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Compactifications on generalized geometries

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We discuss compactifications on generalized geometries and their role as mirror configurations of compactifications with background fluxes.

1 Introduction

Perturbative string theory is a theory where the fundamental building blocks are not point-like objects but one-dimensional extended strings. Unfortunately our current understanding of string theory is rather incomplete and necessitates, for example, the specification of a space-time background the strings are moving in. In order to make contact with Particle Physics one chooses a space-time background of the form

$$M_4 \times Y_6 \,, \tag{1}$$

where M_4 is a four-dimensional ($d = 4$) infinitely extended Minkowski space while Y_6 is a compact Riemannian manifold. It turns out that the geometrical and topological properties of Y_6 are directly related to physical quantities in an effective description in the physical space-time M_4 [1, 2]. For example, the amount of supersymmetry is related to the number of globally defined spinors on Y_6. This notion in turn leads to a number of further geometrical properties of Y_6 all of which turn out to be related to properties of supersymmetric field theories. This close relationship between geometry and supersymmetry has led to a fruitful interplay for many years.

Currently some of the most promising scenarios for new physics in the TeV region are $N = 1$ supersymmetric extension of the Standard Model. Compatibility with the experimental data requires the supersymmetry to appear in its spontaneously broken phase. In string theory this is traditionally arranged by choosing Y_6 to be a Calabi-Yau threefold and then employ non-perturbative effects to spontaneously break the supersymmetry.

In recent years an alternative setup has been proposed where the Standard Model or its generalization lives on a stack of space-time filling D-branes in a type II bulk [2–5]. Such 'Brane World Scenarios' in turn require to replace the product space-time (1) by a warped product and Y_6 by an appropriate orientifold thereof [5]. The $N = 1$ supersymmetry can then be spontaneously broken either by additionally turning on background fluxes or choosing a manifold Y_6 with torsion.

In this talk we do not review the construction and features of Brane World Scenarios and refer the reader to [5] for a recent review. Here we focus on one particular aspect of these scenarios where Y_6 is not a Calabi-Yau manifold but instead a 'manifold with $SU(3)$ structure'. These manifolds admit a globally defined spinor which however is not necessarily covariantly constant as it is for Calabi-Yau manifolds. A particular subclass of these manifolds – so called 'half-flat manifolds' – have been proposed as the mirror

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of Calabi-Yau compactification with non-trivial background fluxes turned on [6]. The purpose of this talk is to review the issue of mirror symmetry in such compactification and the computation of the low energy effective action following [6, 7].

In order to set the stage we start by briefly recalling Calabi-Yau compactifications of type II string theories in Sect. 2. In Sect. 3 we consider turning on non-trivial background fluxes and discuss the issue of mirror symmetry. In Sect. 4 we discuss compactifications on manifolds with $SU(3)$ structure and revisit the question of mirror symmetry. In particular we outline the computation of the resulting low energy effective theory. We conclude in Sect. 5.

2 Calabi-Yau compactification of type II string theory

In order to set the stage let us start by recalling the massless spectrum of type II string theories in $d = 10$ [1,2]. This string theory comes in two versions called type IIA and type IIB. Both theories share the same Neveu-Schwarz (NS) sector which contains the metric G_{MN}, an antisymmetric tensor B_2 with a three-form field strength $H_3 = dB_2$ and the dilaton Φ. They differ in the Ramond-Ramond (RR) sector where type IIA has a one-form C_1 with a two-form field strength $F_2 = dC_1$ and a three-form C_3 with a four-form field strength $F_4 = dC_3$. Type IIB features instead a second scalar l, a second two-form C_2 with a three-form field strength $F_3 = dC_2$ and a four-form C_4 with a five-form field strength $F_5^* = dC_4$ which is self-dual. The fermions arise in the Neveu-Schwarz-Ramond sector (NSR) where one finds two gravitinos $\Psi_M^{1,2}$ and two dilatinos $\lambda^{1,2}$. In type IIA they have opposite chirality while in IIB they have the same chirality. Both theories have 32 real supercharges corresponding to $N = 2$ in $d = 10$ or $N = 8$ in $d = 4$. We summarize the $d = 10$ massless spectrum in Table 1.

Table 1 Massless type II spectrum in $d = 10$.

	IIA	IIB
NS:	G_{MN}, $H_3 = dB_2$, Φ	
RR:	$F_2 = dC_1$, $F_4 = dC_3$	l, $F_3 = dC_2$, $F_5^* = dC_4$
NSR:	$\Psi_M^{1,2}$, $\lambda^{1,2}$	$\Psi_M^{1,2}$, $\lambda^{1,2}$

The next step is to consider type II strings moving in a background (1) with Y_6 being a Calabi-Yau threefold. Such background break 3/4 of the supercharges and thus lead to $N = 2$ supersymmetry in $d = 4$. Let us review a few facts about this class of compactifications.

2.1 Calabi-Yau threefolds

When one considers strings propagating in the space-time background (1) the ten-dimensional Lorentz group $Spin(1, 9)$ decomposes into

$$Spin(1, 9) \to Spin(1, 3) \times Spin(6) . \tag{2}$$

There is an associated decomposition of the spinor representation $\mathbf{16} \in Spin(1, 9)$ according to $\mathbf{16} \to (\mathbf{2}, \mathbf{4}) \oplus (\bar{\mathbf{2}}, \bar{\mathbf{4}})$. In order to achieve the minimal amount of supersymmetry one chooses Y_6 to have a reduced structure group $SU(3) \subset Spin(6)$. This implies a further decomposition of the $\mathbf{4} \in Spin(6)$ under the $SU(3)$ as $\mathbf{4} \to \mathbf{3} \oplus \mathbf{1}$. Manifolds with a reduced structure group $SU(3)$ admit an invariant spinor η (the singlet $\mathbf{1}$) which is nowhere vanishing and globally well defined. Such manifolds are termed 'manifolds with $SU(3)$ structure' in the mathematical literature [8–11]. In Sect. 4 we will learn more about these manifolds but for now we impose the additional constraint that this spinor η is also covariantly constant with respect to the Levi-Civita connection. Geometrically this says that the holonomy of Y_6 is $SU(3)$.

