without time translation symmetry the total energy cannot be defined, hence there is no obvious candidate for a vacuum state.

Even in cases where a spacetime has time translation symmetry the ground state for the total energy might have unwanted properties. This happens e.g. in Schwarzschild spacetime. Schwarzschild spacetime has the metric

$$ds^2 = (1 - \frac{2m}{r})dt^2 - (1 - \frac{2m}{r})^{-1}dr^2 - r^2 d\Omega^2$$

where $d\Omega^2$ is the invariant metric on the 2-sphere S^2 with total area 4π, and r is restricted to values $> 2m$. This metric is invariant under time translations. It has the form

$$ds^2 = a^2 dt^2 - h$$

where the spacetime is $\mathbb{R} \times \Sigma$ with t a variable on $\mathbb{R}$, h a Riemannian metric on the 3-manifold Σ and a a positive smooth function on Σ. Spacetimes of this form are called static. In the case of the Schwarzschild spacetime Σ is given by $(2m, \infty) \times \mathrm{S}^2$. The Klein-Gordon equation on a spacetime with metric g is

$$(\Box_g + m^2)\varphi = 0$$

with the d'Alembertian

$$\Box_g = |\det g|^{-1/2}\partial_\mu g^{\mu\nu}|\det g|^{1/2}\partial_\nu \ .$$

On a static spacetime we obtain

$$g^{00} = a^{-2} \ , \ g^{ij} = -h^{ij} \ , \ g^{0j} = 0 \ , \ |\det g| = a^2 \det h \ ,$$

hence the d'Alembertian takes the form

$$\Box_g = a^{-2}\partial_t^2 - |a^2 \det h|^{-1/2}\partial_i h^{ij}|a^2 \det h|^{1/2}\partial_j \ .$$

Let $\hat{h} = a^{-2}h$ be a new Riemannian metric on Σ, and let $\Delta_{\hat{h}}$ be the Laplacian on Σ with respect to $\hat{h}$,

$$\Delta_{\hat{h}} = \det \hat{h}^{-1/2}\partial_i \hat{h}^{ij} \det \hat{h}^{1/2}\partial_j \ .$$

Then the d'Alembertian is

$$\Box_g = a^{-2}(\partial_t^2 - a^{-1}\Delta_{\hat{h}} a + (\Delta_{\hat{h}} \ln a) + \hat{h}^{-1}(d\ln a, d\ln a)) \ .$$

The Klein-Gordon equation can then be written in the form

$$a^{-2}(\partial_t^2 + A)\varphi = 0$$

with the differential operator $A = -a^{-1}\Delta_{\hat{h}} a + (\Delta_{\hat{h}} \ln a) + \hat{h}^{-1}(d\ln a, d\ln a)) + a^2 m^2$ on Σ. The Laplacian, multiplied by (-1), on complete Riemannian spaces is a selfadjoint positive operator on the Hilbert space $L^2(\Sigma, d\mathrm{vol}_{\hat{h}})$. The same holds true for A on $L^2(\Sigma, a^2 d\mathrm{vol}_{\hat{h}})$ under mild conditions on a which are satisfied in the case of the Schwarzschild metric.. In the case of the Schwarzschild metric the

Riemannian space Σ is not complete with respect to the metric h. Actually, the distance of a point with radial coordinate r from the horizon at $r = 2m$ is

$$d = \int_{2m}^{r} (1 - \frac{2m}{r'})^{-1/2} dr' = 2m \left(\sqrt{\frac{r}{2m}(\frac{r}{2m} - 1)} + \ln \left(\sqrt{\frac{r}{2m}} + \sqrt{\frac{r}{2m} - 1} \right) \right) .$$

The manifold Σ is, however, complete with respect to the metric $\hat{h}$. In fact, the distance to a point at the horizon with respect to $\hat{h}$ is given by the logarithmically divergent integral

$$\hat{d} = \int_{2m}^{r} (1 - \frac{2m}{r})^{-1} = \infty .$$

We can now use the functional calculus for selfadjoint operators for constructing Green's functions for the Klein Gordon operator. The retarded propagator is

$$(G_R f)(t) = \int_{-\infty}^{t} ds \frac{\sin{(t-s)}\sqrt{A}}{\sqrt{A}} a^2 f(s)$$

where $t \mapsto f(t)$ is a continuous, compactly supported function with values in the Hilbert space $L^2(\Sigma, a d\mathrm{vol}_h)$ and a^2 acts by multiplication. Namely,

$$(\partial_t G_R f)(t) = \int_{-\infty}^{t} ds \cos{((t-s)\sqrt{A})} a^2 f(s)$$

and

$$(\partial_t^2 G_R f)(t) = a^2 f(t) - A(G_R f)(t) .$$

Analogously, the advanced propagator is

$$(G_A f)(t) = -\int_{t}^{\infty} ds \frac{\sin{(t-s)}\sqrt{A}}{\sqrt{A}} a^2 f(s)$$

The difference $G_R - G_A =: G$ is the commutator function

$$[\varphi(t,x), \varphi(s,y)] = iG(t,x;s,y) = i\frac{\sin(t-s)\sqrt{A}}{\sqrt{A}}(x,y) .$$

Here the integral kernel of an operator T on $L^2(X, d\mu)$ is defined by

$$\int d\mu(x) \int d\mu(y) \overline{f(x)} T(x,y) g(y) = \langle f, Tg \rangle .$$

The ground state with respect to time translations is obtained as a quasifree state with 2-point function the positive frequency part of the commutator. It is given by

$$\omega_2(\varphi(t,x)\varphi(s,y)) = \frac{e^{-i\sqrt{A}(t-s)}}{2\sqrt{A}}(x,y) .$$

The correlations at equal times in the ground state vanish between points whose $\hat{d}$-distance diverges. So measurements at the horizon are completely uncorrelated with measurements outside of the horizon, in spite of the fact, that the horizon is at finite d-distance. Therefore the ground state has a singular behavior. Actually, one can show that correlations at finite temperature are different, and they give the expected value exactly at the Hawking temperature.

The nonapplicability of the concept of a vacuum also implies that there is no distinguished notion of particles. For a free theory, one can always decompose the field into a sum of two terms, one interpreted as a creation operator for a particle and the other as an annihilation operator. But such a split is highly ambiguous and might have physically unwanted properties, even in cases where a preferred notion exists as on static spacetimes.

Often quantum field theory is presented as a way to compute the S-matrix, which describes the collision of incoming particles in terms of configurations of outgoing particles. But as should be clear form the discussion above the S-matrix looses its meaning in the generic case.

One can add to this list more technical items, as e.g. the absence of a distinguished momentum space formulation or the impossibility of a Wick rotation by which in many cases the light cone singularities of Lorentzian spacetimes are avoided.

The path integral formulation of quantum field theory seems at the first sight to be better behaved, but also there similar problems occur as will be discussed later.

The solution of the difficulties described above is the adoption of the concepts of algebraic quantum field theory. In this formalism, the algebraic relations between the fields of the theory are used as a starting point. These relations make it possible to construct the algebra of local observables as an abstract operator algebra. States are then identified with expectation value functionals on the observables, and the traditional Hilbert space formulation of quantum theory is obtained by the so-called GNS construction.

This approach was originally proposed by Haag [H1957] in order to clarify the origin of the particle structure in quantum field theory on Minkowski space. It was recognized in the late 1970's mainly by Dimock [Di1980] and Kay, that it is the appropriate framework for quantum field theory on curved spacetime.

It took time to get agreement that no distinguished vacuum exist; instead one identified the so-called Hadamard states as states which locally look like a state in the Fock space of a theory on Minkowski space, but are far from being unique.

A breakthrough was obtained by the work of Radzikowski [R1996]. He observed that the Hadamard condition, previously defined by a cumbersome explicit characterization of the singularities of the 2-point function (most precisely in a paper of Kay and Wald [KW1991]), can equivalently be replaced by a positivity condition on the wave front set of the 2-point function. Wave front sets are a crucial concept in the theory of partial differential equations [H2003].

Based on this finding, for the first time composite fields could be defined as operator valued distributions (Brunetti, Fredenhagen and Köhler [BFK1996]). This made it possible to start the program of constructing interacting quantum field theories on generic spacetimes.

For this purpose one had to use a method of renormalization which is formulated algebraically and on position space. Such a method, the so-called causal perturbation theory, was developed by Epstein and Glaser [EG1973], on the basis of older ideas of Stückelberg [St1953] and Bogoliubov [BP1957]. It was further developed by Stora (mainly unpublished) and by Scharf, Dütsch and collaborators. Its generalization to curved backgrounds was achieved in a series of papers by Brunetti and Fredenhagen [BF2000] and by Hollands and Wald [HW2001]. A crucial step in this program was a new concept of covariance adapted to quantum field theory, called *local covariance* [BFV2003]. The inclusion of gauge theories was done after earlier work by Scharf et al. in papers of Dütsch, Fredenhagen, Hollands and Rejzner (2000-2011).

The plan of these lectures is as follows: After a brief review of the algebraic formulation of quantum theory and of Lorentzian geometry we present the axioms of locally covariant quantum field theory.

We then construct in the same spirit classical field theory on curved spacetime. There the main structure is the so-called Peierls bracket, by which a Poisson bracket on classical observables can be introduced.

In a next step we apply deformation quantization to free quantum field theory and discuss the concept of covariant composite fields. For the construction of interacting fields we introduce a modified version of causal perturbation theory.

If time permits I will discuss the generalization to gauge theories and the first steps towards perturbative quantum gravity.

2. Algebraic quantum theory

After these introductory remarks we want now to describe the algebraic formulation of quantum theory. Every quantum system is characterized by two notions: observables and states. In the ordinary formulation of quantum mechanics, one considers observables as self-adjoint operators acting on a Hilbert space, while states are unit vectors of the chosen Hilbert space. However, as we will see below, this turns out to be a special implementation of a much richer algebraic structure.

2.1. Algebraic notion of observables.

It starts from the canonical commutation relation

$$[q,p] = i\hbar$$

and considers the associative unital algebra $\mathfrak{A}$ over the complex numbers generated by p and q with the commutation relation above. This algebra has an involution $A \to A^*$, i.e. an antilinear map with $(AB)^* = B^*A^*$ and $(A^*)^* = A$, uniquely determined by $p^* = p$, $q^* = q$. The algebra is simple, i.e. every nonzero ideal is the whole algebra.

Definition 1. *An ideal* $\mathfrak{I}$ *of an algebra* $\mathfrak{A}$ *is a subspace with the property* $AI, IA \in \mathfrak{A}$ *if* $I \in \mathfrak{I}$ *and* $A \in \mathfrak{A}$*.*

Theorem 1. *The unital algebra* $\mathfrak{A}$ *generated by two elements* p *and* q *fulfilling the canonical commutation relation above is simple.*

Proof. The algebra $\mathfrak{A}$ has the basis $(q^k p^{n-k})_{n\in\mathbb{N}_0, k=0,\ldots n}$. Every element of $A \in \mathfrak{A}$ has a unique expansion in this basis

$$A = \sum_{n,k} \lambda_{nk}(A) q^k p^{n-k} \ .$$

Let now $\mathfrak{I}$ be a nonzero ideal of $\mathfrak{A}$, and let $I \in \mathfrak{I}$ be nonzero. Let (n,k) be maximal such that $\lambda_{nk}(I) \neq 0$ and $\lambda_{n'k'}(I) = 0$ if $n' > n$ or, if $n' = n$ and $k' > k$. We now form first the k-fold commutator of I with p. This annihilates all terms $\lambda_{n'k'} q^{k'} p^{n-k'}$ with $k' < k$. We then form the $(n-k)$-fold commutator with q. This annihilates all terms with $n' < n$. We obtain

$$\underbrace{[q, \ldots [q}_{n-k} \underbrace{[p, \ldots [p}_{k}, I \underbrace{] \ldots]]}_{n} = (-1)^k k!(n-k)! i^n \hbar^n \lambda_{nk}(I) 1 \ .$$

Hence the ideal contains the unit of the algebra and therefore coincides with the whole algebra. □

Unfortunately, there is no algebra norm on $\mathfrak{A}$, i.e. a norm satisfying the inequality $||AB|| \leq ||A|| \, ||B||$. This follows from the iterated canonical commutation relation

$$\underbrace{[p, \ldots [p}_{n}, q^n] \ldots] = n! i^n \hbar^n 1 \ .$$

If there would be an algebra norm $|| \cdot ||$ the norm of the left hand side would be bounded by $2^n ||p||^n \, ||q||^n$ whereas the norm of the right hand side is $n! \hbar^n ||1||$.

