Supersymmetry, Geometry and String Theory

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August 25, 2010

Abstract Lectures notes from the Mathematical Physics Master course given in the summer semester 2010 at the University of Hamburg by Dr. Jan Louis. Lecture notes by B.Sc. Ana Ros Camacho, Paniz Imani and Lisa Bauer.

1 Outline of the lectures

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    - σ-models
    - (Gauged) supergravities
    - Supergravity in arbitrary dimensions
2. Kaluza-Klein compactifications
    - Torus/sphere compactifications
    - Calabi-Yau compactifications
    - Generalized geometries and flux compactifications
3. String Theory
    - Introduction to string theory
    - Dualities in string theory
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2 Supersymmetry in D=4

Supersymmetry in $D = 4$ can be regarded as an extension of the Poincaré algebra. Let us remember some facts about this algebra.

The Poincaré algebra has a set of 10 generators in total, 6 $L_{\mu\nu}$ (also known as Lorentz generators) and 4 P_μ which are the components of the three momentum plus the energy operator [1], satisfying the relations:

$$\begin{aligned}
[L_{\mu\nu}, L_{\rho\sigma}] &= \eta_{\nu\rho} L_{\mu\sigma} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\sigma\mu} L_{\rho\nu} + \eta_{\sigma\nu} L_{\rho\mu} \\
[P_\mu, L_{\rho\sigma}] &= \eta_{\mu\rho} P_\sigma - \eta_{\mu\rho} P_\rho \\
[P_\mu, P_\nu] &= 0
\end{aligned} \tag{1}$$

where $\eta_{\mu\nu} = (-,+,+,+)$.

The representation of this algebra is labelled by the mass m of the particle, and by its spin $s = 0, \frac{1}{2}, 1, ...$

Coleman and Mandula, in 1968, proved that there exist no trivial extension of the Poincaré algebra. A possible extension is the supersymmetric algebra, which was presented by Haag, Sohnius and Lopuszanski in 1973. The importance of it roots in the fact that it is the only graded Lie algebra of symmetries of the S-matrix consistent with relativistic quantum field theory. The SuSy algebra is given by the following relations:

$$\begin{aligned}
\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\} &= 2\sigma^\mu_{\alpha\beta} P_\mu \delta^{IJ} \\
\{Q^I_\alpha, Q^J_\beta\} &= \epsilon_{\alpha\beta} Z^{IJ} \\
\{\bar{Q}^I_{\dot{\alpha}}, \bar{Q}^J_{\dot{\beta}}\} &= \epsilon_{\alpha\beta} \bar{Z}^{IJ} \\
\left[P_\mu, Q^J_\alpha\right] &= 0 \\
\left[L_{\mu\nu}, Q^I_\alpha\right] &= \frac{1}{2} \sigma^{\beta}_{\mu\nu\alpha} Q^J_\beta
\end{aligned} \tag{2}$$

(plus the fact that Z^{IJ} commutes with everything) where $\alpha, \dot{\alpha} = 1, 2$; $I, J = 1, ...N$ (this N symbolizes the extension of the SuSy algebra), $\sigma_{\mu\nu} = 2(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu)$, Q is a 2 component complex Weyl spinor ($\bar{Q}$ is its complex conjugated) and

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Now let us comment the representation of the supersymmetric algebra for $N = 1$:

1. Massive representation: in this case, let us take the rest frame, where $\vec{P} = 0$. Choose the eigenvalue of P_μ by $P_\mu = (-m, 0, 0, 0)$, where m is a positive quantity. The little group of this representation is $SO(3)$. In this representation, the SuSy algebra simplifies a lot, and (susyalgebra) becomes:

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2m\delta_{\alpha\dot{\beta}} \tag{3}$$

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0 \tag{4}$$

[1] $\mu, \nu = 0, ..., 3$

Defining two operators,

$$a_\alpha := \frac{1}{\sqrt{2m}} Q_\alpha \tag{5}$$

$$(a_\alpha)^\dagger := \frac{1}{\sqrt{2m}} \bar{Q}_{\dot{\alpha}} \tag{6}$$

(3) and (4) take the shape:

$$\{a_\alpha, (a_{\dot{\beta}})^\dagger\} = \delta_{\alpha\dot{\beta}} \tag{7}$$

$$\{a_\alpha, a_\beta\} = 0 = \{a^\dagger_{\dot{\alpha}}, a^\dagger_{\dot{\beta}}\} \tag{8}$$

We can easily recognize that this is the algebra of two fermionic harmonic oscillators.

The representations of this algebra are well-known. They are constructed from a Clifford vacuum $|s\rangle$. The Clifford vacuum is defined through the condition

$$a_\alpha |s\rangle = 0 \tag{9}$$

For the present case $N = 1$, the fundamental representation consists of the states $|s\rangle$, $(a_\alpha)^\dagger|s\rangle$ and $(a_1)^\dagger(a_2)^\dagger|s\rangle$. For every of these states, they present a spin and a degeneracy showed in the following table:

state	spin	degeneracy
$\lvert s\rangle$	s	2s+1
$(a_\alpha)^\dagger\lvert s\rangle$	$s \pm \frac{1}{2}$	2(2s+1)
$(a_\alpha)^\dagger(a_\beta)^\dagger\lvert s\rangle$	s	2s+1

Or also,

Spin/State	$\lvert 0\rangle$	$\lvert \frac{1}{2}\rangle$	$\lvert 1\rangle$	$\lvert \frac{3}{2}\rangle$	Degeneracy
0 (scalar field)	1+1	1			1
$\frac{1}{2}$ (Weyl fermion)	1	1+1	1		2
1 (gauge bosons)		1	1+1	1	3
$\frac{3}{2}$ (gravitino)			1	1+1	4
2 (graviton)				1	5

The first column is also known as the chiral multiplet, the second as the massive vector multiplet, the third as the massive gravitino multiplet and the fourth the massive graviton multiplet. Taking into account the degeneracy, the total number of states in each of them of 4, 8, 12 and 20, respectively. Concerning this number of states, we can enunciate some small theorem, which says that the number of fermionic and bosonic states must be the same,

$$n_F = n_B \tag{10}$$

The proof of it is based on the computation of $\mathrm{tr}((-1)^F)$ (see [1]).

2. Massless representation: we can make a similar analysis in this case, but this time taking $P_\mu = (-E, 0, 0, E)$ for instance. This change affects the little group, which this time is only $SO(2)$, and also,

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2E(-\sigma^0 + \sigma^1)_{\alpha\dot\beta} = 2E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot\beta} \tag{11}$$

Defining again the operators,

$$a := \frac{1}{\sqrt{2E}} Q_i \tag{12}$$

$$a^\dagger := \frac{1}{\sqrt{2E}} \bar{Q}_i \tag{13}$$

which satisfy the relations

$$\{a, a^\dagger\} = 1 \tag{14}$$

$$\{a, a\} = \{a^\dagger, a^\dagger\} = 0 \tag{15}$$

we notice that this time, we don't have two fermionic oscillators but only one. The representations are now labelled by the helicity λ, the eigenvalue of $(\vec{P}, \vec{s})$. The Clifford vacuum is then $|\lambda\rangle$ with the similar to (9) defining relation:

$$a|\lambda\rangle = 0 \tag{16}$$

The total number of states for the fundamental representation in this case are $|\lambda\rangle$ and $a^\dagger|\lambda\rangle$ (this last one, with $\lambda + \frac{1}{2}$).