From η one can build two globally defined tensors: a two-form J and a complex three-form Ω

$$\eta_\pm^\dagger \gamma^{mn}\eta_\pm = \pm\frac{i}{2}\, J^{mn}\ , \qquad \eta_-^\dagger \gamma^{mnp}\eta_+ = \frac{i}{2}\,\Omega^{mnp}\ , \qquad \eta_+^\dagger \gamma^{mnp}\eta_- = \frac{i}{2}\,\bar\Omega^{mnp}\ , \tag{3}$$

where $\eta_\pm$ denotes the two chiralities of the spinor. They are normalized as ($\eta_\pm^\dagger\eta_\pm = \frac{1}{2}$) and $\gamma^{m_1\cdots m_p} = \gamma^{[m_1}\gamma^{m_2}\ldots\gamma^{m_p]}$ are anti-symmetrized products of six-dimensional γ-matrices. Using appropriate Fierz identities one shows that with this normalization for the spinors J and Ω are not independent but satisfy

$$J\wedge J\wedge J = \frac{3i}{4}\,\Omega\wedge\bar\Omega\ , \qquad J\wedge\Omega = 0\ . \tag{4}$$

Furthermore, lowering one index of the two-form J with the metric results in a complex structure $\mathcal{J}$ since it satisfies $\mathcal{J}^2 = -\mathbf{1}$ and the associated Nijenhuis-tensor vanishes. For a fixed metric and fixed complex structure J is a closed $(1,1)$-form while Ω is a closed $(3,0)$-form. Thus Y_6 is a Kähler manifold with holonomy $SU(3)$ or in other words it is a Calabi-Yau manifold [1] which we denote as Y.

It turns out that the massless modes of the compactified $d=4$ theory are in one-to-one correspondence with the harmonic forms on Y which in turn are in one-to-one correspondence with elements of the Dolbeault cohomology groups $H^{(p,q)}(Y)$ [1]. Here (p,q) denotes the number of holomorphic and anti-holomorphic differentials of the harmonic forms. The dimensions of $H^{(p,q)}(Y)$ are called Hodge numbers and are denoted as $h^{p,q} = \mathrm{dim}H^{p,q}(Y)$. They are conventionally arranged in a Hodge diamond which on a Calabi-Yau threefold simplifies as follows

$$\begin{array}{ccccccc} & & & h^{(0,0)} & & & \\ & & h^{(1,0)} & & h^{(0,1)} & & \\ & h^{(2,0)} & & h^{(1,1)} & & h^{(0,2)} & \\ h^{(3,0)} & & h^{(2,1)} & & h^{(1,2)} & & h^{(0,3)} \\ & h^{(3,1)} & & h^{(2,2)} & & h^{(1,3)} & \\ & & h^{(3,2)} & & h^{(2,3)} & & \\ & & & h^{(3,3)} & & & \end{array} \; = \; \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{(1,1)} & & 0 & \\ 1 & & h^{(1,2)} & & h^{(1,2)} & & 1 \\ & 0 & & h^{(1,1)} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}\ . \tag{5}$$

Or in other words the $h^{(p,q)}$ satisfy

$$\begin{aligned} &h^{(1,0)} = h^{(0,1)} = h^{(2,0)} = h^{(0,2)} = h^{(3,1)} = h^{(1,3)} = h^{(3,2)} = h^{(2,3)} = 0\ , \\ &h^{(0,0)} = h^{(3,0)} = h^{(0,3)} = h^{(3,3)} = 1\ , \quad h^{(2,1)} = h^{(1,2)}\ , \quad h^{(1,1)} = h^{(2,2)}\ . \end{aligned} \tag{6}$$

We see that $h^{(1,1)}$ and $h^{(1,2)}$ are the only non-trivial, i.e. arbitrary Hodge numbers on a Calabi-Yau threefold.

2.2 The moduli space of Calabi-Yau threefolds

Deformations of the Calabi-Yau metric which preserve the Calabi-Yau condition mathematically define moduli parameters of the metric. After a Kaluza-Klein reduction they correspond to gauge neutral scalar fields in the low energy effective action which are flat direction of the effective potential. It turns out that for Calabi-Yau threefolds these moduli are in one-to-one correspondence with the harmonic $(1,1)$- and $(1,2)$-forms [1]. The $(1,1)$-forms correspond to deformations of the complexified Kähler-form $J_c = B + iJ$ where B is the NS two-form of type II string theories. Choosing a basis of harmonic $(1,1)$-forms $\omega_a \in H^{(1,1)}(Y)$ J_c can be expanded as

$$J_c = J + iB = t^a\,\omega_a\ , \qquad a = 1,\ldots,h^{(1,1)}\ , \tag{7}$$

where t^a are the complex moduli parameters.