2.1.1. *The Weyl Algebra.* If one wants to go beyond the polynomials in p and q, it is convenient to consider the exponential series

$$W(\alpha, \beta) = e^{i(\alpha p + \beta q)}$$

defines as formal power series in α and β. They satisfy the relations (Weyl relations)

$$W(\alpha, \beta) W(\alpha', \beta') = e^{-\frac{i\hbar}{2}(\alpha\beta' - \alpha'\beta)} W(\alpha + \alpha', \beta + \beta") \ .$$

Since the numerical coefficient in the Weyl relation is a convergent power series, we replace it by its limit and interpret this relation as a definition of a product on the linear span of the elements $W(\alpha, \beta)$ for real α and β. The obtained algebra $\mathcal{W}$ is called the Weyl-algebra. It is unital with the unit $1 = W(0,0)$, and for real α and β

$$W(\alpha, \beta)^* = W(-\alpha, -\beta) = W(\alpha, \beta)^{-1} \ .$$

Hence these elements are unitary. Also this algebra is simple.

For later purposes we generalize the notion of a Weyl-algebra in the following way. Let L be a real vector space with a symplectic form σ, i.e. a bilinear form σ on L which is antisymmetric,

$$\sigma(x,y) = -\sigma(y,x) \ ,$$

and nondegenerate,

$$\sigma(x,y) = 0 \ \forall \ y \in L \text{ implies } x = 0 \ .$$

We consider the unital *-algebra $\mathcal{W}(L,\sigma)$ of complex linear combinations of elements $W(x)$ with the product

$$W(x)W(y) = e^{i\sigma(x,y)}W(x+y)$$

the involution

$$W(x)^* = W(-x)$$

and the unit $1 = W(0)$. In the case above $L = \mathbb{R}^2$ and $\sigma(x,y) = -\frac{1}{2}\hbar(x_1y_2 - x_2y_1)$.

To show that $\mathcal{W}(L,\sigma)$ is simple we consider a nonzero ideal $\mathfrak{I}$. If $1 \in \mathfrak{I}$ then $\mathfrak{I} = \mathcal{W}(L,\sigma)$. Now let $A = \sum_{i=1}^n \lambda_i W(x_i) \in \mathfrak{I}$ with $n > 1$, $\lambda_1 \neq 0$ and pairwise different x_i. Then $B = \lambda_1^{-1}W(-x_1)A = 1 + \sum_{i=2}\mu_i W(z_i) \in \mathfrak{I}$ with $\mu_i = \lambda_1^{-1}\lambda_i e^{i\sigma(-x_1,x_i)}$ and $z_i = x_i - x_1 \neq 0$. We now use the fact that σ is nondegenerate. Therefore there exists some $y \in L$ such that $\sigma(y,z_n) = \frac{\pi}{2}$. We conclude that

$$C = W(y)BW(-y) = 1 + \sum_{i=2}^{n-1}\mu_i' W(z_i) - \mu_n W(z_n) \in \mathfrak{I}$$

with $\mu_i' = \mu_i e^{2i\sigma(y,z_i)}$, and hence also $D = \frac{1}{2}(B+C) = 1 + \frac{1}{2}\sum_{i=2}^{n-1}(\mu_i + \mu_i')W(z_i)$. Iterating the argument we finally arrive at $1 \in \mathfrak{I}$, hence $\mathfrak{I} = \mathfrak{A}$.

The Weyl algebra admits the norm

$$||\sum_{i=1}^n \lambda_i W(x_i)||_1 = \sum_{i=1}^n |\lambda_i| \ .$$

This norm satisfies the condition $||AB||_1 \leq ||A||_1||B||_1$ of an algebra norm. Moreover, the involution is isometric, $||A^*||_1 = ||A||_1$.

2.1.2. *C^*-algebra.* We are looking for a so-called C*-norm, i.e. a norm satisfying the condition

$$||A^*A|| = ||A||^2 \ .$$

An example for a C*-norm is the operator norm of Hilbert space operators. In our case, we obtain a C*-norm in the following way. We have the inequality

$$||A^*A||_1 \leq ||A||_1^2 \ .$$

From this we conclude that

$$a_n = ||(A^*A)^{2^n}||_1^{2^{-(n+1)}}$$

is a monotonically decreasing sequence of positive numbers. The C*-norm of A is then defined by

$$||A|| = \lim_{n\to\infty} a_n \ .$$

By construction, $||\cdot||$ satisfies the C*-condition. It is certainly nonzero, since $||1|| = 1$. One can show that it is the unique C*-norm on $\mathcal{W}(L,\sigma)$. The completion makes it to a C*-algebra, often also called the Weyl-algebra over (L,σ).

Another useful algebra based on the canonical commutation relations was recently found by Buchholz and Grundling. It is generated by the resolvents $R(\alpha,\beta,z) = (\alpha p + \beta q - z1)^{-1}, \ z \in \mathbb{C}$.

Physically, a unital C*-algebra is interpreted as the algebra of observables. The selfadjoint elements A represent real valued observables, and their spectrum, i.e. the set of real numbers λ such that $A - \lambda 1$ has no inverse, is interpreted as the set of measurable values.

2.2. **Algebraic notion of states.** States, interpreted as an association of expectation values to each observable, can be defined as linear functionals ω on the algebra with the positivity condition

$$\omega(A^*A) \geq 0$$

and the normalization condition

$$\omega(1) = 1 \ .$$

If $\mathfrak{A}$ is an algebra of Hilbert space operators, with the adjoint as the involution, every unit vector Φ induces via

$$\omega(A) = \langle \Phi, A\Phi \rangle$$

a state.

2.3. **GNS construction.** We will now observe that the usual formulation of quantum mechanics as self-adjoint operators and vectors in a Hilbert space can be realized as a particular representation of the C^*-algebra. In other words, it is gratifying that the converse of the above statement holds as well: every state ω arises from a unit vector in a Hilbert space representation of the algebra. This is the content of the so-called GNS construction.

Theorem 2. *Let ω be a state on the unital *-algebra $\mathfrak{A}$. Then there exists a pre-Hilbert space $\mathfrak{D}$, a representation π of $\mathfrak{A}$ by operators on $\mathfrak{D}$ and a unit vector $\Omega \in \mathfrak{D}$ such that*

$$\omega(A) = \langle \Omega, \pi(A)\Omega \rangle$$

and

$$\mathfrak{D} = \pi(\mathfrak{A})\Omega \ .$$

Proof. By

$$\langle A, B\rangle := \omega(A^* B)$$

we define a positive semidefinite scalar product on $\mathfrak{A}$. The null set

$$N_\omega = \{A \in \mathfrak{A}, \omega(A^* A) = 0\}$$

is a left ideal. Hence the quotient space $\mathfrak{A}/N_\omega$ is a pre-Hilbert space $\mathcal{D}$ on which the algebra acts by left multiplication,

$$\pi(A)(B + N_\omega) = AB + N_\omega \ .$$

The unit vector Ω is the class of the unit. Hence,

$$\langle \Omega, \pi(A)\Omega\rangle = \omega(1^* A 1) = \omega(A) \ .$$

□

The construction is unique (up to a unitary transformation) in the following sense: given an other triple $(\mathfrak{D}', \pi', \Omega')$ with the same properties, then there is a unitary operator $U : \mathfrak{D} \to \mathfrak{D}'$ such that

$$U\pi(A) = \pi'(A)U \ \forall \ A \in \mathfrak{A}$$

and

$$U\Omega = \Omega' \ .$$

In case $\mathfrak{A}$ is a C*-algebra, one can show that the operators $\pi(A)$ are bounded and can therefore uniquely be extended to bounded operators on the completion of the pre-Hilbert space.

2.4. **Examples.** Let us discuss examples of states and the corresponding GNS-representation. We consider the Weyl algebra as the C^*-algebra of observables and study certain states in terms of the symplectic space (L, σ). While many of such states are considered physically pathological in the ordinary formulation of quantum field theory, in the algebraic framework they are mathematically well-defined.

2.4.1. *A trace state* ω_0. Our first example is the linear functional ω_0 on the Weyl-algebra $\mathcal{W}(L, \sigma)$, defined by

$$\omega_0(W(x)) = \begin{cases} 0 & \text{for} \quad x \neq 0 \\ 1 & \text{for} \quad x = 0 \end{cases} \ .$$

To see it is actually a state, let us check continuity and positivity: Consider A a generic element of $\mathcal{W}(L, \sigma)$: $A = \sum_{\text{finite}} \lambda_x W(x)$. Then we have:

$$\omega_0(A) = \lambda_0 \Rightarrow |\omega_0(A)| = |\lambda_0| \leq \sum_x |\lambda_x| = ||A||_1.$$

Also,

$$\omega_0(A^* A) = \sum_x |\lambda_x|^2 > 0.$$

ω_0 is a so-called trace state, i.e. it fulfils the condition

$$\omega_0(AB) = \omega_0(BA) \ .$$

The GNS-Hilbert space is the space of square summable sequences $l^2(L)$, i.e. maps $\Phi : L \to \mathbb{C}$ such that

$$||\Phi||^2 := \sum_{x \in L} |\Phi(x)|^2 < \infty \ ,$$

and $W(x)$ acts on $l^2(L)$ by

$$\pi((W(x))\Phi)(y) = e^{i\sigma(x,y)}\Phi(x+y) \ .$$

The cyclic vector Ω is given by

$$\Omega(x) = \delta_{x0} \ .$$

ω_0 is not pure. Namely let $0 < B < 1$ with $0 < \omega_0(B) < 1$. Then $\omega_B = \omega_0(\cdot B)/\omega_0(B)$ is also a state, and ω_0 can be written as a convex combination of other states,

$$\omega_0 = \omega(B)\omega_B + \omega(1-B)\omega_{1-B} \ ,$$

provided, B is not a multiple of the identity.

2.4.2. *Lagrangian subspaces.* The second example depends on the choice of a Lagrangian subspace $K \subset L$, i.e. a maximal subspace K of L such that $\sigma(x,y) = 0 \ \forall \ x, y \in K$.

Recall that in classical mechanics, the phase space consists of all positions and their conjugate momenta (p,q) which constitute a symplectic space. A Lagrangian subspace of phase space is the space of all coordinates. Of course one could perform a canonical transformation on phase space and mix the ps and qs, but the defining property of such a subspace would still hold which leads to many Lagrangian subspaces. Here, we consider one Lagrangian subspace K and define:

$$\omega_K(W(x)) = \begin{cases} 1 & \text{for } x \in K \\ 0 & \text{for } x \notin K \end{cases} \ .$$

Quantum mechanically, this state corresponds to an extreme case $\Delta q = 0$, $\Delta p = \infty$ since the Weyl operator on the Lagrangian subspace of all qs has the form $e^{i\beta q}$, and setting this equal to 1 means that the particle is at origin.