Helicity/State	$\|-2\rangle$	$\|-\frac{3}{2}\rangle$	$\|-1\rangle$	$\|-\frac{1}{2}\rangle$	$\|0\rangle$	$\|\frac{1}{2}\rangle$	$\|1\rangle$	$\|\frac{3}{2}\rangle$
2								1
$\frac{3}{2}$							1	1
1						1	1	
$\frac{1}{2}$					1	1		
0				1	1			
$-\frac{1}{2}$			1	1				
-1		1	1					
$-\frac{3}{2}$	1	1						
-2	1							

At this point, we shall invoke some result from QFT: massless particles have helicities with values $\pm\lambda$ (due to CPT invariance).Thus, we need to combine massless representations. In this case, the multiplets are:

chiral multiplet	$1[\pm\frac{1}{2}] + 2[0]$
vector multiplet	$[\pm 1] + [\pm\frac{1}{2}]$
gravitino multiplet	$[\pm\frac{3}{2}] + [\pm 1]$
graviton multiplet	$[\pm 2] + [\pm\frac{3}{2}]$

2.1 N-extended sypersymmetries

Let us come back to the massive representation, with $P_\mu = (-m, 0, 0, 0)$, for a more general case. Now the algebra is:

$$\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\} = 2m\delta_{\alpha\dot\beta}\delta^{IJ} \tag{17}$$

$$\{Q^I_\alpha, Q^J_\beta\} = 2\epsilon_{\alpha\beta}Z^{IJ} \tag{18}$$

Notice here the presence of central charges in (18). For simplicity, let us take $N = 2$. Then (17) remains the same and (18) becomes:

$$\{Q^I_\alpha, Q^J_\beta\} = 2\epsilon_{\alpha\beta}\epsilon^{IJ}Z \tag{19}$$

Then, the Weyl operators may be expressed as linear combinations of:

$$a_\alpha = \frac{1}{\sqrt{2}}(Q^1_\alpha + \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger) \tag{20}$$

$$b_\alpha = \frac{1}{\sqrt{2}}(Q^1_\alpha - \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger) \tag{21}$$

These operators satisfy the following relations:

$$\{a_\alpha, a^\dagger_\beta\} = 2\delta_{\alpha\beta}(m + Z) \tag{22}$$

$$\{b_\alpha, b^\dagger_\beta\} = 2\delta_{\alpha\beta}(m - Z) \tag{23}$$

All the other possible commutators between them cancel. From these relations, we see that $M \geq Z$; this is called the BPS bound. In fact, we can classify some of these cases:

- $m > Z$: this corresponds to the massive representation
- $m = Z$: this is the BPS representation, used among others in String Theory
- $m = Z = 0$: this is the massless representation

3 σ-models for $N = 1$

In order to formulate a supersymmetric field theory, we first need to represent the supersymmetry algebra (2) in terms of (classical) fields. We will start with the simplest case, $N = 1$. The chiral multiplet can be expressed in terms of fields as: $(z(x), \chi_\alpha(x))$, where $z(x)$ is a complex scalar field (corresponding to spin $s = 0$) and $\chi_\alpha(x)$ is a Weyl fermion ($s = -\frac{1}{2}$) [2].

Define the supersymmetric variation of the fields with respect to a parameter ϵ as:

$$\delta_\epsilon \equiv \epsilon^\alpha Q_\alpha + \bar\epsilon_{\dot\alpha}\bar{Q}^{\dot\alpha} \tag{24}$$

[2] χ represents 4 degrees of freedom off-shell and 2 on-shell, and Z two in both cases.

where $\alpha, \dot{\alpha} = 1, 2$ and $\bar{Q}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{Q}_{\dot{\beta}}$. The ϵ are the parameters of the SuSy transformation. They are Grassmannian variables satisfying: $\epsilon_\alpha \epsilon_\beta = -\epsilon_\beta \epsilon_\alpha$. Now, apply this variation to every field of the multiplet. One finds [1]:

$$\delta_\epsilon z = \sqrt{2}\epsilon^\alpha \chi_\alpha \tag{25}$$

$$\delta_\epsilon \chi_\alpha = i\sqrt{2}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}\partial_\mu z + \sqrt{2}\epsilon_\alpha F \tag{26}$$

$$\delta_\epsilon F = i\sqrt{2}\bar{\epsilon}\partial\!\!\!/\chi \tag{27}$$

where F is a complex auxiliary field.

For this kind of multiplet, the minimal lagrangian we can write is:

$$L = \partial_\mu z \partial^\mu \bar{z} - i\bar{\chi}\partial\!\!\!/\chi - F\bar{F} \tag{28}$$

The Euler-Lagrange equations for this lagrangian are:

$$\partial_\mu \partial^\mu z = 0 \tag{29}$$

$$\partial\!\!\!/\chi = 0 \tag{30}$$

$$F = 0 \tag{31}$$

We see that F has an algebraic field equation (i.e. no derivative is involved). The variation of this lagrangian gives:

$$\delta_\epsilon L = \partial_\mu j^\mu \tag{32}$$

which tells us that the action

$$S = \int L d^4x \tag{33}$$

is invariant. Any lagrangian satisfying (32) is said to be a *supersymmetric lagrangian.*

If we want to generalize for the case of n_c multiplets, (28) becomes:

$$L = -\delta_{i\bar{j}}\delta_\mu z^i \partial^\mu \bar{z}^{\bar{j}} - i\delta_{i\bar{j}}\bar{\chi}^{\bar{j}}\partial\!\!\!/\chi^i - \delta_{i\bar{j}}F^i\bar{F}^{\bar{j}}, \quad i = 1, \ldots, n_c \tag{34}$$

Till now we have only considered free fields. For introducing interactions, we must add to (34) some extra terms:

$$L_{int} = F^i \frac{\partial W}{\partial z^i} + \frac{\partial^2 W}{\partial z^i \partial z^j}\chi^i \chi^j + c.c. \tag{35}$$

where $W = W(z)$ is an arbitrary holomorphic function, called *superpotential.* A physically relevant expression for W is:

$$W(z) = \frac{1}{2}m_{ij}z^i z^j + \frac{1}{3}Y_{ijk}z^i z^j z^k + \ldots \tag{36}$$

where m_{ij} is the mass matrix and Y_{ijk} is the Yukawa coupling; $m, Y \in \mathbb{C}$. In renormalizable quantum field theories, W can be at most cubic. In string theory, we can choose W to be arbitrary as renormalizability does not have to be imposed. Also notice that if we derive again the Euler-Lagrange equations:

$$\bar{F}^{\bar{j}}\delta_{\bar{j}i} = \frac{\partial W}{\partial z^i} \tag{37}$$

we notice that F is no longer zero, but a function of z. Plugging this back in (34+35), we get a potential term,

$$V = \delta^{i\bar{j}}\frac{\partial W}{\partial z^i}\frac{\partial \bar{W}}{\partial \bar{z}^j} \tag{38}$$

Let us make a couple of remarks:

- The terms in the potential are related to fermionic couplings.
- (62) is not the most general potential we can have, but it is determined in terms of a superpotential.
- $V \geq 0$

If we allow non-renormalizable interactions we get:

$$L_{nr} = -g_{i\bar{j}}(z)\partial_\mu z^i \partial^\mu \bar{z}^j - i g_{i\bar{j}}(z)\bar{\chi}^{\bar{j}} \not{D}\chi^i - \frac{1}{2}(D_i\partial_j W)\chi^i\chi^j + \frac{1}{4}R_{i\bar{j}k\bar{l}}\chi^i\chi^k\bar{\chi}^{\bar{j}}\bar{\chi}^{\bar{l}} - V \tag{39}$$

which is called the *σ-model.* $g_{i\bar{j}}$ has to be positive, for the positivity of the kinetic energy, and it can be interpreted as a metric on some target space M, with coordinates z^i. In order for this lagrangian to be supersymmetric, $g_{i\bar{j}}$ has to be Kähler, i.e.