The harmonic $(1,2)$-forms correspond to deformations of the complex structure or equivalently deformations of the three-form Ω. It is convenient to choose a real symplectic basis of harmonic three-forms $(\alpha_K, \beta^L) \in H^3(Y)$ which is independent of the complex structure and obeys

$$\int_Y \alpha_K\wedge\beta^L = \delta_K^L\ , \quad \int_Y \alpha_K\wedge\alpha_L = 0 = \int_Y \beta^K\wedge\beta^L\ , \quad K,L = 0,\ldots,h^{(1,2)}\ . \tag{8}$$

In terms of this basis Ω can be expanded as

$$\Omega(z) = Z^K(z)\,\alpha_K - \mathcal{F}^\Omega_L(z)\,\beta^L\ , \tag{9}$$

where $Z^K(z)$ and $\mathcal{F}^\Omega_L(z)$ are holomorphic functions of the $h^{(1,2)}$ complex moduli z^k. Furthermore

$$\mathcal{F}^\Omega_L = \frac{\partial \mathcal{F}^{\rm cs}}{\partial Z^L}\ , \tag{10}$$

where the holomorphic prepotential $\mathcal{F}^\Omega$ is a homogeneous function of Z^L of degree two. Therefore one can choose 'special' coordinates where

$$\mathcal{F}^\Omega = (Z^0)^2 f^\Omega(z^k)\ ,\qquad z^k = \frac{Z^k}{Z^0}\ , \tag{11}$$

with $f^\Omega(z^k)$ being an arbitrary holomorphic function of the complex structure deformations z^k. Ω is only defined up to complex rescalings which can be used to choose $Z^0 = 1$.

The t^a and z^k can be viewed as the coordinates of what is called the geometrical moduli space $\mathcal{M}$ of the Calabi-Yau manifolds [12,13]. It is locally is a direct product

$$\mathcal{M} = \mathcal{M}_\Omega^{h^{(1,2)}} \times \mathcal{M}_J^{h^{(1,1)}}\ , \tag{12}$$

where $\mathcal{M}_\Omega^{h^{(1,2)}}$ is the $h^{(1,2)}$-dimensional component spanned by the complex structure deformations z^k while $\mathcal{M}_J^{h^{(1,1)}}$ is the $h^{(1,1)}$-dimensional component spanned by the Kähler deformations t^a. The metric on $\mathcal{M}_\Omega^{h^{(1,2)}}$ is a special Kähler metric with a Kähler potential given by [13]

$$g_{k\bar l} = \partial_{z^k}\partial_{\bar z^{\bar l}}\, K_\Omega\ ,\qquad K_\Omega = -\ln\Big[-i\int_Y \Omega\wedge\bar\Omega\Big] = -\ln i\Big[\bar Z^K \mathcal{F}^\Omega_K - Z^K \bar{\mathcal{F}}^\Omega_K\Big]\ . \tag{13}$$

Manifolds with a Kähler metric whose Kähler potential is in this way entirely determined by a holomorphic prepotential $\mathcal{F}$ are termed special Kähler manifolds [14–17].

$\mathcal{M}_J^{h^{(1,1)}}$ spanned by the coordinates t^a also is a special Kähler manifold with a Kähler potential and prepotential $f^J(t)$ given by [12]

$$K_J = -\ln\int_Y J\wedge J\wedge J\ ,\qquad f^J(t) = \mathcal{K}_{abc}t^a t^b t^c\ , \tag{14}$$

where $\mathcal{K}_{abc} = \int \omega_a\wedge\omega_b\wedge\omega_c$ are topological intersection numbers.

The fact that $f^J(t)$ is a cubic polynomial only holds in the large volume limit. In general world-sheet instanton effects correct the prepotential by terms of the form $\mathcal{O}(e^{-t})$. These terms can be computed using mirror symmetry, a property of Calabi-Yau manifolds we turn to now.

2.3 Mirror symmetry

The status of mirror symmetry is somewhat murky [18]. Originally it was conjectured that for every Calabi-Yau Y there exists a mirror manifold $\tilde Y$ with reversed Hodge numbers, i.e.

$$h^{1,1}(Y) = h^{1,2}(\tilde Y)\ ,\qquad h^{1,2}(Y) = h^{1,1}(\tilde Y)\ . \tag{15}$$

In terms of the Hodge diamond (5) this corresponds to a reflection along the diagonal or in other words the third cohomology $H^{(3)} = H^{(3,0)}\oplus H^{(2,1)}\oplus H^{(1,2)}\oplus H^{(0,3)}$ is interchanged with the even cohomologies $H^{(\rm even)} = H^{(0,0)}\oplus H^{(1,1)}\oplus H^{(1,2)}\oplus H^{(3,3)}$.

Furthermore, the respective (complexified) moduli spaces given in (12) are conjectured to be identical for a mirror pair of Calabi-Yau manifolds

$$\mathcal{M}_\Omega^{h^{(1,2)}}(Y) \equiv \mathcal{M}_J^{h^{(1,1)}}(\tilde{Y}) , \qquad \mathcal{M}_J^{h^{(1,1)}}(Y) \equiv \mathcal{M}_\Omega^{h^{(1,2)}}(\tilde{Y}) , \tag{16}$$

or equivalently

$$f^\Omega(Y) \equiv f^J(\tilde{Y}) , \qquad f^J(Y) \equiv f^\Omega(\tilde{Y}) . \tag{17}$$

Mirror symmetry has been rigorously established on a subspace of Calabi-Yau manifolds [19] but has not been proven in general. In fact so called rigid Calabi-Yau manifolds which have $h^{(1,2)} = 0$ cannot have a Calabi-Yau mirror since the Kähler-form J always exists on Y and thus $h^{(1,1)} > 0$ always holds. The believe is that one has to enlarge the space of manifolds mirror symmetry acts on in order to fully establish the symmetry [20].

Even though mirror symmetry is not yet proven it has been an extremely useful concept in order to compute the holomorphic prepotential of the effective action. For example the instanton corrections to the cubic prepotential f^J of the Kähler moduli (14) have been determined using mirror symmetry [21].

In type II string theory mirror symmetry manifests itself by the equivalence of the two different type II string theories in mirror symmetric background or in other words the following equivalence holds

$$\text{IIA in background } M_4 \times Y \quad \equiv \quad \text{IIB in background } M_4 \times \tilde{Y} . \tag{18}$$

Therefore one can focus the attention on one of the two string theories and infer couplings of the other one by mirror symmetry. However, depending on the precise question it might be easier to perform the computation either in type IIA or in type IIB.