The positivity of ω_K follows from

$$\begin{aligned}\omega_K((\sum_x \lambda_x W(x))^*(\sum_y \lambda_y W(y)) &= \sum_{x,y} \overline{\lambda_x}\lambda_y \omega_K\left(W(-x)W(y)\right) \\ &= \sum_{x,y} \overline{\lambda_x}\lambda_y e^{-i\sigma(x,y)}\omega_K(W(y-x)) \\ &= \sum_{[z]\in L/K}\sum_{y\in[z]} \overline{\lambda_x}\lambda_y e^{-i\sigma(x,y)} \\ &= \sum_{[z]\in L/K} |\sum_{y\in[z]} \lambda_y e^{e^{i\sigma(y,z)}}|^2 \geq 0 \ .\end{aligned}$$

where in the last step, we have used the fact that for $x, y \in [z] \in L/K$ we have $\sigma(x, y) = \sigma(z + (x - z), z + (y - z)) = \sigma(x, z) + \sigma(z, y)$.

The state ω_K is pure. Namely, let $\omega_K = \lambda\omega_1 + (1-\lambda)\omega_2$ with $0 < \lambda < 1$. Then for $x \in K$ we have

$$1 = \lambda\omega_1(W(x)) + (1-\lambda)\omega_2(W(x)) \ .$$

Since the expectation value of a unitary operator is bounded by 1, we must have $\omega_1(W(x)) = \omega_2(W(x)) = 1$.

Now let $x \notin K$. For every $y \in K$ we have

$$|\omega_1(A(W(y)-1)|^2 \leq \omega_1(AA^*)\omega_1(|W(y)-1|^2) = 0 \ ,$$

since $\omega_1(|W(y)-1)|^2) = \omega(2 - W(y) - W(-y)) = 0$. Then

$$\omega_1(W(x)) = \omega_1(W(y)W(x)W(-y)) = e^{2i\sigma(y,x)}\omega_1(W(x)) \ .$$

It follows from the maximality of K that there exists some $y \in K$ with $e^{2i\sigma(x,y)} \neq 1$. Hence $\omega_1(W(x)) = 0$, hence $\omega_1 = \omega_K = \omega_2$. Thus ω_K is pure.

The GNS-Hilbert space is $l^2(L/K)$, the Weyl operators act on this space by

$$\left(\pi(W(x))\Phi\right)([z]) = e^{i\sigma(x,z)}\Phi([x+z])$$

where we had to choose a system of representatives $L/K \to L, [z] \mapsto z$.

2.4.3. *Quasifree (Gaussian) states.* Another class are the quasifree (also called Gaussian) states. They are of the form

$$\omega_\mu(W(x)) = e^{-\frac{1}{2}\mu(x,x)}$$

with a real scalar product μ on L. The positivity condition on the state requires the bound

$$\sigma(x,y)^2 \leq \mu(x,x)\mu(y,y)$$

Namely, we have

$$\omega_\mu(|\sum_x \lambda_x W(x)|^2) = \sum_{x,y\in L} \overline{\lambda_x}\lambda_y e^{-i\sigma(x,y)-\frac{1}{2}\mu(x-y,x-y)} \ .$$

We define a complex scalar product on the complex vector space $L_{\mathbb{C}} = L \oplus iL$ by

$$\langle x, y\rangle = \mu(x, y) + i\sigma(x, y) \ .$$

The Cauchy-Schwarz inequality then requires the condition above on μ.

ω_μ is a pure state if and only if the map $L \to L_{\mathbb{C}}/\mathrm{Ker}(\langle ., .\rangle)$ is surjective.

The GNS-Hilbert space turns out to be the bosonic Fock space:

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} (\mathcal{H}_1^{\otimes n})_{\mathrm{symm}} \ ; \ \mathcal{H}_1 = \overline{L_{\mathbb{C}}/\mathrm{Ker}(\langle ., .\rangle)}$$

3. Lorentzian geometry

3.1. **Globally hyperbolic space-times.** According to the principles of General Relativity, our spacetime is a 4 dimensional manifold M equipped with a metric g with signature (+ - - -). On the tangent space T_pM at a point $p \in M$ we distinguish timelike ($g(\xi, \xi) > 0$), lightlike ($g(\xi, \xi) = 0$) , and spacelike ($g(\xi, \xi) < 0$) vectors $\forall \xi \in T_pM$. A smooth curve with timelike or lightlike tangent vectors is called *causal*.

A *Cauchy surface* is a smooth hypersurface with spacelike tangent vectors such that every nonextendible causal curve hits it exactly once. A standard example of a Cauchy surface is the time zero hyperplane in Minkowski space. A spacelike hypersurface which fails to be a Cauchy surface is the hyperboloid $g(x, x) = 1$ in Minkowski space.

Spacetimes M with a Cauchy surface are called *globally hyperbolic*. They have a number of nice properties, some of them were derived only recently by Bernal and Sanchez. In particular, they are diffeomorphic to $\mathbb{R} \times \Sigma$, such that $(\{t\} \times \Sigma)_{t \in \mathbb{R}}$ is a foliation of M by Cauchy surfaces.

For our application, most important is that normally hyperbolic differential equations have a well posed initial value problem on globally hyperbolic spacetimes. Here a second order differential operator on a Lorentzian manifold is called *normally hyperbolic* if its principal symbol is the inverse metric (i.e. the term with the highest order in a given coordinate system is $g^{\mu\nu}\partial_\mu\partial_\nu$). In particular, there exist unique retarded and advanced Green's functions,

3.2. **Microlocal analysis.** These Green's functions are distributions, and it will be important to understand their singularity. The framework in which the singularity structure of Green's functions are studied systematically is called the microlocal analysis. An appropriate concept for doing this is the wave front set.

3.2.1. *Wave front set.* Let t be a distribution on $\mathbb{R}^n$. We are interested in understanding the singularity of t in the neighborhood of a point $x \in \mathbb{R}^n$. For this purpose we multiply t by a test function f with compact support and $f(x) \neq 0$.

Then ft is a distribution with compact support and therefore has a smooth Fourier transform

$$\widehat{ft}(k) = t(fe^{ik\cdot}) \ .$$

ft is a smooth function if and only if its Fourier transform is rapidly decreasing.

We now call (x, k) a *regular point* of t, if $\widehat{ft}$ is rapidly decreasing in an open cone containing k for some f with $f(x) \neq 0$. The complement of the set of regular points in $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ is called the *wave front set* $\mathrm{WF}(t)$ of t:

$$\mathrm{WF}(t) = \{(x, k), k \neq 0, x \in \mathbb{R}^n | (x, k) \text{ is not a reg. pt. of } t\}$$

3.2.2. *Examples.* Let us illustrate the concept of the wave front set in two simple but important examples.

The first one is the δ-function. We find

$$\int dx f(x)\delta(x)e^{ikx} = f(0)$$

hence $\mathrm{WF}(\delta) = \{(0, k), k \neq 0\}$.

The second one is the function

$$x \mapsto (x + i\epsilon)^{-1}$$

in the limit $\epsilon \downarrow 0$. We find

$$\lim_{\epsilon\downarrow 0} \int dx \frac{f(x)}{x + i\epsilon} e^{ikx} = -i \int_k^\infty dk' \hat{f}(k') \ .$$

$$\begin{aligned}
\lim_{\epsilon\downarrow 0} \int dx \frac{f(x)}{x + i\epsilon} e^{ikx} &= i \int dx f(x) \frac{e^{ik(x+i\epsilon)}}{i(x + i\epsilon)} e^{k\epsilon} \\
&= -i \int dx f(x) \int_k^\infty dk' e^{ik'(x+i\epsilon)} \\
&= -i \int_k^\infty dk' \int dx f(x) e^{ik'x} e^{(k'-k)\epsilon} \\
&= -i \int_k^\infty dk' \hat{f}(k') e^{(k'-k)\epsilon} \to -i \int_k^\infty dk' \hat{f}(k')
\end{aligned}$$

Since the Fourier transform $\hat{f}$ of a test function f is strongly decreasing, also $\int_k^\infty dk' \hat{f}(k')$ is strongly decreasing for $k \to +\infty$; but for $k \to -\infty$ we obtain

$$\lim_{k\to-\infty} \int_k^\infty dk' \hat{f}(k') = 2\pi f(0) \ ,$$

hence

$$\mathrm{WF}(\lim_{\epsilon\downarrow 0}(\mathrm{x} + \mathrm{i}\epsilon)^{-1}) = \{(0, k), k < 0\} \ .$$

For distribution on a manifold, one can perform the same construction within a given chart. But the property of rapid decrease turns out to be independent of the choice of a chart, therefore the regular points can be understood as elements of the cotangent bundle. We thus obtain the wave front set of a distribution as a closed subset of the cotangent bundle, with the zero section removed.

3.2.3. *Applications.* For our analysis, two propositions are important. Recall that in functional analysis one can multiply a distribution and a function, and obtain another distribution. However, there is no notion of pointwise product of two distributions. Nevertheless, with the aid of the wave front set one can define multiplication of distributions in the following sense:

Proposition 1. *Let s and t be distributions such that the sum of their wave front sets*

$$\mathrm{WF}(s) + \mathrm{WF}(t) := \{(p, k + k')|(p, k) \in \mathrm{WF}(s), (p, k') \in \mathrm{WF}(t)\}$$

does not intersect the zero section. Then the pointwise product of the distributions st can be defined in the following way:

Let f and g be test functions with sufficiently small compact support. Then

$$(st)(fg) := \int dk \widehat{fs}(k)\widehat{gt}(-k)$$

The integrand is strongly decreasing since at least one factor is strongly decreasing and the other factor is polynomially bounded.

Proof. For a generic test function h with compact support we choose a sufficiently fine, but finite covering of the support of h, choose a subordinate partition of unity

$$1 = \sum \chi_j$$

and write each summand $h\chi_j$ as a product of test functions $f_j g_j$ with sufficiently small compact support. We then set

$$(st)(h) := \sum_j (st)(f_j g_j) \ .$$

(see [H2003] for more details.) □

The second proposition we need characterizes the propagation of singularities. Before stating it, we need to define our notation to write down a differential operator on $\mathbb{R}^n$.

Notation 1. *Let α be a multi-index i.e. $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0$. Then,*

$$\partial^\alpha := \frac{\partial^{\alpha_1 + \cdots + \alpha_n}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$$

$$|\alpha| = \alpha_1 + \cdots + \alpha_n$$

$$\alpha! = \alpha_1! \dots \alpha_n!$$

.

A differential operator on $\mathbb{R}^n$ *can then be written as* $D = \sum_\alpha a_\alpha \partial^\alpha$. *We call* $\max_{a_\alpha \neq 0} |\alpha|$ *the order of* D. *The principal symbol of* D *is defined by*

$$\sigma_p(x,k) = \sum_{|\alpha|=r} a_\alpha(x)(ik)^\alpha,$$

with $k^\alpha := k_1^{\alpha_1} \dots k_n^{\alpha_n}$ *and where* r *is the order of* D

On manifolds, differential operators can be defined in terms of coordinates within a given chart. First order operators are vector fields $X = X^\mu \partial_\mu$ and can be interpreted as functions on the cotangent bundle

$$X(x,k) = X^\mu(x) k_\mu$$

This is no longer true for higher order operators. If we consider e.g. a second order differential operator on manifolds, then under coordinate transformations one finds additional terms of first order. The highest order term, however, have an invariant meaning. We have the following proposition:

Proposition 2. *Under a diffeomorphism, the principal symbol of a differential operator transforms as a function on the cotangent bundle.*

The cotangent bundle has a natural Poisson structure. Namely, the identity map on the tangent bundle may be interpreted as a 1-form on the cotangent bundle,

$$\theta = \sum k_i \otimes dx^i$$

where $k_i \equiv \partial_{x_i}$ is the function $k_i(dx^j) = \delta_{ij}$ on the cotangent bundle. The differential of θ is a symplectic form

$$\omega = d\theta = \sum dk_i \wedge dx^i \ .$$

The inverse is the bivector

$$\omega^{-1} = \sum \left(\partial_{x^i} \otimes \partial_{k_i} - \partial_{k_i} \otimes \partial_{x_i}\right)$$

The Poisson bracket of functions on the cotangent bundle is

$$\{f,g\} = m \circ \omega^{-1}(f \otimes g)$$

with the pointwise multiplication m of two functions.