$$g_{i\bar{j}} = \partial_i \partial_j K \tag{40}$$

where K is a Kähler potential. The proof for this statement was given between Zumino in 1979 and by Álvarez-Gaumé and Freedmann in 1981 ([3], [4]). The other couplings in (39) are:

$$V = g^{i\bar{j}}\partial_i W \partial_{\bar{j}} W \tag{41}$$

$$D_i\partial_j W = \partial_i\partial_j W - \Gamma^k_{ij}\partial_k W \tag{42}$$

$$D_\mu \chi^i = \partial_\mu \chi^i + \Gamma^i_{jk}\partial_\mu z^j z^k \tag{43}$$

Chiral multiplets carry charge under some gauge (finite dimensional Lie) group G. This means that:

$$\delta_\Lambda z^i = \Lambda^a k^{ai}(z), \quad a = 1, \ldots, n_v \tag{44}$$

where n_v is the dimension of the Lie algebra asocciated to the Lie group G, Λ is the parameter of the gauge transformation and k^{ai} is a holomorphic Killing vector. Demanding $\delta_\Lambda g_{i\bar{j}} = 0$, implies

$$\nabla_i k^a_j + \nabla_j k^a_i = \nabla_i k^a_{\bar{j}} + \nabla_{\bar{j}} k^a_i = 0 \tag{45}$$

which are the Killing equations, where for every Killing vector $k_j^a = g_{j\bar{k}} k^{\bar{k}a}$. The solution of (45) is $k_i^a = i\partial_i P^a$. P^a is a real parameter, called the Killing prepotential, D-term (for N=1) or also the moment map. Defining:

$$k^a(z) := k^{ai} \frac{\partial}{\partial z^i} \tag{46}$$

$$\bar{k}^a(\bar{z}) := \bar{k}^{a\bar{j}} \frac{\partial}{\partial \bar{z}^{\bar{j}}} \tag{47}$$

One has:

$$\left[k^a, k^b\right] = -f^{abc} k^c \tag{48}$$

$$\left[\bar{k}^a, \bar{k}^b\right] = -f^{abc} k^c \tag{49}$$

$$\left[k^a, \bar{k}^b\right] = 0 \tag{50}$$

where f^{abc} are the structure constants of the Lie group G.

Thus, we can proceed and give the coupling to vector multiplets $\left(V_\mu^a, \lambda_\alpha^a\right)$ (λ is called the "gaugino"):

$$\begin{aligned} L = & -g_{i\bar{j}}(D_\mu z^i D^\mu \bar{z}^j + i\chi^i \not{D}\bar{\chi}^j) \\ & - \frac{1}{4} Re(f_{ab}(z) F_{\mu\nu}^a F^{\mu\nu b} - \frac{i}{4} Im(f_{ab}) \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a \\ & - Re(f_{ab}) \lambda^a \not{D} \lambda^b \\ & + \sqrt{2} g_{i\bar{j}} k^{ia} \bar{\chi}^j \bar{\lambda}^a + h.c. \\ & - \frac{1}{2}(D_i D_j W) \chi^i \chi^j + h.c \\ & + \frac{1}{4} R_{i\bar{j}k\bar{l}} \chi^i \chi^k \bar{\chi}^{\bar{j}} \bar{\chi}^{\bar{l}} - V \end{aligned} \tag{51}$$

where

$$V = g^{i\bar{j}} \frac{\partial W}{\partial z^i} \frac{\partial W}{\partial z^j} + \frac{1}{2}(Re(f_{ab}))^{-1} P^a P^b \tag{52}$$

with

$$D_\mu z^i = \partial_\mu z^i - V_\mu^a k^{ia} \tag{53}$$

$$D_\mu \chi^i = \partial_\mu z^i + \Gamma_{jk}^i D_\mu z^j \chi^k - V_\mu^a \frac{\partial k^{ia}}{\partial z^j} \chi^j \tag{54}$$

$$D_\mu \lambda^a = \partial_\mu \lambda^a - f^{abc} V_\mu^b \lambda^c \tag{55}$$

$$F_{\mu\nu}^a = \partial_\mu V_\nu^a - \partial_\nu V_\mu^a - f^{abc} V_\mu^b V_\nu^c \tag{56}$$

The gauge transformations for z^i and V^a are finally given by:

$$\delta_\epsilon z^i = \lambda^a(x) K^{ai}(z) \tag{57}$$

$$\delta_\epsilon V^a = \partial_\mu \lambda^a + f^{abc} \lambda^b V_\mu^c \tag{58}$$

4 $N = 1$ gauged supergravity

Our aim now will be to couple $N = 1$ supersymmetric field theory to gravity. For that,

- we need to add the massless gravity multiplet $(g_{\mu\nu}, \Psi_{\mu\alpha})$, where $g_{\mu\nu}$ is the metric (graviton, with helicity $\lambda = \pm 2$) and $\Psi_{\mu\alpha}$ is the gravitino field ($\lambda = \pm\frac{3}{2}$).
- we need to make supersymmetric transformation local through promotion of the Grassmann variables (this is analogous to local Yang-Mills gauge transformations (57), (58) where however the gauge parameter Λ is a Lorentz scalar): $\epsilon_\alpha \to \epsilon_\alpha(x)$

The gravitino field $\Psi_{\mu\alpha}$ is the gauge field of local supersymmetry, obeying the transformation law:

$$\delta\Psi_{\mu\alpha} = D_\mu \epsilon_\alpha + \dots \tag{59}$$

Considering only the bosonic terms, the lagrangian is in this case [3]:

$$L = -\frac{\sqrt{-g}}{2\kappa^2} R + \sqrt{-g}\bar{\Psi}_\mu \sigma_\nu D_\rho \Psi_\sigma \epsilon^{\mu\nu\rho\sigma} + L_{mat}(V_\mu^a, \lambda^a, z^i, x^i, g_{\mu\nu}, \Psi_\mu) \tag{60}$$

where $\kappa^2 = \frac{8\pi}{M_{Pl}^2}$, R is the Hilbert-Einstein action, and

$$\begin{aligned} L_{mat} = & -G_{i\bar{j}}(z,\bar{z})\, g^{\mu\nu} D_\mu z^i D_\nu \bar{z}^j - V(z,\bar{z}) - \frac{1}{4} Re\left(f_{ab}(z)\right) F^a_{\mu\nu} F^{\mu\nu b} \\ & - \frac{i}{4} Im\left(f_{ab}\right) \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^b_{\rho\sigma} + \text{fermionic terms} \end{aligned} \tag{61}$$

with $i = 1, \dots, n_c$, $a = 1, \dots, n_v$. $G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$ where K is the Kähler potential, and the covariant derivatives are given by (53).

The potential is given by:

$$V(K,W,P) = e^{\kappa^2 K}\left[(D_i W)\left(G^{i\bar{j}}\right)^{-1}(D_{\bar{j}}\bar{W}) - 3\kappa^2 |W|^2 \right] + \frac{1}{2} Re\left(f^{-1}_{ab} P^a P^b \right) \tag{62}$$

where $W(z)$ is the holomorphic superpotential and $f_{ab}(z)$ the holomorphic gauge kinetic matrix, with the Kähler covariant derivative:

$$D_i W = \frac{\partial W}{\partial z^i} + \kappa^2 \left(\frac{\partial K}{\partial z^i} \right) W \tag{63}$$

As we can see, the matter lagrangian is determined by the 4 coupling functions (K, W, f, P).