2.4 The low energy effective action

For both type IIA and type IIB the low energy effective action is a $N = 2$ supergravity coupling the gravitational multiplet to vector-, tensor- and hypermultiplets. For massless tensor fields (which is the case in Calabi-Yau compactifications) one can dualize the tensor to a scalar and express the action in terms of vector- and hypermultiplets only. An $N = 2$ vector multiplet contains a one-form V and a complex scalar ϕ as bosonic components. A hypermultiplet instead features four real scalars q^A. The most general action for these multiplets coupled to $N = 2$ supergravity reads [14,22,23]

$$\begin{aligned} S_{IIB}^{(4)} = \int & -\frac{1}{2} R *\mathbf{1} + \frac{1}{4} \mathrm{Re}\mathcal{M}_{KL}(\phi,\bar\phi)\, F^K \wedge F^L + \frac{1}{4} \mathrm{Im}\mathcal{M}_{KL}(\phi,\bar\phi)\, F^K \wedge *F^L \\ & - g_{k\bar l}(\phi,\bar\phi)\, d\phi^k \wedge *d\bar\phi^{\bar l} - h_{AB}(q)\, dq^A \wedge *dq^B , \end{aligned} \tag{19}$$

where $F^K = dV^K, K = 0, \ldots, n_V$ is the field strength of the vectors $V^k, k = 1, \ldots, n_V$ in the vector multiplets together with the graviphoton denoted by V^0. $\mathcal{M}(\phi,\bar\phi)$ are ϕ-dependent gauge couplings functions which can be expressed in terms of the holomorphic prepotential $\mathcal{F}(\phi)$ (see [23] for an explicit formula).

$N = 2$ supersymmetry constrains the metric $g_{k\bar l}$ for the scalars ϕ^k to be a special Kähler metric [14]. The metric h_{AB} of the $4n_H$ scalars in the hypermultiplets is constrained to be quaternionic [22]. Thus the $N = 2$ moduli space has the form

$$\mathcal{M} = \mathcal{M}_Q^{n_H} \times \mathcal{M}_{SK}^{n_V} , \tag{20}$$

where $\mathcal{M}_Q^{n_H}$ is the quaternionic component spanned by the scalars in the hypermultiplet while $\mathcal{M}_{SK}^{n_V}$ is the special Kähler component spanned by the scalars in the vector multiplet.

For Calabi-Yau compactifications of type II theories the geometrical moduli space discussed in Sect. 2.2 is a subspace of the $N = 2$ moduli space. Explicitly one finds for the two cases

$$\text{IIA}: \qquad \mathcal{M}_{SK}^{n_V} = \mathcal{M}_J^{h^{(1,1)}} , \qquad \mathcal{M}_{QK}^{n_H} \supset \mathcal{M}_\Omega^{h^{(1,2)}} ,$$

$$\text{IIB}: \qquad \mathcal{M}_{SK}^{n_V} = \mathcal{M}_{\Omega}^{h^{(1,2)}} \,, \qquad \mathcal{M}_{QK}^{n_H} \supset \mathcal{M}_{J}^{h^{(1,1)}} \,.$$

We see that the geometrical moduli space of Calabi-Yau threefolds has to be the product of two special Kähler manifolds in order to be consistent with $N = 2$ supergravity combined with mirror symmetry. The quaternionic component on the other hand is not the most general manifold allowed by $N = 2$ in that for Calabi-Yau compactification they have a special Kähler submanifold. This class of quaternionic manifolds have been termed 'dual quaternionic manifold' [24]. Exactly as special Kähler manifolds they are entirely characterized by a holomorphic prepotential f. The additional scalars in $\mathcal{M}_{QK}$ arise from a Kaluza-Klein reduction of the gauge potentials in the RR-sector and the explicit form of h_{AB} can be found in [25].

3 Background fluxes

3.1 General discussion

If localized sources such as D-branes are present it is possible to turn on background fluxes on the Calabi-Yau manifold [5,26,27]. Generically background fluxes e_I arise from integrating a p-form field strength F_p over a set of p-cycles γ_p^I in Y

$$\int_{\gamma_p^I \in Y} F_p \;=\; e_I \;\neq\; 0 \,. \tag{21}$$

In order to keep the Bianchi identity and the equation of motion intact one insists that $dF_p = 0 = d^\dagger F_p$ holds. This implies that the background fluxes e_I have to be constants. Equivalently one can expand F_p in terms of harmonic forms ω_p^I with constant coefficients e_I

$$F_p \;=\; e_I \,\omega_p^I \,, \qquad \omega_p \in H^p(Y) \,, \tag{22}$$

such that the ω_p^I are dual to the cycle γ_p^I.

Due to a Dirac quantization condition the e_I are quantized in string theory [26]. However in the low energy/large volume approximation we are considering here they appear as continuous parameters which deform the low energy supergravity. If one keeps the e_I as small perturbations the light spectrum does not change. Instead the low energy supergravity turns into a gauged or massive supergravity where the fluxes e_I appear as additional gauge couplings or as mass parameters. As a consequence a potential is generated which at least partially lifts the vacuum degeneracy of string theory. Furthermore at the minimum of this potential supersymmetry is generically spontaneously broken.

3.2 Fluxes in type IIB

Let us be slightly more specific and consider background fluxes in IIB compactifications. In this case one can turn on three-form flux for $G_3 \equiv F_3 - \tau H_3$ where $\tau \equiv l + i\mathrm{e}^{-\phi}$. Expanded into the symplectic basis one has

$$G_3 \;=\; m^K(\tau)\,\alpha_K + e_K(\tau)\,\beta^K \,, \tag{23}$$

where

$$e_K(\tau) \;=\; e_K^{\mathrm{RR}} - \tau\, e_K^{\mathrm{NS}} \,, \qquad m^K(\tau) \;=\; m^{\mathrm{RR}\,K} - \tau\, m^{\mathrm{NS}\,K} \,. \tag{24}$$

Altogether these are $2(h^{(1,2)}+1)$ RR-flux parameters and $2(h^{(1,2)}+1)$ NS-flux parameters.