For any function f on the cotangent space one defines the Hamiltonian vector field X_f by

$$X_f g := \{f,g\}$$

Now, using proposition 2, we can consider the principal symbol as our function f on the cotangent bundle (Hamiltonian) and look at the flow induced by this Hamiltonian which will characterize the propagation of singularity.

Theorem 3. *Let D be a differential operator with real principal symbol σ_P, and let u be a distributional solution of the equation $Du = f$ with a smooth function f. Then the wave front set of u is contained in the zero set of σ_P and is a union of orbits of the flow of the Hamiltonian vector field associated to σ_P.*

The points of cotangent bundle move in time subject to Hamilton's equations. The above theorem states that if one point is in the wavefront set then all points which can be reached by the Hamiltonian flow must also be within the wavefront set.

We apply this theorem to the case of normally hyperbolic differential operators. Their principal symbol is the inverse metric, the associated Hamiltonian flow is the geodesic flow on the cotangent bundle. , i.e. a union of a set Γ of nonextendible null geodesics γ, together with their cotangent vectors,

$$\mathrm{WF}(u) = \bigcup_{\gamma\in\Gamma} \{(\gamma(t), g(\dot{\gamma}(t), \cdot)), t \in \mathbb{R}\}.$$

We now discuss the Green's functions of a normally hyperbolic operator D. We assume that the spacetime M is globally hyperbolic. The crucial property of a globally hyperbolic spacetime is the existence of a unique solution of the initial value problem ("the Cauchy problem is well posed"). In particular, one finds unique retarded and advanced Green's function. The retarded Green's function is a linear operator

$$G_R : \mathcal{D}(M) \to \mathcal{E}(M)$$

mapping compactly supported smooth functions f to smooth functions G_R such that

$$G_R \circ Df = D \circ G_R f = f \ , \ f \in \mathcal{D}(M) \ ,$$

and

$$\mathrm{supp}\, G_R f \subset J_+(\mathrm{supp} f)$$

where $J_+(N)$ (the future of the set $N \subset M$) is the closure of the set of all points which can be reached by future directed causal curves starting in N.

The advanced Green's function G_A is analogously defined by replacing the future J_+ by the past J_-. The difference

$$G = G_R - G_A$$

is a distributional solution of the differential equations

$$D_x G(x, y) = 0 = D^t_y G(x, y)$$

where D^t is the transpose differential operator. In case of a formally selfadjoint differential operator ($D^t = D$), the retarded Green's function is the transposed of the advanced Green's function, hence G is antisymmetric.

The wave front set of G is

$\mathrm{WF}\, G = \{(x, y; k, k')|\exists$ a null geodesic γ connecting x and y, k and k' are coparallel to γ such that the parallel transport $P_\gamma k$ of k along γ satisfies $P_\gamma k + k' = 0\}$.

Exercise 1. *Check Theorem 3 by comparing the wave front set of the distributional solution to the wave equation on Minkowski space-time obtained as a result of the proposition and by explicit calculation.*

$$\Box G_R(x,y) = \delta(x,y); G_R(x,y) = \frac{1}{4\pi|\vec{x}|}\delta(t-|\vec{x}|)$$

$$\Box G_A(x,y) = \delta(x,y); G_A(x,y) = \frac{1}{4\pi|\vec{x}|}\delta(t+|\vec{x}|)$$

$$\Box G(x,y) = \delta(x,y); G(x,y) = G_R(x,y) - G_A(x,y)$$

4. Free scalar field

As the first case in studying a field theory, we begin with the simplest one, namely a free scalar field. The free scalar field on a globally hyperbolic manifold M with metric g is a solution of the Klein-Gordon equation

$$P\varphi = 0$$

with the Klein-Gordon operator $P = \Box_g + m^2 + \xi R$. Here R is the scalar curvature, and ξ and m are real valued constants. A quantum field is a distribution $\mathcal{D}(M) \ni f \mapsto \varphi(f) \in \mathfrak{A}(M)$ with values in the algebra of observables, hence we understand the field equation in the sense of distributions,

$$\varphi(Pf) = 0 \ .$$

The canonical commutation relations are obtained from G, the difference of the retarded and the advanced Green's function for the Klein-Gordon equation, and has the form

$$[\varphi(f), \varphi(h)] = i\langle f, Gh\rangle \ 1 = i\hbar \int dxdy f(x)G(x,y)h(y)1.$$

In this part we illustrate two strategies to study free scalar field. The first one, introduced in section 4.1 is a generalization of notion of algebra of observables discussed in part 2. We start from the algebra of canonical commutation relations and construct an abstract algebra. The second, presented in section 4.3 is to give explicit construction of the algebra as a set of linear functionals, and then defining a suitable product which captures the more computational features of QFT.

4.1. Algebra of observables.

We have the following choices for the algebra of observables. The first choice is the algebra generated by elements $\varphi(f)$, $f \in \mathcal{D}(M)$ which depend linearly on f and are subject to the relations above. This algebra is an infinite dimensional version of the algebra of canonical commutation relations. The antisymmetric bilinear form on $\mathcal{D}(M)$

$$\sigma(f,h) = \langle f, Gg\rangle$$

is degenerate. It vanishes for $h = Ph_0$ with $h_0 \in \mathcal{D}(M)$. On the quotient space $L = \mathcal{D}(M)/P(\mathcal{D}(M))$ it becomes a (weak) symplectic form σ. As a consequence, the algebra of the free field is simple.

The elements of the quotient space can be identified with a smooth solution with compactly supported Cauchy data, since the map

$$G : \mathcal{D}(M) \to \mathcal{E}(M)$$

has as the kernel the image of P (restricted to $\mathcal{D}(M)$ and as the image the kernel of P.

Recall that on a globally hyperbolic space-time, each solution of the Klein-Gordon equation is characterized by it's Cauchy data on a Cauchy surface Σ:

$$\text{f is a solution} \leftrightarrow (f|_\Sigma, \partial_n f|_\Sigma) \equiv (f_1, f_2),$$

where, ∂_n is the normal derivative on Σ ($\partial_n f = n^\mu \partial_\mu f$, $n^\mu \xi_\mu = 0$ for $\xi \in T\Sigma$, $n^\mu n_\mu = 1$.) We can also express the canonical commutation relations in terms of Cauchy data:

$$\langle f, Gh \rangle \; = \int_\Sigma dx (f_1 h_2 - f_2 h_1),$$

where f_1, f_2 and h_1, h_2 are Cauchy data of Gf and Gh respectively. This suggest to choose the algebra of observables as Cauchy data.

Theorem 4. *Let M be a globally hyperbolic space-time. Let Σ be a Cauchy surface of M. Let $f_1, f_2 \in \mathcal{D}(\Sigma)$. Then, there exists a unique solution f of the equation $Pf = 0$, such that $f|_\Sigma = f_1$, $(\partial_n f)|_\Sigma = f_2$.*

Corollary 1. *f, the unique solution to $Pf = 0$, vanishes outside of $J_+(Suppf_1 \cup Suppf_2) \cup J_-(Suppf_1 \cup Suppf_2)$.*

To sum up, there are three different choices to construct a symplectic (L, σ) vector space for real scalar field.

(1)

$$L = \mathcal{D}(M)/ImP = P\mathcal{D}(M),$$
$$\sigma(f, g) = \langle f, Gh \rangle$$

(2) We can consider it as the space of solutions with compact support on each Cauchy surface.

$$L_1 = \{f \in \mathcal{C}^\infty(M), Pf = 0 \text{ s.t. initial data have cmpt. Supp.}\}$$
$$\sigma(f, g) = \int_\Sigma dvol_\Sigma (f(\partial_n g) - (\partial_n f) g).$$

(3)

$$L_2 = \{(f_1, f_2) \in \mathcal{C}^\infty(\Sigma) \times \mathcal{C}^\infty(\Sigma)\}$$
$$\sigma_2((f_1, f_2), (g_1, g_2)) = \int_\Sigma \mathrm{dvol}_\Sigma (f_1 g_2 - f_2 g_1).$$

Theorem 5. *There exist the following isomorphisms between the above choices for symplectic space, which preserve the symplectic form:*

$$\alpha : L \to L_1; f \mapsto Gf,$$

$$\beta : L_1 \to L_2; f \mapsto (f|_\Sigma, \partial_n f|_\Sigma).$$

Proof. We show the surjectivity of α. Let f be a solution, $\chi \in \mathcal{C}^\infty(M)$, and Σ_1, Σ_2 be Cauchy surfaces such that $\Sigma_1 \cap J_+(\Sigma_2) = \emptyset$. Assume $\chi(x) = 0$ for $x \in J_-(\Sigma_1)$ and $\chi(x) = 1$ for $x \in J_+(\Sigma_2)$. Then $P\chi f = 0$ outside Σ_1, Σ_2 ($\chi =$const. there) which implies $P\chi f$ has compact support. Hence,

$$GP\chi f = G_R P\chi f + G_A P(1-\chi)f = f.$$

□

4.1.1. *Construction of Retarded Propagator.* Let $\mathbb{R} \times \Sigma \to M$ be a foliation of M by Cauchy surfaces ($\Sigma_t = \{t\} \times \Sigma$). Let $G_t : \mathcal{D}(\Sigma_t \to \mathcal{C}^\infty(M))$ be a solution of $Pf = 0$ with initial data $f|_{\Sigma_t} = 0$, $(\partial_n f)|_{\Sigma_t} = f_2$, $G_t f_2 = f$.

Set $G_R : \mathcal{D}(M) \to \mathcal{C}^\infty(M)$ be defined by

$$(G_R h)(t,x) = \int_{-\infty}^{t} dt^{'} (G_t^{'} h_{t'})(x),$$

with $h_t(x) = h(t,x)$.

Exercise 2. (1) *Prove that $PG_R = G_R P = id$.*
(2) *Construct the advanced propagator G_A.*

Requiring the additional conditions:

$$Supp(G_R h) \subset J_+(Supp h),$$

$$Supp(G_A h) \subset J_-(Supp h),$$

would make G_R and G_A unique.

Let $\chi \in \mathcal{C}^\infty(M)$, and Σ_1, Σ_2 be Cauchy surfaces such that $\Sigma_1 \cap J_+(\Sigma_2) = \emptyset$. Assume $\chi(x) = 1$ for $x \in J_-(\Sigma_1)$ and $\chi(x) = 0$ for $x \in J_+(\Sigma_2)$. Then $G_\chi = G_R(1-\chi) + G_A\chi$ is a Green's function.

4.2. **Hadamard function.** As in section 2.1.1, it would be more convenient to introduce the Weyl algebra over the symplectic space (L, σ) instead of the algebra of commutation relations.

$$(L, \sigma) \to \mathcal{W}(L, \sigma).$$

It is a simple algebra with a unique C*-norm. We can now choose as our states the quasifree state. Let μ be a real scalar product on L, satisfying

$$\mu(f,f)\mu(g,g) \leq \frac{1}{4}(\sigma(f,g))^2.$$

Then, quasifree states would be of the form $\omega_\mu(W(f)) = e^{-\frac{1}{2}\mu(f,f)}$, for $W \in \mathcal{W}$. Now, $\mu + \frac{i}{2}\sigma$ is a complex scalar product on $L_{\mathbb{C}}$, which can be used to construct the one particle Hilbert space $\mathcal{H}_1 = \overline{(L_{\mathbb{C}}, \mu + \frac{i}{2}\sigma)/\text{null space}}$.