[3] This lagrangian was developed simulanously by Deser, Zumino, Freedman, Ferrara and van Nieuwenhuizen, see [5], [6], [7]

Let us check the limit $\kappa^2 \to 0$, i.e. the limit of decoupling gravity: in this case, the space-time metric will tend to the Minkowski one, i.e. $g_{ab} \to \eta_{ab}$, and the potential becomes

$$V \to |\frac{\partial W}{\partial z^i}|^2 + \frac{1}{2}P^2 \tag{64}$$

consistently with global supersymmetry.

The Kähler geometry is projetive (Kähler-Hodge). This can be seen from two points of view, either from superconformal constraints [8], or from fermionic interacting terms, which are

- $(D_i\partial_j W)\,\chi^i\chi^j + \text{h.c.}$ in the global case
- $e^{\frac{K}{2}}\,(D_iD_jW)\,\chi^i\chi^j + \text{h.c.}$ in the local case

The Kähler invariance is given by:

$$K \to K + F\,(z) + F\,(\bar{z}) \tag{65}$$

$$W \to We^{-F} \tag{66}$$

$$D_iW \to e^{-F}\,(D_iW) \tag{67}$$

So imposing this invariance, we notice that

$$e^{\frac{K}{2}}\,(D_iD_jW) \to e^{\frac{1}{2}(\bar{F}-F)}e^{\frac{K}{2}}\,(D_iD_jW) \tag{68}$$

$$\chi \to e^{-\frac{1}{4}(\bar{F}-F)}\chi \tag{69}$$

Then, the kinetic term of the fermion is $\bar{\chi}\bar{\sigma}D\chi$, with $D_\mu\chi = \partial_\mu\chi + \frac{1}{4}A_\mu\chi + ...$, where

$$A_\mu = \frac{\partial K}{\partial z^i}\partial_\mu z^i - \frac{\partial K}{\partial \bar{z}^i}\partial_\mu \bar{z}^i \tag{70}$$

Let us finally focus on spontaneous SuSy breaking. This means that the SuSy lagrangian is invariant under local SuSy transformations, but the background (e.g. the minimum of the potential) is not invariant. Under SuSy transformations one has generically

$$\delta_\epsilon \text{fermion} = \text{boson} \tag{71}$$

$$\delta_\epsilon \text{boson} = \text{fermion} \tag{72}$$

Analyzed in a Lorentz-invariant background, (71) and (72) become:

$$\langle\delta_\epsilon \text{fermion}\rangle = \langle\text{spin-0}\rangle \tag{73}$$

$$\langle\delta_\epsilon \text{boson}\rangle = 0 \tag{74}$$

In $N=1$, the spin-0 part of the SuSy transformations of the fermions are:

$$\delta_\epsilon \chi^i \sim \langle F^i \rangle \epsilon + \dots \tag{75}$$

$$\delta_\epsilon \lambda^a \sim \langle P^a \rangle \epsilon + \dots \tag{76}$$

$$\delta_\epsilon \psi_\mu \sim D_\mu \epsilon + i \langle e^{\frac{\kappa}{2}} W \rangle \sigma_\mu \epsilon + \dots \tag{77}$$

If $\langle F^i \rangle \neq 0$ or $\langle P^a \rangle \neq 0$, SuSy is broken spontaneously. $\langle F^i \rangle$ and $\langle P^a \rangle$ are called the order parameters. The potential (62) then can be rewritten as:

$$V = F^i \bar{F}^{\bar{j}} G_{i\bar{j}} - 3e^K |W|^2 + \frac{1}{2} \left(Re\left(f \right) \right)^{-1}_{ab} P^a P^b \tag{78}$$

$\langle F^i \rangle \neq 0$ is surprisingly difficult to arrange (see [9]). Some short remark: if $\langle F^i \rangle = \langle P^a \rangle = 0$, then the potential evaluated at the minimum is zero, or negative: $\langle V \rangle = -3 \langle e^K |W|^2 \rangle \leq 0$. $\langle V \rangle$ plays the role of a cosmological constant, so that

1. $\langle V \rangle > 0$: de Sitter background
2. $\langle V \rangle = 0$: Minkowski background
3. $\langle V \rangle < 0$: Anti de Sitter background

Recent cosmological observations suggest that our universe is a de Sitter universe.

Let us consider the stability of background with spontaneous supersymmetric breaking (see [10], [11]). Consider the Hessian matrix of the potential:

$$\begin{pmatrix} \partial_i \partial_i V & \partial_i \partial_j V \\ \partial_j \partial_i V & \partial_j \partial_j V \end{pmatrix} \tag{79}$$

which for Minkowski and de Sitter universes is bigger than zero, and obeys BF-bound for Anti de Sitter universes. A necessary condition required for stability is that

$$R\left(F\right) \geq -\frac{2}{3} \frac{1}{1+\gamma}, \quad \gamma > 0 \tag{80}$$

or

$$R\left(F\right) \geq -\frac{2}{3} \frac{1-\frac{9}{8}}{1+\gamma}, \quad -1 \leq \gamma < 0 \tag{81}$$

where

$$R\left(F, \bar{F}\right) = -R_{i\bar{j}k\bar{l}} \frac{F^i \bar{F}^{\bar{j}} F^k \bar{F}^{\bar{k}}}{\left(F\bar{F}\right)^2} \tag{82}$$

is the holomorphic sectional curvature, with $\gamma = \frac{\langle V \rangle}{3 \langle W^2 e^K \rangle}$.

5 $N=2$ supersymmetry

For the case of $N=2$, the supersymmetry algebra we obtain is analogous to (2), but with $I,J=1,\ldots,N=2$. Let us mention one of the equations, which becomes:

$$\{Q_\alpha^I, Q_\beta^J\} = \epsilon_{\alpha\beta}\epsilon^{IJ} Z$$

where Z is the central charge (it commutes with everything). The algebra has an automorphism group, called the R-symmetry (symmetry of rotation of the supercharges) $U(N)$: $Q^I \to Q^{I'} = U_J^I Q^J$, where U_J^I is a unitary matrix. This group can be decomposed as: $U(N) = U(1) \times SU(N)$.

The construction of the representation proceeds as in section 2, and one distinguishes the massive representation ($M > Z$), the BPS representation $M = Z$ and the massless representations ($M=0$). For the massive representation, there exists 4 fermionic creation operators (in general, $2N$); for the BPS or the massless representations, there exists 2 fermionic creation operators (in general, N).

Let us focus on the massless multiplets of $N=2$. If they are CPT complete, they consists of one state $|\lambda\rangle$, 2 states $a_\alpha^{\dagger I}|\lambda\rangle$ and one $a_{[\alpha}^{\dagger[I} a_{\beta]}^{\dagger J]}|\lambda\rangle$. If not CPT complete, there are 8 states in total. Let us display various multiplets in $N=2$ in the following table:

Helicity/State	$\lvert -\frac{1}{2}\rangle$	$\lvert -1\rangle$ $\lvert 0\rangle$	$\lvert -2\rangle$ $\lvert 1\rangle$
0	2	1 1	
$\pm\frac{1}{2}$	1 1	2 2	
± 1		1 1	1 1
$\pm\frac{3}{2}$			2 2
± 2			1 1
	Half-hypermultiplet	Vector multiplet	Gravity multiplet

The half-hypermultiplet is CPT complete, but the others are not, so we have to combine appropiately representations. Both the half-hyper and the vector multiplet can also be BPS massive multiplets.