The electric fluxes gauge a translational isometry of the quaternionic manifold $\mathcal{M}_Q$ in that the ordinary derivatives are replaced by covariant derivatives [28–30]

$$\partial_\mu q^{1,2} \to D_\mu q^{1,2} = \partial_\mu q^{1,2} + e_K^{\mathrm{NS,RR}} A_\mu^K \,. \tag{25}$$

Here $q^{1,2}$ denote the dual scalars of the two space-time two-forms B_2 and C_2 which are scalar fields in the hypermultiplets. For the magnetic fluxes the situation is slightly more involved in that B_2 and C_2 become massive with m^K being related to the mass parameters [30,31]. In both cases the induced scalar potential reads [32,33]

$$V(z,\tau) = -(\bar{e} - \bar{\mathcal{M}} \cdot \bar{m})_K (\mathrm{Im}\mathcal{M})^{-1KL} (e - \mathcal{M} \cdot m)_L \ , \tag{26}$$

where $\mathcal{M}(z,\bar{z})$ is the matrix of gauge couplings appearing in (19). In terms of Calabi-Yau data the potential $V(z,\tau)$ depends on the quantity [32,34]

$$W = \int_Y \Omega \wedge G_3 \ . \tag{27}$$

3.3 Fluxes in IIA and mirror symmetry

In type IIA compactified on the mirror Calabi-Yau $\tilde{Y}$ one can turn on the RR-fluxes [30,35]

$$F_2 \ = \ -\tilde{m}^{\mathrm{RR}\,a} \omega_a \ , \qquad F_4 \ = \ \tilde{e}^{\mathrm{RR}}_a \, \tilde{\omega}^a \ , \tag{28}$$

and the NS-fluxes

$$H_3 \ = \ \tilde{m}^{\mathrm{NS}\,K} \alpha_K - \tilde{e}^{\mathrm{NS}}_K \, \beta^K \ . \tag{29}$$

Two additional RR-flux arises from the dual of the four-dimensional space-time three-form C_3 and from the mass parameter of the ten-dimensional massive IIA supergravity [36]. Thus altogether we have $2(h^{(1,1)}+1)$ RR-fluxes and $2(h^{(1,2)}+1)$ NS-fluxes in type IIA.

An interesting question is the fate of mirror symmetry in the presence of fluxes. Just by counting the flux-parameters we immediately see that in the RR-sector the numbers nicely match. In this case one also finds perfect agreement of the corresponding effective actions S if one identifies the fluxes. Or in other words one finds [30]

$$S^{\mathrm{IIB}}(Y, e^{\mathrm{RR}}, m^{\mathrm{RR}}) \ \equiv \ S^{\mathrm{IIA}}(\tilde{Y}, \tilde{e}^{\mathrm{RR}}, \tilde{m}^{\mathrm{RR}}) \ , \tag{30}$$

if one identifies $e^{\mathrm{RR}} = \tilde{e}^{\mathrm{RR}}$, $m^{\mathrm{RR}} = \tilde{m}^{\mathrm{RR}}$.

However, for NS-fluxes the situation is considerably more complicated. In this case there is no obvious mirror symmetry since in both theories the three-form H_3 is expanded in terms of the third cohomology $H^{(3)}$ and thus $2(h^{(1,1)}+1)$ flux parameters are missing on both sides. Since we are in the NS-sector these missing fluxes can only come from the internal metric or in other words must arise from geometrical quantities. Technically one needs a NS two-form and a NS four-form which complexify the RR-fluxes (28) and which then could map to the fluxes of the complex type IIB three-form G_3 (23) under mirror symmetry. It has been suggested in [37] to compactify on a 'non-Calabi-Yau' manifold Y_6 where an NS-four-form arises from the non-integrability of the complex structure. This proposal was made more concrete in ref. [6] where Y_6 was identified as a 'half-flat manifold' considered before in the mathematical literature [11,38]. Therefore we discuss such manifold in the next section.

4 Compactifications on manifolds with $SU(3)$-structure

4.1 Mathematical properties

In the study of space-time backgrounds of the form (1) one needs to distinguish two conditions. First of all for phenomenological reasons one is interested to choose Y_6 in such a way that the effective four-dimensional theory has the minimal amount of supersymmetry. Therefore, as reviewed in Sect. 2.1, one needs to demand that Y_6 admits a globally defined spinor η or equivalently one needs to choose Y_6 to be a manifold with $SU(3)$ structure. If one further insists that this supersymmetry is unbroken an additional condition has to be imposed. Since all spinorial quantities vanish in a ground state which preserves four-dimensional Poincaré invariance, one has to examine the supersymmetry transformation of the spinors (which are bosonic quantities) and in particular the supersymmetry transformation of the gravitino Ψ_M. Schematically it reads

$$\delta\Psi_M = \nabla_M \eta + \sum_p (\gamma \cdot F_p)_M \, \eta + \dots \,, \qquad M = 0, \dots, 9 \,, \tag{31}$$

where η is the parameter of the supersymmetry transformations. In (31) we have written the contribution of all p-form field strength appropriately contracted with (anti-symmetrized) products of γ-matrices symbolically as $(\gamma \cdot F_p)_M$. For the argument here the precise form of these terms is irrelevant but they can be found for example in [7, 39]. What we see immediately from (31) is that if all background fluxes vanish unbroken supersymmetry requires the existence of a covariantly constant spinor η or in other words demands that Y_6 is a Calabi-Yau manifold. If on the other hand the background fluxes are non-zero one has two choices. Either one still insists on keeping some fraction of the supercharges unbroken. This requires $\nabla_M \eta \neq 0$ or in other words the geometry back-reacts to the presence of the fluxes. If one does not require the existence of unbroken supercharges the fluxes and/or $\nabla_M \eta$ can be non-zero without an exact cancellation and as a consequence supersymmetry is broken spontaneously. In this section we do not specify which case occurs but consider the generic situation that η exists but is not covariantly constant $\nabla_M \eta \neq 0$. This situation has first been studied in refs. [40,41] while the more recent developments are reviewed in [5] where also a more complete list of references can be found.