The GNS triplet $(\mathcal{H}, \pi, \Omega)$ becomes:

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \left(\mathcal{H}_1^{\otimes n}\right)_{\text{symm.}} \text{ Fock Space,}$$

$$a(f)\Omega = 0, [a(g), a(f)^*] = \langle g, f\rangle, f, g \in \mathcal{H}, a(f) \text{an anti-linear function of } f,$$
$$\pi(\phi(f)) = a(f) + a(f)^*.$$

Now the natural question which arises would be how to find such a scalar product μ? In Minkowski space it exists and explicitly can be given.

$$\begin{aligned} G(t, x, t', x') &= \Delta(t - t', x - x') \\ &= \frac{1}{(2\pi)^3} \int \frac{d^3\vec{p}}{\sqrt{m^2 + p^2}} \sin(\sqrt{p^2 + m^2}(t - t'))e^{i\vec{p}.(\vec{x} - \vec{x}')} \end{aligned}$$

$$(\mu + \frac{i}{2}\sigma)(f, g) = \frac{1}{(2\pi)^3} \int \frac{d^3\vec{p}}{2\sqrt{m^2 + p^2}} \overline{\hat{f}(\sqrt{p^2 + m^2}, \vec{p})}\hat{g}(\sqrt{p^2 + m^2}, \vec{p}).$$

$$\Delta_+ = \frac{1}{(2\pi)^3} \int \frac{d^3\vec{p}}{\sqrt{m^2 + p^2}} e^{\sqrt{p^2+m^2}(t-t')} e^{i\vec{p}.(\vec{x} - \vec{x}')}$$

Notice that while the wave front set of Δ is the full light cone (because of the sin function, two signs of mass shell are present: $\vec{p}||\vec{x}, p_0 > 0, p_0 < 0$), the wave front set of Δ_+ constitutes only V_+, the future light cone (where only the positive sign of mass shell is present: $\vec{p}||\vec{x}, p_0 > 0$). In the case of Minkowski, we encounter a happy coincidence. $(\mu + \frac{i}{2}\sigma)$ not only defines a positive definite scalar product, but also satisfies the positive energy condition (recall that these two have totally different meanings: the positivity of scalar product implies the that of probability, while the positivity of energy implies stability of the quantum system).

We now intend to generalize this to the case of arbitrary curved space-times. In fact, we want to find some other function H with the following property:

$$WF(H + \frac{i}{2}G) = \{(x, k; x', k') \in WF(G), k \in V_+\}$$

H, the so-called *Hadamard function*, annihilates the part of WF(G) which lies in the past light cone. Given such an H, we can now define a real scalar product μ:

$$\mu(f, g) = \int dxdyf(x)H(x, y)g(y).$$

In this regard, H can be seen as an integral kernel which must satisfy the following:

(1) H is a bi-solution of the Klein-Gordon equation,
(2) H is symmetric (in order for μ to form a scalar product),

(3) (first positivity condition) H is positive, in the sense that

$$\int dxdy\overline{f(x)}(H(x,y)+\frac{i}{2}G(x,y))f(y)\geq 0,$$

(4) (second positivity condition) The wave front set should be chosen in such a way that compensates the negative part of WF(G).

$$WF(H+\frac{i}{2}G)=\{(x,k;x^{'},k^{'})\in WF(G),k\in V_{+}\}.$$

Exercise 3. *Let $H^{'}$ be another Hadamard function. Show that $H-H^{'}$ is a smooth bi-solution of the Klein-Gordon equation.*

We will sketch the proof of existence of such an H, which can be found in [].

Consider a deformation of a space-time M to an ultra-static one N in the early past:

$$g_M=a(t)dt^2-h_t\rightarrow g_N=dt^2-h.$$

Now we insert an intermediate space-time L ($g_L=a^{'}(t)dt^2-h_t^{'}$) between M and N such that:

$$\text{for } t>t_1:\begin{cases}a^{'}(t)=a(t)\\ h^{'}(t)=h_t\end{cases},\text{for } t<t_2:\begin{cases}a^{'}(t)=1\\ h^{'}(t)=h\end{cases}\ (t_1>t_2).$$

On N, we can choose $H(t,x;s,y)=\frac{cos\sqrt{A}(t-s)}{2\sqrt{A}}(x,y)$, where $A=-\triangle_h+m^2+\xi R$, and $x,y\in\Sigma$.

$$(H+\frac{i}{2}G)(t,x;s,y)=\left(\frac{e^{i(t-s)\sqrt{A}}}{2\sqrt{A}}\right)(x,y)$$

$$\int dtdsdxdy\overline{f(t,x)}\left(\frac{e^{i(t-s)\sqrt{A}}}{2\sqrt{A}}\right)(x,y)f(s,y)=\int dtds\langle\overline{f}_t,\frac{e^{i(t-s)\sqrt{A}}}{2\sqrt{A}}f_s\rangle,$$

$$\Rightarrow\langle\int dt\frac{e^{-it}}{\sqrt{2\sqrt{A}}}f_t,\int ds\frac{e^{-is}}{\sqrt{2\sqrt{A}}}f_s\rangle\geq 0.$$

As an ansatz for Hadamard function, one can use

$$H=\frac{u}{\sigma}+v\ln\sigma,$$

where $u,v\in\mathcal{C}^\infty(M^2)$, and σ is the "squared geodesic distance": $\sigma(x,y)=\pm\left(\int_{t_1}^{t_2}dt\sqrt{|g(\dot\gamma,\dot\gamma)|}\right)^2$ with γ being a geodesic from x to y and $+$, $-$ signs denote time-like and space-like geodesics respectively. Note that this is well-definedd on a geodesically convex set (every two points inside the set can be joined by a unique geodesic).

4.3. **Functional formalism.** In this section, we introduce the functional formulation of field theory. The functional formalism makes the comparison of classical and quantum theory more clear, and it opens up the possibility to compare the algebraic and the path integral formulation of quantum mechanics. Moreover, the functional formalism has the advantage that it is less abstract than algebraic one, and makes it possible to do more concrete calculations; it is a concretization of algebraic framework but still does not make use of the Hilbert space representation. We begin with stating basic definitions.

Space of Field Configurations. We admit all field configuration $\phi \in \mathcal{C}^\infty(M)$ which are smooth functions of space-time (not necessarily solutions).

Functionals. Observables of the theory, made out of fields, can be represented as functionals. They associate to each field configuration a number:

$$F(\phi) = \sum_{n=0}^{N} \int dx_1 \dots dx_n f_n(x_1, \dots, x_n)\phi(x_1)\dots\phi(x_n).$$

Based on different choices for f_n, we have the following types of functionals:

- F is called *regular* if $f_n \in D(M^n)_{symm}$
- F is called *local* if supp $f_n \subset \mathrm{diag}_n(M) = \{(x, \dots, x) \in M^n, x \in M\}$

For example, consider the functional $F(\phi) = \int dx f(x)\phi(x)^2$. This can be written as $\int dxdy f_2(x, y)\phi(x)\phi(y)$, with $f_2(x, y) = f(x)\delta(x - y)$ which has support only where $x = y$, and hence F is a local functional.

Note that F can be both regular and normal only if $n = 0$ or $n = 1$.

$\star$-product. We aim to construct the algebra of observables. To this end, we need to introduce a notion of product on the space of functionals. Such a $\star$-product is defined via,

$$(F_1 \star F_2)(\phi) := \sum_{n=0}^{\infty} \left(\frac{i\hbar}{2}\right)^n \frac{1}{n!} \langle \frac{\delta^n F_1}{\delta\phi^n}[\phi], G^{\otimes n} \frac{\delta^n F_2}{\delta\phi^n}[\phi] \rangle \,,$$

with the notation,

$$\begin{aligned} \langle \frac{\delta^n F_1}{\delta\phi^n}[\phi], G^{\otimes n} \frac{\delta^n F_2}{\delta\phi^n}[\phi] \rangle \quad &= \quad \int dx_1 \dots dx_n dy_1 \dots dy_n \frac{\delta^n F_1[\phi]}{\delta\phi(x_1)\dots\delta\phi(x_n)} \\ &\quad \times G(x_1, y_1)\dots G(x_n, y_n) \frac{\delta^n F_2[\phi]}{\delta\phi(y_1)\dots\delta\phi(y_n)}. \end{aligned}$$

$\frac{\delta^n F}{\delta\phi^n}[\phi]$, the n-th functional derivative of F with respect to ϕ, is a distribution of n variables, therefore must be evaluated at test functions of n variables:

$$\begin{aligned}\langle\frac{\delta^n F_1}{\delta\phi^n}[\phi],\psi^{\otimes n}\rangle &= \int dx_1\dots dx_n \frac{\delta^n F(\phi)}{\delta\phi(x_1)\dots\delta\phi(x_n)}\psi(x_1)\dots\psi(x_n)\\ &:= \frac{d^n}{d\lambda^n}F(\phi+\lambda\psi)|_{\lambda=0}.\end{aligned}$$

Examples.

(1) Lets F be a regular functional of the form

$$F(\phi)=\int dx_1,\dots dx_k f(x_1,\dots x_k)\phi(x_1)\dots\phi(x_k)$$

$$\begin{aligned}\langle\frac{\delta^n F_1}{\delta\phi^n}[\phi],\psi^{\otimes n}\rangle &= \frac{d^n}{d\lambda^n}|_{\lambda=0}\int dx_1,\dots dx_k f(x_1,\dots x_k)\phi(x_1+\lambda\psi(x_1))\dots\phi(x_k+\lambda\psi(x_k))\\ &= \sum_{j=0}^{k}\binom{k}{j}\int dx_1\dots dx_k f(x_1\dots x_k)\phi(x_1)\dots\phi(x_j)\\ &\quad\times\psi(x_{j+1})\dots\psi(x_k)\frac{d^n}{d\lambda}\lambda^{k-j}|_{\lambda=0}\\ &= \binom{k}{n}n!\int dx_1\dots dx_k f(x_1\dots x_k)\phi(x_1)\dots\phi(x_{k-n})\psi(x_{k-n+1})\dots\psi(x_k).\end{aligned}$$

(2) Lets calculate the $\star$-product of two functionals which are both regular and local,

$$F_1(\phi)=\int dx f_1(x)\phi(x), F_2(\phi)=\int dx f_2(x)\phi(x)$$

$$(F_1\star F_2)(\phi)=F_1(\phi)F_2(\phi)+\frac{i\hbar}{2}\langle f_1, Gf_2\rangle.$$

In order to show that the algebra of functionals generated by $\star$-product is the algebra of observables, we must show that the $\star$-product is associative. Here, we present two arguments to do so.

First argument: Define the Weyl functional $W(f)$ by

$$W(f)[\phi]=e^{i\int dx f(x)\phi(x)}, f\in\mathcal{D}(M).$$

Then for this functional we have:

$$\begin{aligned}\langle \frac{\delta W(f)}{\delta \phi}[\phi], h\rangle &= \frac{d}{d\lambda}\left(W(f)(\phi+\lambda h)\right)|_{\lambda=0} \\ &= \frac{d}{d\lambda} e^{i\int dx f(x)(\phi(x)+\lambda h(x))}\Big|_{\lambda=0} \\ &= \left(i\int dx f(x)h(x)\right) W(f)[\phi].\end{aligned}$$

$$\langle \frac{\delta^n W(f)}{\delta \phi^n}[\phi], h^{\otimes n}\rangle = \left(i\int dx f(x)h(x)\right)^n W(f)[\phi].$$

Inserting this into the $\star$-product formula, we find,

$$\begin{aligned}W(f)\star W(g) &= \sum_{n=0}^{\infty}\left(\frac{i\hbar}{2}\right)^n \frac{(-1)^n}{n!}\left(dxdyG(x,y)g(y)\right)^n .W(f+g) \\ &= e^{-\frac{i\hbar}{2}\langle f, Gg\rangle} W(f+g),\end{aligned}$$

which are precisely the Weyl relations defined in 2.1.1. Now the associativity of $\star$-product can be checked easily:

$$\begin{aligned}(W(f_1)\star W(f_2))\star W(f_3) &= e^{-\frac{i\hbar}{2}(\langle f_1, Gf_2\rangle + \langle f_1+f_2, Gf_3\rangle)} W(f_1+f_2+f_3) \\ &= W(f_1)\left(W(f_2)\star W(f_3)\right).\end{aligned}$$

Weyl functionals generates all regular functionals; we can take the derivatives of this functional with respect to test functions and get polynomials in ϕ. And all such polynomials satisfy the associativity of product.