5.1 Geometry of $N=2$ vector multiplets

The vector multiplet consists in one vector V_μ, two fermions λ_α^I ($I=1,2$) and two spin-zero objects (complex scalars) z. For n_v vector multiplets we use the notation $\left(V_\mu^a, \lambda_\alpha^{aI}, z^a\right)$ with $a=1,\ldots,n_v$. In terms of $N=1$ multiplets, we have the decomposition: $\left(V_\mu^a, \lambda_\mu^{a1}\right) \oplus \left(\lambda_\alpha^{a2}, z^a\right)$, where the first one is the vector multiplet at $N=1$ and the second the chiral multiplet $N=1$. The bosonic lagrangian is:

$$\begin{aligned} L = &-G_{a\bar{b}} D_\mu z^a D^\mu \bar{z}^b - V(z,\bar{z}) \\ &- \mathrm{Im}\left(N(z)\right)_{ab} F_{\mu\nu}^a F^{b\mu\nu} - \frac{i}{2}\mathrm{Re}\left(N(z)\right)_{ab} F_{\mu\nu}^a F_{\rho\sigma}^b \epsilon^{\mu\nu\rho\sigma} \end{aligned} \tag{83}$$

where

$$
\begin{gathered}
G_{a\bar{b}} = \partial_a \partial_{\bar{b}} K \\
K = i\left(\bar{F}_a Z^a - F_a \bar{Z}^a\right) = 2\mathrm{Im} F_{ab} \\
F_a = \partial_a F(z) \\
N_{ab} = \bar{F}_{ab} = \bar{\partial}_{\bar{a}} \bar{\partial}_{\bar{b}} \bar{F}(\bar{z})
\end{gathered}
$$

where $F(z)$ is the holomorphic prepotential. An special requirement in order to prevent ghost propagators and other physical inconsistencies, is $\mathrm{Im}(N) > 0$. The rigid special Kähler geometry is determined by a single holomorphic function, $F(z)$ [4]. It is determined by Seiberg-Witter theory [13]. Also,

$$
\begin{gathered}
D_\mu Z^a = \partial_\mu Z^a - V_{\mu b} k^{ab}(z) \\
k^a_{\bar{b}} = G_{\bar{b}c} k^{ca}(z) = i\partial_b P^a \\
V = G_{a\bar{b}} k^a_{\bar{c}} k^{\bar{b}}_d \bar{Z}^{\bar{c}} Z^d
\end{gathered}
$$

where $k_b = k^{ba}\partial_a$ satisfying $\left[k^b, k^c\right] = -f^{bc}_d k^d$, and P^a is the Killing prepotential or the moment map.

Now we will couple this multiplet to gravity. The gravity multiplet is given by $\left(g_{\mu\nu}, \psi^I_{\mu\alpha}, V^0_\mu\right)$. Thus, we have $n_v + 1$ vector bosons. V^0_μ is called the graviphoton. The lagrangian is given by:

$$
\begin{aligned}
\frac{L}{\sqrt{-g}} = & -\frac{1}{2}R - G_{a\bar{b}} D_\mu z^a D^\mu \bar{z}^b - V(z,\bar{z}) \\
& - \mathrm{Im}(\mathcal{N})_{AB} F^A_{\mu\nu} F^{B\mu\nu} - \frac{i}{2}\mathrm{Re}(\mathcal{N})_{AB} F^A_{\mu\nu} F^B_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma}
\end{aligned} \tag{84}
$$

where $a = 1, \ldots, n_v$, $A = 0, \ldots, n_v$ and

$$
\begin{gathered}
G_{a\bar{b}} = \partial_a \partial_{\bar{b}} K \\
K = -\ln i\left[\bar{X}^A(z) F_A(X(z)) - X^A \bar{F}_A\left(\bar{X}(\bar{z})\right)\right]
\end{gathered} \tag{85}
$$

We have $F_A = \frac{\partial F}{\partial X^A}$ and $X^A F_A = 2F$, so that F is homogeneous of degree 2. As $X^A = X^A(Z^a)$,we can define the special coordinates as $Z^a = \frac{X^a}{X^0}$, so that $F = \left(X^0\right)^2 \mathcal{F}(z)$. Then, (85) can be also expressed as:

$$
K = -\ln 2i\left(\mathcal{F} - \bar{\mathcal{F}}\right) - \left(\mathcal{F}_a + \bar{\mathcal{F}}_a\right)(z^a - \bar{z}^a) - \ln X^0 - \ln \bar{X}^0 \tag{86}
$$

$\mathcal{N}$ is given by:

$$
\mathcal{N}_{ab} = \bar{F}_{AB} - \frac{(\mathrm{Im}F)_{AC} X^C (\mathrm{Im}F)_{BD}}{X^C (\mathrm{Im})_{CD} X^D} \tag{87}
$$

[4] A mathematical discussion of special geometry can be found in [12]

where the second term is due to the graviphoton. Finally, the potential is given by:

$$V = e^K G_{a\bar{b}} K^a_A \bar{K}^{\bar{b}}_{\bar{B}} X^A(z) X^B(z)$$

There exists a Kähler invariant, given by $K \to K + f(z) + \bar{f}(z)$. This implies the transformation: $X^A \to X^A e^{-f}$, and $F_A \to F_A e^{-f}$ for F_A.

5.2 Geometry of $N = 2$ hyper multiplets

A hypermultiplet is buildt from two half-hypermultiplets. For n_H hypermultiplets we use the notation: $\left(\chi^i_\alpha, q^u\right)$, where χ^i_α, $i = 1, \ldots, 2n_H$, are the fermions, the q_n are real scalars, and $n = 1, \ldots, 4n_H$. The following ansatz for the SuSy gauge transformations is assumed:

$$\delta q^u = f(q)^u_{Ii} \epsilon^I \chi^i \tag{88}$$

The lagrangian is in this case:

$$L = h_{uv}(q) \partial_\mu q^u \partial^\mu q^v + h_{ij}(q) \chi^i \partial\!\!\!/ \bar{\chi}^j \tag{89}$$

(h_{uv} is the metric on M_{HK}, on a hyper Kähler manifold). As shown in [4], three complex structures exist given by

$$J^x_u{}^v = -i f^{iI}_u \sigma^{Xj}_i f^v_{Ij} \tag{90}$$

where the f's satisfy $f^{iI}_u f^v_{iI} = \delta^v_u$ and $f^{iI}_u f^u_{jI} = \delta^i_j \delta^J_I$; x can take the values $1, 2, 3$, and the complex structures satisfy:

$$J^x J^x = -1 \tag{91}$$

$$J^x J^y = i J^z \tag{92}$$

$$D_w J^{xv}_u \tag{93}$$

where D_w is the Levi-Civita connection. Using the metric, we can lower indices: $K^x_{uv} = h_{uw} J^{xw}_v . K^x = K^x_{uv} dq^u dq^v$ form a triplet of hyper Kähler forms such that they satisfy $dK^x = 0$. Thus the scalar field space is a hyper-Kähler manifold with h_{ab} as its metric.

The coupling to gravity proceeds as before. In this case, one finds

$$D_w J^{xu}_v = 0 \tag{94}$$

where the derivative includes both Levi-Civita connection and a $SU(2)$ connection w. This tells us that we do not have a hyper-Kähler manifold but quaternionic Kähler manifold. Furthermore, in terms of K^x we have

$$DK^x = dK^x + \epsilon^{xyz} w^y \wedge K^z = 0 \tag{95}$$

points out that at the same time we are also lying in a quaternionic Kähler manifold.