In general manifolds which admit a G-invariant tensor or spinor are called 'manifolds with G-structure' in the mathematical literature [8–11].[1] Even though generically $\nabla\eta \neq 0$ for the Levi-Civita connection one can show that there always is a different connection with torsion which satisfies $\nabla^{(T)}\eta = 0$.

Once an invariant spinor η exists one can use it to define a two-form J and a three-form Ω in exact analogy with the construction reviewed in Sect. 2.1 and explicitly given in eqs. (3) and (4). This construction never used the fact that η is covariantly constant and therefore it goes through as long as η is well defined. However, in general the associated tensor $\mathcal{J}$ is merely an almost complex structure in that it continues to satisfy $\mathcal{J}^2 = -\mathbf{1}$ but the associated Nijenhuis-tensor no longer vanishes. Similarly both J and Ω are no longer closed precisely due to the presence of torsion.

One decomposes dJ and $d\Omega$ according to their $SU(3)$ representation and in this way defines five irreducible torsion classes $W_\alpha, \alpha = 1, \dots, 5$ [11]. More precisely one has

$$\begin{aligned} dJ &= \frac{3i}{4}\left(W_1\bar{\Omega} - \bar{W}_1\Omega\right) + W_4 \wedge J + W_3 \,, \\ d\Omega &= W_1 \, J \wedge J + W_2 \wedge J + \bar{W}_5 \wedge \Omega \,, \end{aligned} \tag{32}$$

where W_1 is a zero-form $(\mathbf{1} \oplus \bar{\mathbf{1}})$, W_4, W_5 are one-forms $(\mathbf{3} \oplus \bar{\mathbf{3}})$, W_2 is a two-form $(\mathbf{8} \oplus \bar{\mathbf{8}})$ and finally W_3 is a three-form $(\mathbf{6} \oplus \bar{\mathbf{6}})$ where in brackets we give their $SU(3)$ representation. As a consequence of (4) the W_α further satisfy

$$W_3 \wedge J \;=\; W_3 \wedge \Omega \;=\; W_2 \wedge J \wedge J \;=\; 0 \,. \tag{33}$$

We summarize the torsion classes in Table 2.

Generically manifolds with $SU(3)$ structure are neither complex, nor Kähler, nor Ricci-flat. Only for a particular choice of the torsion where some of the W_α vanish one has manifolds with additional properties. For example Calabi-Yau manifolds are manifolds of $SU(3)$ structure where all five torsion classes vanish $W_\alpha = 0$.

[1] This notion was first introduced in a physics context in [42].

Table 2 $SU(3)$ torsion classes.

component	interpretation	$SU(3)$-representation
W_1	$J \wedge d\Omega$ or $\Omega \wedge dJ$	$\mathbf{1} \oplus \mathbf{1}$
W_2	$(d\Omega)_0^{2,2}$	$\mathbf{8} \oplus \mathbf{8}$
W_3	$(dJ)_0^{2,1} + (dJ)_0^{1,2}$	$\mathbf{6} \oplus \bar{\mathbf{6}}$
W_4	$J \wedge dJ$	$\mathbf{3} \oplus \bar{\mathbf{3}}$
W_5	$d\Omega^{3,1}$	$\mathbf{3} \oplus \bar{\mathbf{3}}$

4.2 Half-flat manifolds as mirrors of electric NS 3-form flux

Let us now return to the question of mirror symmetry for Calabi-Yau compactifications with NS three-form flux. In Sect. 3.3 we argued that the mirror symmetric compactification has to feature a different geometrical manifold and in the previous section we identified manifolds with $SU(3)$ structure as promising candidates.

In [6] we proposed that the mirror of the electric fluxes defined in (23) are found among the manifolds termed 'half-flat manifolds' [11, 38]. These are manifolds within the class of $SU(3)$ structure manifolds discussed in the previous section which in addition satisfy

$$W_4 = W_5 = \mathrm{Im}W_1 = \mathrm{Im}W_2 = 0 \, , \tag{34}$$

or equivalently

$$d(\mathrm{Im}\Omega) \;=\; 0 \;=\; d\,(J \wedge J) \, . \tag{35}$$

In this case the 'missing' NS 4-form is $F_4^{\mathrm{NS}} \sim d\,\mathrm{Re}\Omega$ which when expanded in a basis of $(2,2)$ forms provides for the mirror of the electric fluxes

$$F_4 \sim \; d(\mathrm{Re}\Omega) = e^i_{NS}\,\omega^i_4 \, . \tag{36}$$

This proposal is supported by the following facts. First of all one can take a limit in moduli space – called the SYZ limit – where a Calabi-Yau can be viewed as a T^3 fibration over some base manifold B [43]. In this limit mirror symmetry corresponds to T-duality of the T^3 which can be performed explicitly using the Buscher rules [44]. Starting from type IIB with electric fluxes one indeed derives a geometry satisfying (35) with no H-flux turned on [6].

A second check can be performed by matching the type IIB $N = 1$ BPS-domain-wall solution of [45] with the type IIA solution of the half-flat geometry [38,46]. Finally one can compute the low energy effective action for type IIA compactified on Y_6 [6,7] and compare it to its type IIB mirror action. This computation is the topic of the next section.