Second Argument: We can redefine the $\star$-product as:

$$F_1 \star F_2 = m\circ e^{\Gamma}(F_1\otimes F_2)$$

where,

- $(F_1\otimes F_2)(\phi_1,\phi_2) := F_1(\phi_1)F_2(\phi_2)$
- $m(F_1\otimes F_2)(\phi) := F_1(\phi)F_2(\phi)$ (pointwise product)
- $\Gamma := \frac{i\hbar}{2}\int dxdyG(x,y)\frac{\delta}{\delta\phi(x)}\otimes\frac{\delta}{\delta\phi(y)} \equiv \Gamma_{12}$

To check the associativity, note that the Leibniz rule states

$$\frac{\delta}{\delta\phi}\circ m = m\circ\left(\frac{\delta}{\delta\phi}\otimes \mathrm{id} + \mathrm{id}\otimes\frac{\delta}{\delta\phi}\right)$$

Therefore,

$$\Gamma\circ(m\times \mathrm{id}) = (m\otimes\mathrm{id})\left(\Gamma_{13}+\Gamma_{23}\right),$$

and

$$e^{\Gamma}\circ(m\times\mathrm{id}) = (m\otimes\mathrm{id})e^{(\Gamma_{13}+\Gamma_{23})}.$$

Now we have,

$$\begin{aligned}(F_1 \star F_2) \star F_3 &= m \circ e^{\Gamma}(m \circ e^{\Gamma} \otimes \mathrm{id})(F_1 \otimes F_2 \otimes F_3)\\ &= m \circ (m \otimes \mathrm{id}) e^{(\Gamma_{13}+\Gamma_{23})} \circ e^{\Gamma_{12}}(F_1 \otimes F_2 \otimes F_3)\\ &= m \circ (\mathrm{id} \otimes m) e^{\Gamma_{13}} \circ e^{(\Gamma_{23}+\Gamma_{12})}(F_1 \otimes F_2 \otimes F_3)\\ &= F_1 \star (F_2 \star F_3).\end{aligned}$$

What is the advantage of the functional formalism compared to the abstract algebra of observables? There is one basic problem with the abstract algebra of commutation relation, namely the most interesting observables, such as $\phi(x)^2$, are not in this algebra, simply because such observables are singular. However, in the functional formalism such objects are prototypes of local functional: $F(\phi) = \frac{1}{2}\int dx f(x)\phi(x)^2$. Nevertheless, the difficulty in working with such objects is now manifest in defining the product of them:

$$\begin{aligned}F(\phi) \star F(\phi) &= F(\phi)F(\phi) + \frac{i\hbar}{2}\int dxdy f(x)\phi(x)f(y)\phi(y)G(x,y)\\ &\quad + \left(\frac{i\hbar}{2}\right)^2 . \frac{1}{2}\int dx f(x)G(x,y)^2 f(y),\end{aligned}$$

while $G(x,y)^2$ is ill-defined. Circumventing such a problem is normally done by representing the fields on a Fock space, decomposing them into annihilation and creation operators, and applying the normal ordering prescription, meaning to subtract the singular terms, which leads to the normal ordered fields. Apparently, this method depends on the choice of the Fock space representation which in general is not unique.

In order to overcome such pathological behavior in a general framework, we can change the definition of $\star$-product in such a way that leads to an isomorphic algebra. This can be done by using the Hadamard function in definition of product.

$$(F_1 \star_H F_2)(\phi) = \sum_{n=0}^{\infty} \langle \frac{\delta^n F_1}{\delta\phi^n}, (H + \frac{i\hbar}{2}G)^{\otimes n} \frac{\delta^n F_2}{\delta\phi^n} \rangle\,,$$

where H is the Hadamard function introduced in section 4.2. By methods discussed above, one can check that $\star_H$ is associative as well. Below, we first prove that the two algebras $\mathfrak{A} = (\mathcal{F}_{reg}, \star)$ and $\mathfrak{A}' = (\mathcal{F}_{reg}, \star_H)$ are isomorphic, with $\mathcal{F}_{reg}$ being the space of regular functionals.

Theorem 6. *$\star_H$ is equivalent to $\star$, i.e. there exists a linear isomorphism $e^{\gamma} : \mathcal{F}_{reg} \to \mathcal{F}_{reg}$, such that $e^{\gamma}(F_1 \star F_2) = e^{\gamma}F_1 \star_H e^{\gamma}F_2$.*

Proof. Take $\gamma = \frac{\hbar}{2}\int dxdyH(x,y)\frac{\delta^2}{\delta\phi(x)\delta\phi(y)}$ and $\gamma_{jk} = \frac{\hbar}{2}\int dxdyH(x,y)\frac{\delta^2}{\delta\phi_j(x)\delta\phi_k(y)}$. Then

$$\begin{aligned} F_1 \star_H F_2 &= e^{\gamma} \circ m \circ e^{\Gamma}(e^{-\gamma} \otimes e^{-\gamma})(F_1 \otimes F_2) \\ &= m \circ e^{\gamma_{11}+\gamma_{12}+\gamma_{21}+\gamma_{22}} \circ e^{\Gamma} \circ e^{-\gamma_{11}-\gamma_{22}} \\ &= m \circ e^{(2\gamma_{12}+\Gamma)} \\ &= F_1 \star_H F_2. \end{aligned}$$

where we have made use of the symmetry of the Hadamard function in its arguments which implies $\gamma_{12} = \gamma_{21}$, and $2\gamma_{12} + \Gamma \equiv \hbar\int dxdy(H + \frac{i}{2}G)\frac{\delta^2}{\delta\phi_1(x)\delta\phi_2(y)}$. □

We can now extend this product to local functionals.

Exercise 4. *Show that the algebra generated by local functionals is independent of the choice of H. In addition, show that $\star_H$ and $\star_{H'}$ are equivalent on this algebra.*

4.3.1. *States.* Recall that states of a theory associate to observables real numbers which can be interpreted as the expectation value of measuring them. In functional formulation of scalar field, observables are linear functionals of field, hence using the *star*-product, we can construct different states satisfying its defining conditions introduced in 2.2. There exists an interesting connection between the new $\star_H$-product and the notion of states. Since $H + \frac{i}{2}G$ is of positive type (i.e. $\int dxdy\overline{f(x)}(H + \frac{i}{2}G)(x,y)f(y) \geq 0$), it induces a scalar product on space of test functions. Below, we introduce two types of such states .

Gaussian states. We define a state directly related to the Hadamard function as

$$\omega_H(F) = F(0).$$

It satisfies:

(1) $F \mapsto \omega_H(F) = F(0)$ is linear;
(2) $\omega_H(1) = 1$;
(3) $\omega_H(F^\dagger \star_H F) = \sum_{n=0}^{\infty}\frac{\hbar^n}{n!}\langle\overline{F^{(n)}[0]}, (H + \frac{1}{2}G)^{\otimes n}F^n[0]\rangle \geq 0$.

This is very similar to states in classical physics which are the value of observables (functions of phase space) at a certain point of phase space. Such states could not have been realized according to the original $\star$-product, since quantum uncertainties avoid us of evaluating F at a specific value of fields. Such states are indeed the Gaussian (quasi-free) states we have already encountered. To see this, consider a Weyl function $W(f) = \exp_{\star_H}(i\int dxf(x)\phi(x))$ defined with respect to $\star_H$-product,

$$\begin{aligned} W(f)[\phi] &= e^{\hbar\gamma} o \exp_{\star} o e^{-\hbar\gamma}(i\int dxf(x)\phi(x)) \\ &= e^{\hbar\gamma} o \exp_{\star}(i\int dxf(x)\phi(x)), \end{aligned}$$

where in the last step, we have used $\gamma(i\int dx f(x)\phi(x)) = 0$. Now using the anti-symmetry of $G(x,y)$, we have:

$$\int dx\phi(x)f(x)\star\int dy\phi(y)f(y) = \int dxdy\phi(x)f(x)\phi(y)f(y) + \underbrace{\frac{i\hbar}{2}\int dxdyG(x,y)f(x)f(y)}_{0},$$

and similarly for higher powers. Thus, we get

$$\begin{aligned}\exp_\star\left(i\int dx\phi(x)f(x)\right) &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(i\int dx\phi(x)f(x)\right)^{\star n}\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(i\int dx\phi(x)f(x)\right)^{n}\\ &= \exp\left(i\int dx\phi(x)f(x)\right).\end{aligned}$$

This leads to,

$$\begin{aligned}\omega_H(W(f)) &= e^{\hbar\gamma}e^{i\int dxf(x)\phi(x)}\Big|_{\phi=0}\\ &= e^{\frac{1}{2}\int dxdyH(x,y)f(x)f(y)},\end{aligned}$$

which is a Gaussian state. Hence once we have chosen a suitable $\star$-product, the Gaussian states will emerge naturally.

Coherent states. Let ψ be a real-valued solution of the Klein-Gordon equation. Then,

$$\omega_{H,\psi}(F) = F(\psi)$$

is called a *coherent* state. Acting on a Weyl functional, we have

$$\omega_{H,\psi}(W(f)) = e^{-\frac{1}{2}\int dxdyH(x,y)f(x)f(y)+i\int dxf(x)\psi(x)}.$$

Now we want to see How Gaussian states change if we use another Hadamard functional $H^{'}$?

Consider $H^{'} = H + w$ be a different Hadamard functional, wehre $w(x,y)$ is a symmetric, smooth bi-solution. To see why w is smooth, we look at the wavefront set $WF(H^{'} - H)$. Since H and $H^{'}$ are symmetric, $WF(H^{'} - H)$ is symmetric as well.

$$WF(H^{'}_{+} - H_{+}) \subset WF(H_{+})_{\text{symm}} = \{(x,k,x^{'},k^{'})\in WF(G), k\in\overline{V_{+}(x)}, k^{'}\in\overline{V_{-}(x)}\}_{\text{symm}} = \emptyset,$$

and thus $H^{'} - H$ is smooth. Now we, have the isomorphism

$$e^{\gamma w}:\mathfrak{a}_{H^{'}}\to\mathfrak{a}_H,$$

with $\gamma w = \frac{1}{2}\int dxdyw(x,y)\frac{\delta^2}{\delta\phi(x)\delta\phi(y)}$. Now, $\omega_{H^{'}}oe^{\gamma w}$ is a state on $\mathfrak{a}_H$,

$$\omega_{H^{'}}e^{\gamma w}(F) = (e^{\gamma w}F)(0)$$

To check that H' is of positive type, a sufficient condition would be the positivity of w. As a simple example of w, consider $w(x,y) = \psi(x)\psi(y)$ with ψ being a real valued smooth solution. Then,

$$\omega_{H'}(W(f))\omega_{H,\psi}(W(f)) = e^{-\frac{1}{2}\int dxdyH(x,y)f(x)f(y)+i\int dxf(x)\psi(x)}.$$

This observation has a generalization.

Exercise 5. *Show that ω_H can be obtained as a (continuous) convex combination of the coherent states $\omega_{H,\lambda\psi}$, $\lambda \in \mathbb{R}$:*

$$\omega_{H'}(W(f)) = \int d\lambda h(\lambda)\omega_{H,\lambda\psi}(W(f))$$

This exercise shows that the Gaussian state can be written as a mixture of coherent states.