The scalar geometries are summarized in the following table:

N	cases	manifold
1	global	M_K: Kähler
	local	M_{KH}: Kähler-Hodge
2	global	$M_{SK} \times M_{HK}$: special Kähler $\times$ hyper Kähler
	local	$M_{SKM} \times M_{QK}$: special Kähler-Hodge $\times$ quaternionic Kähler
4	local	$\frac{SO(6,n_v)}{SO(6)\times SO(n_v)}$
8	local	$\frac{E_{7(7)}}{SU(8)}$

Finally, let us comment the massless multiplets for $N = 4$ and $N = 8$. For the case $N = 4$, we have the maximal N possible compatible with $|\lambda| \leq 1$. In the massless representations, we have $N = 4$ fermionic creation operators. The multiplets for this case are displayed in the following table:

Helicity/States	$\|-1\rangle$	$\|-2\rangle$	$\|0\rangle$
0	6	1	1
$\pm\frac{1}{2}$	4	4	4
± 1	1	6	6
$\pm\frac{3}{2}$		4	4
± 2		1	1
	vector multiplet	graviton multiplet	

As we can see, in the vector multiplet we have 6 scalars, 4 Weyl fermions and one vector, and it is CPT complete. In the gravity multiplet we have the metric, 4 gravitinos, 6 graviphotons, 4 Weyl fermions and 2 scalars. It is not CPT complete. The target space of the σ-model, in rigid supersymmetry, is flat, and in local,

$$M = \frac{SO\,(6,n)}{SO\,(6) \times SO\,(4)} \times \frac{SU\,(1,1)}{U\,(1)}$$

where n is the number of vector multiplets. The first component of the product is spanned by the scalars of the vector multiplet and the second by the scalars of the gravity multiplet.

For the case of $N = 8$ (the maximum N possible with $|\lambda| \leq 2$), we observe the following multiplet:

Helicity/State	$\|2\rangle$	
0	70	70 scalars
$\pm\frac{1}{2}$	56	56 Weyl fermions
± 1	28	28 vectors
$\pm\frac{3}{2}$	8	8 gravitinos
± 2	1	1 graviton

which is CPT complete. The target space in this case is $M = \frac{E_{7(7)}}{SU(8)}$.

6 Supersymmetry in arbitrary dimensions

Take the space-time to be $R_{1,D-1}$ with Lorentz metric: $\eta_{MN} = diag\,(-1,+1,\ldots,+1)$ with $D-1$ "+1"s. The first step in order to generalize the supersymmetric models is to discuss the spinor representation of $SO\,(1,D-1)$ [5]. The Dirac algebra is given by:

$$\{\gamma^M,\gamma^N\} = 2\eta_{MN} \tag{96}$$

with $M,N = 0,\ldots,D-1$. $\Sigma^{MN} := \frac{1}{4}\left[\gamma^M,\gamma^N\right]$ is the generator of $SO\,(1,D-1)$ in the spinor representation, and satisfies (1).

Let us analyze the different cases depending on the value of D:

1. $\underline{D = 2l+2,\ l = 0,1,2,\ldots}$: for this case, define

$$\gamma^{0\pm} := \frac{1}{2}\left(\pm\gamma^0 + \gamma^1\right) \tag{97}$$

$$\gamma^{a\pm} := \frac{1}{2}\left(\gamma^{2a} \pm i\gamma^{2a+1}\right) \quad a = 1,\ldots,l \tag{98}$$

$$\gamma^{A\pm} := \left(\gamma^{0\pm},\gamma^{a\pm}\right) \quad A = 0,\ldots,l \tag{99}$$

Inserting these definitions into (96), we obtain the relations:

$$\{\gamma^{A+},\gamma^{B-}\} = \delta^{AB} \tag{100}$$

$$\{\gamma^{A\pm},\gamma^{B\pm}\} = 0 \tag{101}$$

Thus, we obtain $l+1$ fermionic creation and annihilation operators (oscillators). From them, we can define a Clifford vacuum $|\Omega\rangle$ by demanding $\gamma^{A-}|\Sigma\rangle = 0$, $\forall A$. The states are given by $|\Omega\rangle$, $\gamma^{A+}|\Omega\rangle$,... The real dimension of the Dirac representation is given by:

$$dim_R(\text{Dirac rep}) = 2\sum_{i=1}^{l+1}\binom{l+1}{i} = 2^{l+2} \tag{102}$$

It is also possible to define $\gamma_{D+1} := i^l\gamma^0\gamma^1\ldots\gamma^{D-1}$, which satisfies

$$\{\gamma_{D+1},\gamma^M\} = 0 \tag{103}$$

$$\left[\gamma_{D+1},\Sigma^{MN}\right] = 0 \tag{104}$$

$$\gamma^2_{D+1,M} = 1 \tag{105}$$

It is possible to define two projection operators, $1 \pm \gamma_{D+1}$, that split the Dirac representation into 2 Weyl representations with eigenvalues ± 1.Furthermore, the dimension of the Weyl representation is: $dim_R(\text{Weyl rep}) = 2^{l+1}$. Then, $\left(\gamma^M\right)^* = B\gamma^M B^{-1}$ and $\gamma^*_{D+1} = (-1)^l\gamma_{D+1}$. If l is even ($D = 2,6,10,\ldots$), the Weyl representation

[5]This discussion can be found in detail in [14]

is its own conjugate (self-conjugate). If l is odd ($D = 4, 8, \ldots$), the Weyl representations are conjugated to each other. At this point, for $D = 2, 4, 8, 10$, we can define the Majorana spinor as: $\psi^* = \hat{B}\psi$ (which is called the reality condition). The dimension of the Majorana representation is: dim_R(Majorana rep) $= 2^{l+1}$. A Majorana-Weyl representation is only possible if the Weyl one is self-conjugated, i.e. $D = 2, 10, \ldots$, and its dimension is: dim_R(M-W rep) $= 2^l$

2. $\underline{D = 2l+1}$: in this case, there is no Weyl representation, and the Majorana one is only possible in $D = 1, 3, 9, 11, \ldots$. Also, dim_R(Majorana rep) $= 2^l$. All the possibilites of representations are displayed in the following table:

D	l	M	W	M-W	dim
4	1	ok	c.c.	-	4
5	2	-	-	-	8
6	2	-	s.c.	-	8
7	3	-	-	-	16
8	3	ok	c.c.	-	16
9	4	ok	-	-	16
10	4	ok	s.c.	ok	16
11	5	ok	-	-	32
12	5	ok	c.c.	-	32

The supersymmetry algebra in arbitrary D depends on the spinor representation of $SO(1, D-1)$. Schematically, (2) becomes:

$$\begin{aligned} \{Q^I, \bar{Q}^J\} &\sim \gamma^M P_M \delta^{IJ} \\ \{Q^I, Q^J\} &\sim Z^{IJ} \\ \left[L_{MN}, Q^I\right] &\sim \Sigma_{MN} Q^I \\ \left[P_M, Q^I\right] &= 0 \end{aligned} \tag{106}$$

If we display all the possible theories compatible with both the dimension D and number of real supercharges q, we obtain the table below:

D/q	4	8	12	16	20	24	28	32	64
4	$\diamond$	$\diamond$	$\diamond$	$\diamond$	$\diamond$	$\diamond$	$\diamond$	$\diamond$	$\diamond$
5		$\diamond$		$\diamond$		$\diamond$		$\diamond$	$\diamond$
6		$\diamond_{(1,0)}$		$\diamond_{(1,1)}$ $\diamond_{(2,0)}$		$\diamond$		$\diamond_{(2,2)}$	$\diamond$
7				$\diamond$				$\diamond$	$\diamond$
8				$\diamond$				$\diamond$	$\diamond$
9				$\diamond$				$\diamond$	$\diamond$
10				$\diamond_I$				$\diamond_{II_A}$ $\diamond_{II_B}$	$\diamond$
11								$\diamond_M$	$\diamond$
12									$\diamond_F$