Before we continue let us note that the mirror geometry of magnetic fluxes is not yet clear. In ref. [7] we conjectured that is arises from an even more generalized class of manifolds called 'manifolds with $SU(3) \times SU(3)$ structure'. However it is also possible that one needs to further enlarge the concept of compactifications and also allow for the possibility of non-commutative or other non-geometrical structures [47–50].

Finally, we have identified half-flat manifolds a possible mirror compactifications of Calabi-Yau compactifications with background fluxes and the torsion as the geometrical equivalent of the fluxes. Two questions immediately come to mind: What is the role played by compactifications on manifolds with $SU(3)$ structure which are not half-flat and what is the low-energy/supergravity meaning of the torsion. Both questions we address (and answer) in the next section.

4.3 Low energy effective action for compactifications on manifolds with $SU(3)$ structure

In order to compute the low energy effective action for compactifications on manifolds with $SU(3)$ structure one needs to perform a Kaluza-Klein reduction in a space-time background (1) where Y_6 is not a Calabi-Yau manifold but instead a manifold with SU(3) structure. The subtlety is that in order to make sense of this reduction one has to keep the light modes and integrates out the heavy ones. This is straightforward for a Calabi-Yau compactification where only the massless modes corresponding to harmonic forms on Y are kept. However, backgrounds with a generic Y_6 do not necessarily have a flat Minkowskian ground state and the distinction between heavy and light is not straightforward. Guided by mirror symmetry the Kaluza-Klein expansion was performed in [6] in terms of the same basis of forms as in the mirror Calabi-Yau. This ensured the same light spectrum but, as a consequence of (32), this basis also contains non-harmonic forms corresponding to the modes which become massive. The deviation from being harmonic is precisely measured by the torsion which thus plays the 'mirror role' of the fluxes.

In [7] a slightly different approach inspired by [51] was pursued. Without any Kaluza-Klein reduction one first rewrites the ten-dimensional theory in a space-time background of the form (1) with Y_6 being a manifold with $SU(3)$ structure. This breaks the Lorentz group to $Spin(1,3) \times SU(3)$ and one can decompose all ten-dimensional fields given in Table 1 group theoretically. For simplicity we insisted that only the two gravitinos in the gravitational multiplet survive this decomposition. One way to ensure this is to truncate away all $\mathbf{3} + \bar{\mathbf{3}}$ representations of $SU(3)$ or equivalently all one-forms of Y_6. This in turn constrains the class of manifolds under consideration since it implies

$$W_4 = W_5 = 0 \quad \Leftrightarrow \quad d\,(J \wedge J) \;=\; 0\,, \quad d\Omega^{(3,1)} \;=\; 0\,. \tag{37}$$

Note that this class of manifolds is more general than half-flat manifolds discussed in the previous section since there is no condition on $W_{1,2}$ or equivalently $d(\mathrm{Im}\Omega) \neq 0$.

After decomposing the ten-dimensional fields given in Table 1 under the Lorentz group $Spin(1,3) \times SU(3)$ they can be arranged in $N = 2$ multiplets of $Spin(1,3)$. For concreteness we give the results of this decomposition for type IIA in Table 3. The indices $\mu, \nu = 0, \ldots, 3$ indicate the representation of $Spin(1,3)$ while $m, n, p = 1, \ldots, 6$ label the vector representation of $SO(6) \supset SU(3)$. Let us stress that all fields still do depend on all ten space-time coordinates and no Kaluza-Klein reduction has been performed yet.

Table 3 $SU(3)$ decomposition of type IIA.

multiplet	$SU(3)$ rep.	field content
gravity multiplet	$\mathbf{1}$	$(g_{\mu\nu}, C_\mu, \psi_\mu)$
tensor multiplet	$\mathbf{1}$	$(B_{\mu\nu}, \Phi, C_{mnp}, \lambda)$
vector multiplets	$\mathbf{8} + \mathbf{1}$	$(C_{\mu np}, g_{mn}, B_{mn}, \psi_m)$
hypermultiplets	$\mathbf{6}$	$(g_{mn}, C_{mnp}, \psi_m)$

Inserting this decomposition into ten-dimensional action of the NS sector results in

$$\begin{aligned} S_{\mathrm{NS}} &= \int d^{10}x \sqrt{g}\, e^{-2\Phi} \left[R + 4(\partial\Phi)^2 - \frac{1}{12} H^2 \right] \\ &= \int d^{10}x \sqrt{g^{(4)}} \Big[R^{(4)} - 2\partial_\mu \Phi^{(4)} \partial^\mu \Phi^{(4)} - \frac{1}{12} e^{-4\Phi^{(4)}} H_{\mu\nu\rho} H^{\mu\nu\rho} \\ &\qquad - \frac{1}{4} g^{mp} g^{nq} (\partial_\mu g_{mn} \partial^\mu g_{pq} + \partial_\mu B_{mn} \partial^\mu B_{pq}) + \ldots \Big]\,, \end{aligned}$$

where we defined

$$g_{\mu\nu}^{(4)} = e^{-2\Phi_4} g_{\mu\nu} \; , \qquad \Phi^{(4)} = \Phi - \frac{1}{4} \ln \det g_{mn} \; . \tag{38}$$

The terms omitted in (38) correspond to terms without any derivative ∂_μ or in other words in (38) we only kept terms corresponding to kinetic terms from a four-dimensional point of view. The last term in (38) can be interpreted as the metric on the space $\mathcal{M}$ of metric/B-field deformations. It is shown in [7,52] that $\mathcal{M}$ is product of two special geometries

$$\mathcal{M} = \mathcal{M}_J \times \mathcal{M}_\Omega \tag{39}$$

with Kähler potentials

$$e^{-K_J} = J \wedge J \wedge J \; , \qquad e^{-K_\Omega} = \Omega \wedge \bar{\Omega} \; . \tag{40}$$

Let us stress one more time that this is derived without any truncation but instead contains an infinite number of modes – the entire Kaluza-Klein tower. Nevertheless this ten-dimensional theory strongly resembles a four-dimensional $N = 2$ theory in that it has a product of special geometries governing the kinetic terms. Furthermore the form of the Kähler potential coincides with the form of the Kähler potential for Calabi-Yau threefolds given in (13) and (14). The reason being that the computation of the metric on $\mathcal{M}$ exactly parallels the Calabi-Yau computation [12,13] since the derivatives ∂_m along Y_6 can never enter. Therefore the torsion components dJ and $d\Omega$ do not appear.