5. Locally covariant field theory

5.1. **Haag-Kastler axioms.** After the great success of renormalization theory for QED in the late 1940' the difficulties in extending the theory to strong and weak interactions motivated several attempts to formulate axioms for QFT. These axioms should summarize the essential properties a theory should have. Among these systems of axioms are the LSZ (Lehmann-Symanzik-Zimmermann) framework by which the S-matrix is expressed in terms of vacuum expectation values of time ordered products, the Wightman framework, which characterizes the quantum fields (considered as operator valued distributions) by the vacuum expectation values of their products (the so-called Wightman functions), and the Osterwalder-Schrader axioms which were formulated somewhat later and characterize the theory in terms of the analytic extension of Wightman functions to imaginary times (the so-called Schwinger functions). More recently, mathematicians formulated axioms for 2d conformal field theory and for topological field theory. The axiomatic system which is best suited for the extension of the theory to curved spacetimes is the algebraic framework which was developed by Haag, Araki, Schroer and Kastler and was formalized in a programmatic paper by Haag and Kastler. Later it was generalized to globally hyperbolic spacetimes by Dimock.

5.1.1. *Algebras of local observables.* Crucial for this approach is that the principle of locality is incorporated in an new and very general way by just considering for every bounded region of spacetime the algebra $\mathfrak{A}(\mathcal{O})$ of observables which can be measured within $\mathcal{O}$. One may think of this algebra as the algebra generated by fields $\varphi(x)$ with $x \in \mathcal{O}$ but this plays no role for the general structure. $\mathfrak{A}(\mathcal{O})$ is assumed to be a unital C*-algebra.

5.1.2. *Isotony.* Observables measurable in a region $\mathcal{O}_1$ are also measurable in every region $\mathcal{O}_2 \supset \mathcal{O}_1$. In the formalism this is incorporated by requiring the existence of embeddings (i.e. injective homomorphisms) $\iota_{\mathcal{O}_2\mathcal{O}_1} : \mathfrak{A}(\mathcal{O}_1) \to \mathfrak{A}(\mathcal{O}_2)$ with the compatibility condition

$$\iota_{\mathcal{O}_3\mathcal{O}_2} \circ \iota_{\mathcal{O}_2\mathcal{O}_1} = \iota_{\mathcal{O}_3\mathcal{O}_1}$$

if $\mathcal{O}_1 \subset \mathcal{O}_2 \subset \mathcal{O}_3$. The bounded regions form a directed set with respect to inclusions (i.e. for two bounded regions there exists a third one containing both). The system of algebras $(\mathfrak{A}(\mathcal{O}))$ therefore is called a net, usually called the local net or the Haag-Kastler net.

Given such a net, one can construct the algebra of all observables as a so-called inductive limit of the net. Roughly speaking, the inductive limit is the completion of the union of all local algebras. More precisely, the inductive limit of a net of unital C*-algebras $(\mathfrak{A}_i)_{i\in I}$ where I is a directed set, is a unital C*-algebra $\mathfrak{A}$ together with embeddings $\iota_i : \mathfrak{A}_i \to \mathfrak{A}$ such that

$$\iota_i \circ \iota_{ij} = \iota_j$$

for $i \geq j$ and with the following universality condition: If $\mathfrak{B}$ is another unital C*-algebra with embeddings $\kappa_i : \mathfrak{A}_i \to \mathfrak{B}$ fulfilling $\kappa_i \circ \iota_{ij} = \kappa_j$ for $i \geq j$ then there exists an embedding $\kappa : \mathfrak{A} \to \mathfrak{B}$ such that $\kappa_i = \kappa \circ \iota_i$ for all $i \in I$.

The universality condition guarantees that the inductive limit is unique up to isomorphy, provided it exists.

The existence can be shown as follows: Consider sequences $(A_i)_{i\geq i_0}$ with $A_i \in \mathfrak{A}_i$ and $A_i = \iota_{ij}(A_j)$ for $i \geq j \geq i_0$. Consider sequences as equivalent if they coincide for sufficiently large i. The classes of sequences get the structure of a unital *-algebra by pointwise operations. The algebra is equipped with a unique C*-norm $||(A_i)_{i\geq i_0}|| = ||A_j||$ for some $j \geq i_0$ (the norm is independent of the choice of j since the embeddings are norm preserving). The inductive limit is then the completion $\mathfrak{A}$ of this algebra. The embeddings ι_i are given by

$$\iota_i(A)_j = \iota_{ji}(A) \ .$$

The universality condition is satisfied by setting $\kappa((A_i)_{i\geq i_0}) = \kappa_j(A_j)$ where the right hand side is independent of the choice of $j \geq i_0$.

5.1.3. *Covariance.* Let G be the group of isometries of the spacetime which preserve time orientation and orientation. Then there should be a representation α of G by automorphisms of $\mathfrak{A}$ such that

$$\alpha_g(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(g\mathcal{O})$$

One may wonder about the appropriate continuity condition on α. A natural condition would be to require that the maps $G \ni g \mapsto \alpha_g(A)$ are continuous for all $A \in \mathfrak{A}$ (strong continuity). Unfortunately, it turns out, that this condition is not satisfied in all examples of interest. Another condition would be that the maps $G \ni g \mapsto \omega(\alpha_g(A))$ are continuous for all A and all states ω (weak continuity). Even this condition is not always fulfilled. In typical cases one requires the continuity only for a subset of states.

5.1.4. *Einstein causality (local commutativity).* According to the principles of relativity, no signal can travel faster than light. This is taken into account by requiring that observables localized in spacelike separated regions commute. Sending a signal from a region $\mathcal{O}_1$ may be described by the application of a unitary $U \in \mathfrak{A}(\mathcal{O}_1)$. An observable $A \in \mathfrak{A}(\mathcal{O}_2)$ is transformed by this operation to U^*AU. The axiom then states that

$$U^*AU = A$$

if the regions $\mathcal{O}_1$ and $\mathcal{O}_2$ cannot be connected by a causal curve.

The anticommutativity of fermionic fields cannot be directly interpreted as a consequence of causality. On the contrary, the argument for commutativity shows that fermionic fields do not correspond to observables.

5.1.5. *Timeslice axiom.* Up to now there is no specification of dynamics. A weak version of the existence of a dynamical law is the timeslice axiom. It states that for a globally hyperbolic region $\mathcal{O}$ with a Cauchy surface Σ the observables in any neigbourhood $\mathcal{O}_1$ of Σ already contain the information about the observables in $\mathcal{O}$,

$$\mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}(\mathcal{O}_1) \ .$$

5.1.6. *Stability (spectrum condition).* This is the only axiom which has no natural formulation on general spacetimes. On Minkowski space it says that there exists a faithful representation π of $\mathfrak{A}$ on some Hilbert space $\mathfrak{H}$ and a unitary strongly continuous representation U of the translation group on $\mathfrak{H}$ such that $U(x)\pi(A)U(x)^* = \pi \circ \alpha_x(A)$ and such that the spectrum of U is contained in the closed forward lightcone.

5.2. **Local covariance.** On a curved spacetime, there does not exist a meaningful version of the spectrum condition. One may, however, find a weaker version in terms of conditions on wave front sets. While this can be done for expectation values of products of fields (”microlocal spectrum condition”), a purely algebraic version is not known.

But also the covariance axiom is problematic since in the generic case the symmetry group is trivial. This turns out to be a problem when one wants to remove singularities by renormalization. One would like to do the “same” subtraction everywhere, but it is not clear how to formulate such a condition.

This problem was discussed e.g. by Wald for the definition of the energy momentum tensor. It became urgent when renormalization of interacting quantum field theory was performed [BF2000]. It was solved by developing a new concept for quantum field theory, termed *local covariance.*

5.2.1. *QFT as a functor.* The basic ansatz is that one should no longer try to formulate QFT on a specific spacetime. Instead, one should construct it simultaneously on all spacetimes of a given class by requiring appropriate coherence conditions. The basic idea is that a globally hyperbolic subregion of a given spacetime should be considered as a spacetime in its own right. In particular, the observables localized within this subregion should not depend on the fact that the region is part of a larger spacetime.

In the formulation, one must be precise with the *admissible* embeddings of a spacetime into another spacetime. When we embed a spacetime M isometrically into another one N by means of χ, it is in general possible that two different points x and y of a Cauchy surface of M can be connected by a causal curve γ in N. To avoid such a situation, we require the embedding to be causality preserving, in the sense that every causal curve γ in N connecting $\chi(x)$ and $\chi(y)$ must be contained in $\chi(M)$.

We now want to formulate postulates a quantum field theory on generic globally hyperbolic spacetimes should fulfil. We assume that to every contractible oriented and time oriented globally hyperbolic spacetime M we can associate a unital C*-algebra $\mathfrak{A}(M)$. Moreover, if $\chi : M \to N$ is an admissible embedding of manifolds, i.e. it is isometric and orientation and causality preserving as described above, we require the existence of an injective homomorphism $\alpha_\chi : \mathfrak{A}(M) \to \mathfrak{A}(N)$. This expresses the idea that measurements done within $\chi(M)$ cannot see any difference to the corresponding measurements done in M. Moreover, if we first embed M into N by a map χ and then N into L by ψ we require that the measurements in the subregion $\psi \circ \chi(M)$ of L are unaffected by the split of the embedding into two subsequent maps, hence we impose the condition

$$\alpha_{\psi\circ\chi} = \alpha_\psi \circ \alpha_\chi \ .$$

The structure described above is that of a functor between a category of spacetimes $\mathfrak{Loc}$ and a category of observable algebras $\mathfrak{Obs}$. The category of spacetimes has as its objects the spacetimes with the properties listed above and as its morphisms the admissible embeddings. The category of observable algebras has as its objects unital C*-algebras and injective homomorphisms as its morphisms. A quantum field theory is a functor

$$\mathfrak{A} : \mathfrak{Loc} \to \mathfrak{Obs}.$$

between these categories with the action $\mathfrak{A}\chi = \alpha_\chi$ on morphisms.

The functorial picture of quantum field theory generalizes the concept of a local net of observable algebras. To see the connection we restrict our category $\mathfrak{Loc}$ to those objects which are subsets $\mathcal{O}$ of a given spacetime M such that the inclusions $\iota_{M\mathcal{O}}$ are morphisms. We then define the local algebras by

$$\mathfrak{A}_M(\mathcal{O}) = \alpha_{\iota_{M\mathcal{O}}}(\mathfrak{A}(\mathcal{O}))$$

We first observe that the axiom of isotony is fulfilled. Namely, let $\mathcal{O}_1 \subset \mathcal{O}_2 \subset M$. Then

$$\mathfrak{A}_M(\mathcal{O}_1) = \alpha_{\iota_{M\mathcal{O}_1}}(\mathfrak{A}(\mathcal{O}_1)) = \alpha_{\iota_{M\mathcal{O}_2}\circ\iota_{\mathcal{O}_2\mathcal{O}_1}}(\mathfrak{A}(\mathcal{O}_1))$$
$$= \alpha_{\iota_{M\mathcal{O}_2}} \circ \alpha_{\iota_{\mathcal{O}_2\mathcal{O}_1}}(\mathfrak{A}(\mathcal{O}_1)) \subset \alpha_{\iota_{M\mathcal{O}_2}}(\mathfrak{A}(\mathcal{O}_2)) = \mathfrak{A}_M(\mathcal{O}_2)$$

While this was to be expected, it is somewhat surprising that also the axiom of covariance is automatically satisfied. Let $g \in G$ be a symmetry of M. Then g is a morphism from M to M and α_g is an automorphism of $\mathfrak{A}(M)$. Moreover, the functoriality of $\mathfrak{A}$ implies that $g \mapsto \alpha_g$ is a group homomorphism. Let us now look at the action of α_g on the local subalgebras of $\mathfrak{A}(M)$. Let $\mathcal{O}$ be an object in $\mathfrak{Loc}_M$. Then

$$\alpha_g(\mathfrak{A}_M(\mathcal{O})) = \alpha_g \circ \alpha_{\iota_{M\mathcal{O}}}(\mathfrak{A}(\mathcal{O})) = \alpha_{g\circ\iota_{M\mathcal{O}}}(\mathfrak{A}(\mathcal{O})) = \alpha_{\iota_{Mg\mathcal{O}}\circ g\restriction_{\mathcal{O}}}(\mathfrak{A}(\mathcal{O}))$$
$$= \alpha_{\iota_{Mg\mathcal{O}}} \circ \alpha_{g\restriction_{\mathcal{O}}}(\mathfrak{A}(\mathcal{O})) = \alpha_{\iota_{Mg\mathcal{O}}}(\mathfrak{A}(g\mathcal{O})) = \mathfrak{A}_M(g\mathcal{O})$$

The axiom of Einstein causality is related to a tensor structure of our functor. Namely we extend the category $\mathfrak{Loc}$ of spacetimes to finite disjoint unions. The extended category $\mathfrak{Loc}^{\otimes}$ has a tensor (also called monoidal) structure, i.e. a bi-functor

$$\otimes : \mathfrak{Loc}^{\otimes} \times \mathfrak{Loc}^{\otimes} \to \mathfrak{Loc}^{\otimes}$$

with $\otimes(M, N) = M \sqcup N$ and $\chi \in \mathfrak{Mor}(M, N)$ if χ is a map from $M \to N$ such that the restriction to a connected component of M is an admissible embedding into one of the connected components of N and such that embedded components within one component of N are mutually spacelike to each other.