The corresponding scalar geometries are:

D/q	4	8	16	32
4	M_K	$M_{SK} \times M_{QK}$	$SO_{6,n}$	$\frac{E_{7(7)}}{SU(8)}$
5		$M_{RSK} \times M_{QK}$	$SO_{5,n}$	$\frac{E_{6(6)}}{U_{sp}(8)}$
6		$\mathbb{R}^+ \times M_{QK}$	$SO_{4,n}/SO_{5,21}$	$\frac{E_{5(5)}}{U_{sp}(4)\times U_{sp}(4)}$
7			$SO_{3,n}$	$\frac{E_{4,4}}{U_{sp}(4)}$
8			$SO_{2,n}$	$\frac{E_{3,3}}{U(2)}$
9			$SO_{1,n}$	$\frac{GL(2)}{SO(2)}$
10			$\mathbb{R}^+$	$\mathbb{R}^+, \frac{SU(1,1)}{U(1)}$
11			-	-
		"M_{SK}" $\times M_{QK}$	$SO_{10-D,n}$	$\frac{E_{11-D}}{H_R}$

where

$$SO_{m,n} \equiv \frac{SO(m,n)}{SO(m) \times SO(n)} \times \frac{SU(1,1)}{U(1)}$$

if $D = 4$, and

$$SO_{m,n} \equiv \frac{SO(m,n)}{SO(m) \times SO(n)} \times \mathbb{R}^+$$

otherwise.

7 Kaluza-Klein Compactification

The basic idea of Kaluza-Klein theory is to formulate gauge symmetries as space-time symmetries of a higher dimensional space-time.

7.1 Circle Compactification

Consider a five dimensional spacetime $\mathbb{R}_{1,3} \times S_1$. S_1 is a circle with coordinates:

$$\begin{aligned} x^M &= (x^\mu, y), \\ M &= 0, ..., 4, \\ \mu &= 0, ..., 3. \end{aligned} \tag{107}$$

y is the periodic coordinate of the circle i.e. $y = y + 2\pi R$ and R is the radius of the circle. Scalar field can be expanded into Fourier modes according to:

$$\Phi(x^M) = \sum_{n=-\infty}^{n=+\infty} \phi^{(n)}(x^\mu) e^{iny/R} + c.c.$$

Φ satisfies the massless Klein Gordon equation:

$$\begin{aligned}\Box_5\Phi &= \eta^{MN}\partial_M\partial_N\Phi = (\eta^{\mu\nu}\partial_\mu\partial_\nu + {\partial_y}^2)\Phi \\ &= \sum_n(\Box_4\phi^{(n)} - m^2_{(n)}\phi^{(n)}) = 0\end{aligned}$$

where $m^2_{(n)} = n^2/R^2$. Each Fourier mode $\phi^{(n)}$ satisfies $\Box_4\phi^{(n)} - m_n^2\Phi^{(n)} = 0$. From an $\mathbb{R}_{1,3}$ perspective $\phi^{(n)}$ form an infinite tower of massive scalar fields called the *Kaluza-Klein tower*. One can estimate the size of S_1 by experimentally testing the validities of the two $\frac{1}{r}$ potentials:

$$\begin{aligned}V_{Coulomb} &\sim \frac{e_1\cdot e_2}{r} \qquad r\ confirmed\ for\ r > 10^{-18}m \\ V_{Newton} &\sim \frac{m_1\cdot m_2}{r} \qquad r\ confirmed\ for\ r > 10^{-4}m\end{aligned}$$

Therefore we need to choose generically $R \leqq 10^{-18}m$ and then the extra dimensions are not visible. However there is an exception where we choose $R \leqslant 10^{-4}m$. As was suggested by Rubakov, Shaposhnikov: They assumed that electromagnetism is localized on a 3 dimensional hyper plane inside a higher dimensional spacetime. In string theory this scenario is realized by D-brains (See [15]).

Here we do not consider this possibilitiy and therefore study the massless modes or zero modes of $\Box_5$; i.e. $n = 0$ and $\phi^{(0)}(x)$, when there is no y dependence :

$$\Phi(x^\mu, y) \to \Phi^{(0)}(x^\mu).$$

Next we consider 5-dimensional geometry with a metric:

$$g_{MN} = \begin{pmatrix}\hat{g}_{\mu\nu} & \hat{g}_{\mu 4} \\ \hat{g}_{4\nu} & \hat{g}_{44}\end{pmatrix}$$

From a 4 dimensional perspective we identify:

	Spin	particle
$\hat{g}_{\mu\nu}$	2	graviton
$\hat{g}_{\mu 4}$	1	gauge boson
$\hat{g}_{44}$	0	Scalar field

A convenient parametrization of the metric is:

$$g_{MN} = \begin{pmatrix}g_{\mu\nu} + r^2 V_\mu V_\nu & r^2 V_\mu \\ r^2 V_\nu & r^2\end{pmatrix}$$

$$\begin{aligned}\hat{g}_{\mu\nu} &= g_{\mu\nu} + r^2 V_\mu V_\nu \\ \hat{g}_{\mu 4} &= r^2 V_\mu \qquad (108) \\ \hat{g}_{44} &= r^2\end{aligned}$$

$$(109)$$

which implies:

$$det(g_{MN}) = r^2 det(g_{\mu\nu}) \tag{110}$$
$$ds^2 = g_{MN}dx^M dx^N = g_{\mu\nu}dx^\mu dx^\nu + r^2(V_\mu dx^\mu + dy)^2 \tag{111}$$
$$\tag{112}$$

We expand the metric around $SO(1,3)$ invariant background for some fluctuation $\delta g_{\mu\nu}$:

$$g_{MN} = g_{MN}|_{background} + \delta g_{MN}$$

where
$g_{MN}|_{background} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & < r^2 > \end{pmatrix}$ and $\delta g_{MN} = \begin{pmatrix} \delta g_{\mu\nu} + r^2 V_\mu V_\nu & r^2 V_\mu \\ r^2 V_\nu & r^2 - < r^2 > \end{pmatrix}$

For zero modes we have in addition:

$$g_{MN}(x,y) \to g^{(0)}_{MN}(x, \cancel{y}).$$

Therefore there are three metrics:

$$g_{MN}(x,y),\;\; g_{MN}|_{background}\;\; and\;\; g^{(0)}_{MN}(x)$$

General coordinate transformation are of the form:

$$x^M \to x'^M = x^M - \xi^M(x) \tag{113}$$

And the zero modes of the transformation are:

$$x^\mu \to x'^\mu = x^\mu - \xi^\mu(x, \cancel{y}) \tag{114}$$
$$y \to y' = y - \xi^4(x, \cancel{y}) \tag{115}$$

which are the 4 dimensional coordinate transformations and as x^μ-dependent isometries of S_1.The metric transforms:

$$\delta_\xi g_{MN} = -\xi^L \partial_L g_{MN} - (\partial_M \xi^L) g_{LN} - (\partial_N \xi^L) g_{LM} \tag{116}$$