Similarly one can compute the scalar potential in this approach. It turns out that it is easiest to first compute the Killing prepotential from the supersymmetry transformations of the gravitinos and then use the $N = 2$ formula which expresses the scalar potential in terms of the Killing prepotential [23]. The supersymmetry transformation of the ten-dimensional gravitinos can also be written in a form resembling the $d = 4, N = 2$ form. It reads [23]

$$\delta\psi_{A\,\mu} = D_\mu \varepsilon_A + i\gamma_\mu S_{AB} \varepsilon^B + \dots \; , \qquad A, B = 1, 2 \; , \tag{41}$$

where

$$S_{AB} = \frac{i}{2} e^{\frac{1}{2} K_V} \vec{\sigma}_{AB} \vec{P} \; , \tag{42}$$

The vector $\vec{P}$ is called the Killing prepotential. Starting from the ten-dimensional gravitino transformation and inserting the decomposition of Table 3 one finds for IIA

$$P^1 + iP^2 \;=\; e^{\frac{1}{2} K_\Omega + \Phi^{(4)}} \big[e^{-J_c} \wedge d\Omega \big] \; , \qquad P^3 \;=\; e^{2\Phi^{(4)}} \big[e^{-J_c} \wedge F_A \big] \; , \tag{43}$$

where $F_A \equiv \sum_k F_{2k}$ is the sum of the RR-forms. For type IIB one obtains

$$P^1 + iP^2 = e^{\frac{1}{2} K_J + \Phi^{(4)}} \big[\Omega \wedge de^{-J_c} \big] \; , \qquad P^3 = e^{2\Phi^{(4)}} \big[\Omega \wedge F_B \big] \; , \tag{44}$$

with $F_B \equiv \sum_k F_{2k+1}$. As expected the fluxes and torsion enter in that $\vec{P}$ depends on F, dJ and $d\Omega$. For Calabi-Yau threefolds without fluxes one thus has $\vec{P} = 0$.

We see that (43) and (44) are mirror symmetric under the exchange

$$e^{-J_c} \leftrightarrow \Omega \; , \qquad F_A \leftrightarrow F_B \; , \tag{45}$$

as long as $d(J_c \wedge J_c) = 0$. This latter condition expresses the fact that one obtains only an 'electric mirror symmetry'. In order to see a 'magnetic mirror symmetry' it is necessary to consider a yet more general class of compactifications.

The next step is to perform the Kaluza-Klein reduction or in other words truncate the infinite-dimensional space of modes to a finite subspace. The idea is to truncate in such a way that all the $N = 2$ structures are preserved. Or in other words we insist on a special Kähler geometry on this subspace and demand that K and $\vec{P}$ descend to the subspace. This in turn requires that the space of even/odd forms is non-degenerate. (The precise condition is given in [7].) On this subspace one then has

$$e^{-K_J} = \int_{Y_6} J \wedge J \wedge J \,, \qquad e^{-K_\Omega} = \int_{Y_6} \Omega \wedge \bar{\Omega} \tag{46}$$

and

$$P^1 + iP^2 \;=\; e^{\frac{1}{2}K_\Omega + \Phi^{(4)}} \int_{Y_6} \left[e^{-J_c} \wedge d\Omega \right] , \qquad P^3 \;=\; e^{2\Phi^{(4)}} \int_{Y_6} \left[e^{-J_c} \wedge F_A \right] , \tag{47}$$

for type IIA and

$$P^1 + iP^2 = e^{\frac{1}{2}K_J + \Phi^{(4)}} \int_{Y_6} \left[\Omega \wedge de^{-J_c} \right] , \qquad P^3 = e^{2\Phi^{(4)}} \int_{Y_6} \left[\Omega \wedge F_B \right] , \tag{48}$$

for type IIB.

5 Conclusions

Let us conclude by briefly summarizing our results and by stating some of the open questions. We argued that manifolds with $SU(3)$ structure enlarge the space of supersymmetric compactifications. A particular subclass of such manifolds called half-flat manifolds are mirror symmetric to compactification of Calabi-Yau compactifications with electric NS three-form flux. However mirror symmetry acts more generally on the entire space of manifolds with $SU(3)$ structure with mirror symmetry for Calabi-Yau manifolds being only a subset.

The computation of the effective action for these generalized compactifications is not entirely straightforward. We argued that one way to proceed is a reorganization of the ten-dimensional supergravity which makes the $N = 2$ couplings manifest. In particular on the infinite-dimensional space of metric deformations a special Kähler geometry does appear. Truncating to a (non-degenerate) finite subspace results in a four-dimensional $N = 2$ effective action with a Kähler potential which takes exactly the same form as in Calabi-Yau compactifications. The reason is that it only depends on the existence of a globally defined spinor but not on its derivatives. Instead the derivatives – which are related to the intrinsic torsion – appear in the Killing prepotential or equivalently in the scalar potential.

One of the main open issues is the extension of mirror symmetry to also include magnetic NS-fluxes. This requires a further enlargement of the class on backgrounds. Manifolds with $SU(3) \times SU(3)$ structure and/or non-geometric background have already been advocated [7,47–50].

A second interesting open question is the precise mathematical definition of the deformation or moduli space of manifolds with $SU(3)$ structure. Mirror symmetry suggests that this moduli space is closely related to Calabi-Yau moduli space. Such a relation has already been suggested in [53].

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