The time slice axiom, implying the existence of dynamics, in this functorial picture of QFT expresses the diffeomorphism invariance of the theory. To see this, we look at the relative Cauchy evolution of two Cauchy surfaces Σ_- and Σ_+ embedded into spacetimes M_1 and M_2 by means of χ_{i-}, χ_{i+} respectively for $i = 1, 2$. The dynamics of $\chi_{1-}(\Sigma_-)$ as it evolves in M_1 amounts to the automorphism

$$\beta = \alpha_{\chi_{1+}} \circ \alpha^{-1}_{\chi_{2+}} \circ \alpha_{\chi_{2-}} \circ \alpha^{-1}_{\chi_{1-}} \in Aut(\mathfrak{A}(M_1)).$$

In particular, a realization of such a situation could be a change in the metric of M_1:

$$M_1 = (M, g), M_2 = (M, g + h),$$

with h having sufficiently small support. Then, one can show that the change of $\beta = \beta_h$ with respect to $h_{\mu\nu}$ is a covariantly conserved quantity:

$$\nabla_\mu \left(\frac{\delta \beta_h}{\delta h_{\mu\nu}} \right) = 0,$$

which reflects the fact that the automorphism β_h is independent of the choice of coordinates.

5.2.2. *Locally covariant QFT.* Having formulated QFT on an arbitrary spacetime as a functor, we now may define the concept of a *locally covariant* quantum field A as a natural transformation between two functors:

$$A : \mathcal{D} \to \mathfrak{A}.$$

Here $\mathcal{D} : \mathfrak{Loc} \to \mathfrak{Vec}$ is a functor to the category of vector spaces which associates to each spacetime its space of test functions and acts on morphisms as $\mathcal{D}\chi = \chi_*$, where χ_* denotes the pushforward of test functions. The requirement on A of being a natural transformation means that there is for each spacetime M a quantum field (i.e. an $\mathfrak{A}(M)$-valued distribution) A_M, and these fields on different spacetimes are related by the commutative diagram:

$$\begin{array}{ccc} \mathcal{D}(M) & \xrightarrow{A_M} & \mathfrak{A}(M) \\ \chi_* \downarrow & & \downarrow \alpha_\chi \\ \mathcal{D}(N) & \xrightarrow{A_N} & \mathfrak{A}(N) \end{array} .$$

This means

$$\alpha_\chi(A_M(f)) = A_N(\chi_* f),$$

or representing the distribution A by an integral $A(f) = \int dx A(x) f(x)$,

$$\alpha_\chi(A_M(x)) = A_N(\chi(x)), x \in M.$$

In Minkowski spacetime, one may insert for χ a Lorentz transformation. Then, the above requirement simply stands for the covariance of scalar fields under Lorentz transformations (with obvious generalizations for other types of fields). This means that the notion of a locally covariant quantum field theory contains the usual notion of fields transforming covariantly under some spacetime symmetry, however it still makes sense even if there is no single symmetry. Using this new concept, we can say what it means to have the "same" field on different spacetimes. In particular, when we have two points on the same spacetime, which are not connected by any symmetry, we can give meaning to having the same observation at both points.

5.3. **Locally covariant free scalar field.** We now reformulate the quantum field theory of a free scalar field on generic spacetimes (discussed in Section 4) as a functor. As the algebra of observables $\mathfrak{A}(M)$, we choose the Weyl algebra $\mathcal{W}_M$ over the symplectic space (L_M, σ_M), where $L_M = \mathcal{D}(M)/P\mathcal{D}(M)$, and $\sigma_M(f,g) = \langle f, G^M g\rangle$. The superscript M on the commutator function G^M indicates that it is the difference between the unique retarded and advanced Green's functions on the spacetime M: $G^M = G^M_R - G^M_A$.

Now consider the functor $\mathfrak{A} : \mathfrak{Loc} \to \mathfrak{Obs}$, with $\chi : M \to N$, $\chi \in Mor_{\mathfrak{Loc}}(M,N)$, and $\mathfrak{A}\chi \equiv \alpha_\chi : \mathcal{W}_M \to \mathcal{W}_N$, $\alpha_\chi \in Mor_{\mathfrak{Obs}}(\mathcal{W}_M, \mathcal{W}_N)$. The requirement of local covariance takes the form

$$\alpha_\chi(W_M(f)) = W_N(\chi_* f),$$

for $f \in \mathcal{D}(M)$.The condition above is in fact the requirement of a (nonlinear) natural transformation of $\mathcal{D} \to \mathfrak{A}$. We want to check whether α_χ is a homomorphism, i.e. to verify

$$\alpha_\chi\left(W_M(f_1)W_M(f_2)\right) = \alpha_\chi\left(W_M(f_1)\right)\alpha_\chi\left(W_M(f_2)\right).$$

The left hand side reads

$$\alpha_\chi\left(e^{-\frac{i\hbar}{2}\sigma^M(f_1,f_2)}W_M(f_1+f_2)\right) = e^{-\frac{i\hbar}{2}\sigma^M(f_1,f_2)}W_N(\chi_*f_1+\chi_*f_2),$$

while for the right hand side we get

$$W_N(\chi_*f_1)W_N(\chi_*f_2) = e^{-\frac{i\hbar}{2}\sigma^N(\chi_*f_1,\chi_*f_2)}W_N(\chi_*f_1+\chi_*f_2).$$

Thus we should show

$$\langle f_1, G^M f_2\rangle = \langle \chi_*f_1, G^N\chi_*f_2\rangle\,.$$

As a result of uniqueness of $G^{M,N}$, the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{D}(M) & \xrightarrow{G^M} & \mathcal{E}(M) \\ \chi_*\big\downarrow & & \big\uparrow\chi^*\,. \\ \mathcal{D}(N) & \xrightarrow{G^N} & \mathcal{E}(N) \end{array}$$

Here $\mathcal{E}(M), \mathcal{E}(N)$ are spaces of smooth functions on the respective spacetimes. They are objects of the category of vector spaces $\mathfrak{Vec}$, and $\mathcal{E}$ is a contravariant functor which acts on morphisms by pullbacks $\mathcal{E}\chi \equiv \chi^* : \mathcal{E}(N) \to \mathcal{E}(M)$, $\phi \mapsto \phi\circ\chi$, $\phi \in \mathcal{E}(N)$ (in other words, the above commutative diagram can be seen as the requirement of G being a natural transformation between $\mathcal{D} : \mathfrak{Loc} \to \mathfrak{Vec}$ and $\mathcal{E} : \mathfrak{Loc}^{\mathrm{op}} \to \mathfrak{Vec}$). It leads to

$$G^M = \chi^*\circ G^N\circ\chi_*,$$

and consequently

$$\begin{aligned} \langle f_1, G^M f_2\rangle &= \langle f_1, \chi^*\circ G^N\circ\chi_*f_2\rangle \\ &= \langle \chi_*f_1, G^N\chi_*f_2\rangle\,, \end{aligned}$$

as desired.

As discussed in Section 4.3, the algebra of the free scalar field can also be constructed out of functionals on the configuration space $\mathcal{C}^\infty(M)$ equipped with the $\star$-product as introduced in Section 4.3. Consider the spaces of all regular functionals on M, $\mathcal{F}(M)$, as objects of the category of vector spaces $\mathfrak{Vec}$. $\mathcal{F}$ is a functor $\mathcal{F} : \mathfrak{Loc} \to \mathfrak{Vec}$, $\mathcal{F}\chi \equiv \alpha_\chi$, $(\alpha_\chi(F))(\phi) = F(\phi\circ\chi)$. The $\star$-products associated to the different spacetimes fulfil the condition $\alpha_\chi(F_1\star F_2) = \alpha_\chi(F_1)\star\alpha_\chi(F_2)$ as follows from the above commutative diagram. Thus $\mathfrak{A} = (\mathfrak{F},\star)$ where $\mathfrak{A}(M) = (\mathfrak{F}(M),\star$ is a vector space equipped with the $\star$-product, is a functor to the category $\mathfrak{Obs}$ which satisfies all axioms of a locally covariant quantum field theory up to the time slice axiom.

5.3.1. *Extended algebra.* Recall that in order to define products of more singular functionals, in particular on the nonlinear local functionals, we introduced a new star product $\star_H$, by means of a Hadamard function (see section 4.2). In the standard formulation of quantum field theory on Minkowski spacetime, this amounts to normal ordering of the fields. Now, we intend to incorporate this in the locally covariant formulation of a free scalar field.

Let H be a Hadamard function on N. Then, the corresponding Hadamard function on M is

$$H_\chi = \chi^* \circ H \circ \chi_*.$$

Therefore the following relation holds:

$$\alpha_\chi \left(F_1 \star_{H_\chi} F_2\right) = \alpha_\chi(F_1) \star_H \alpha_\chi(F_2).$$

But there are many Hadamard functions on a given spacetime. It is not possible to choose on every spacetime M a Hadamard function H_M in such a way that $H_M = \chi^* \circ H_N \circ \chi_*$ holds for all admissible embeddings between spacetimes (i.e. there is no *natural* Hadamard function).

This problem is directly related to the nonexistence of a vacuum. To circumvent it we take a radical step and use all Hadamard functions.

Let $\underline{\mathcal{F}}(M)$ be the set of consistent families of functionals on M, labeled by Hadamard functions, i.e. $F = (F_H)_H \in \underline{\mathcal{F}}(M)$, such that the consistency condition $F_{H'} = e^{\gamma_{H'-H}} F_H$ is satisfied (note that $H' - H$ is smooth). We define the $\star$-product on $\underline{\mathcal{F}}(M)$ by

$$(F_1 \star F_2)_H = F_{1,H} \star_H F_{2,H}$$

The product is again a consistent family, due to the equivalence between the $\star_H$ products as discussed in Section 4.3. $\underline{\mathcal{F}}$ becomes a functor by setting $\underline{\mathcal{F}}\chi = \underline{\alpha}_\chi$ with

$$\underline{\alpha}_\chi(F)_H(\phi) := F_{H_\chi}(\phi \circ \chi).$$

In this way we obtain a locally covariant quantum field theory where now also the local functionals are included. In concrete calculations we usually work with a specific Hadamard function. The formalism described above tells us what happens if we change the Hadamard function.

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