Inserting (111) and (112) one obtains:

$$\delta V_\mu = -\partial_\mu \xi^4(x) \tag{117}$$

which is a local U(1) gauge transformation. Let us now reduce the Einstein-Hilbert action

$$S = \frac{-1}{2\kappa_5^2} \int d^5x \sqrt{-g^{(5)}} \mathcal{R}^{(5)} \tag{118}$$

where $\mathcal{R}$ is the Ricci scalar. Inserting $g^{(0)}$ one finds:

$$S = \frac{1}{2\kappa_5^2} \int d^4x \sqrt{-g^{(4)}} \sqrt{g_{44}} (\mathcal{R}^{(4)} + ...) \int_{S_1} dy \tag{119}$$

Where $\int_{S_1} dy = 2\pi R$.The Einstein-Hilbert term is noncanonical and therefore the Weyl transformation has to be applied to the metric:

$$g_{\mu\nu} = \Lambda(x)\tilde{g}_{\mu\nu}(x). \tag{120}$$

The corresponding Ricci scalar is:

$$\mathcal{R}^{(4)} = \Lambda^{-1}\tilde{\mathcal{R}}^{(4)} + \cdots \tag{121}$$

where $\tilde{\mathcal{R}}^{(4)} = \mathcal{R}^{(4)}(\tilde{g})$ and the ellipsis depends on $\partial\Lambda$. Now choose $\Lambda^{-1} = r(x)$ and then the action will turn into:

$$S = \int d^4x\sqrt{\tilde{g}^{(4)}}\left(\frac{-1}{2\kappa_4^2}\tilde{\mathcal{R}}^{(4)} - \frac{r^2}{4}\tilde{F}_\mu\tilde{F}^{\mu\nu} - \frac{\partial_\mu^\nu\partial_\nu^\mu}{\kappa_4^2 r^2}\right), \tag{122}$$

where

$$\kappa_4^{-2} = \frac{2\pi R}{-\kappa_5^2},$$
$$\tilde{F}_{\mu\nu} = \partial_\mu\tilde{V}_\nu - \partial_\nu\tilde{V}_\mu$$
$$\tilde{V}_\mu = \kappa_4 V_\mu$$

This shows that the 5 dimensional Einstein-Hilbert action decomposes into the 4 dimensional Einstein-Hilbert action and $U(1)$ gauge theory plus a neutral scalar under the Kaluza-Klein reduction.

7.2 Generalization

As a first generalization consider a spacetime of the form:

$$M_4 \times T^d \tag{123}$$

where T^d is a d-dimensional form. The metric is:

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} + g_{mn}V_\mu^m V_\nu^n & g_{np}V_\mu^p \\ g_{np}V_\nu^p & g_{mn} \end{pmatrix}, \tag{124}$$
$$m, n = 1, ..., d$$

For a 4-dimensional perspective we have

Zero modes	Spin	Multiplicity
$g_{\mu\nu}$	2	1
V_μ^p	1	d
g_{mn}	0	$\frac{1}{2}d(d+1)$

The isometries of T^d:

$$y^m \longrightarrow y^m - \xi^m(x) \tag{125}$$
$$V_\mu^m \longrightarrow V_\mu^m - \partial_\mu \xi^m \tag{126}$$

Corresponds to a $[U(1)]^d$ gauge theory. Then the action of the KK reduction will take the following form:

$$S = \int d^4x \sqrt{-\tilde{g}^{(4)}} \left(\frac{1}{2\kappa_4^2} \tilde{\mathcal{R}}^{(4)} + g_{mn} \tilde{F}^m_{\mu\nu} \tilde{F}^{n\mu\nu} - (\partial_\mu g_{mn})(\partial^m g^{mn}) \right) \tag{127}$$

Remarks. The potential is zero and the σ-model target space is:

$$M = GL(d)/SO(d) \tag{128}$$

7.3 Fermions in Kaluza Klein theory

Consider a spacetime:

$$\mathbb{R}^{(1,3)} \times X^d \tag{129}$$
$$(x^\mu, y^m)$$
$$\tag{130}$$

where (x^μ, y^m) are coordinates, $\mu = 0, ..., 3$ and $m = 1, ..., d$. Including fermions into KK theories we write down the massless Dirac equation:

$$i\gamma^M D_M \Psi(x,y) = 0 \tag{131}$$

where γ^M are Dirac matricies and we denote $i\gamma^M D_M$ by $\not{D}$.

KK ansatz:

$$\Psi(x,y) = \sum_I \psi^I(x)\eta^I(y) \tag{132}$$

The Dirac operator decomposes according to $(\not{D}^{(4)} + \not{D}^d)\Psi = 0$ so that the massless modes obey:

$$\not{D}^d \eta^I = 0. \tag{133}$$

More generally we can demand $\not{D}\eta^I = m^I \eta^I$, where there is no sum on I We call η^I "*Eigen-Spinors*" and require η^I to be normalizable:

$$||\eta^I|| = 1 \tag{134}$$

Let us discuss the constraint of the theory. The Dirac algebra is:

$$\{\gamma^M, \gamma^N\} = 2\eta^{MN} \Leftrightarrow \begin{cases} \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \\ \{\gamma^\mu, \gamma^4\} = 0 \\ \{\gamma^4, \gamma^4\} = 2 \end{cases} \tag{135}$$

where the later implies $(\gamma^4)^2 = 1$ and thus $\gamma^4 = \gamma_5$ of $\mathbb{R}_{1,3}$. As a result the projection operators $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$ break 5-dim Lorentz invariance. Therefore chiral theories are not Lorentz invariant in higher dimensional spacetime.

7.4 Supersymmetry In KK-theories

Consider the $\mathbb{R}^{(1,3)} \times X^d$ background with KK-Ansatz:

$$Q(x, y) = Q^I(x)\eta^I(y) \tag{136}$$

where $Q^I(x)$ and $\eta^I(y)$ are the spinor on $\mathbb{R}^{1,3}$ and the eigen spinor on X^d, respectively. The number of eigen spinors η^I is qual to the number of supersymmetries. We also require the followings conditions on η^I

- It is normalizable i.e. nowhere vanishing on X^d.
- It should be a singlet of the structure group which implies it is globally well-defined.

The second requirement implies a reduction of the structure group. Therefore implies constraints on X^d. With these two requirements we can construct a well-defined supersymmetry. X^d is called a manifold with the G-structure ($G \subset GL(d, \mathbb{R})$) which leaves η^I invariant.

7.5 Examples With $d = 6$

- 1)$X^6 = T^6$ or Torus fiberation. Thus has an identity structure and four η exist. This leads to $N = 4$ supersymmetry.
- 2)X^6 with $SU(2)$. In this case two η exist. Therefore we have $N = 2$ supersymmetry.
- 3)X^6 with $SU(3)$ structure. One η exists. Therefore we have $N = 1$ supersymmetry.

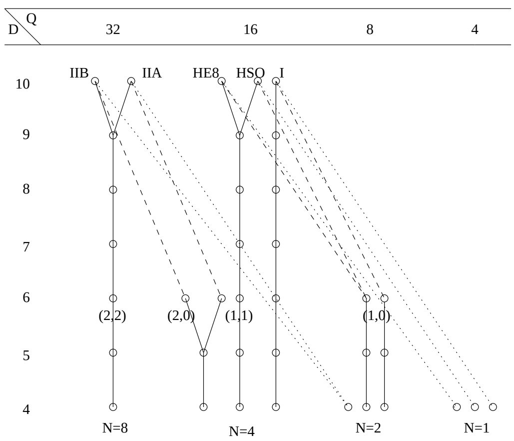

Figure 1: The super gravity

Here the solid, dashed and dotted lines show S_1 compactification, $su(2)$ and $su(3)$ structure respectively.

7.6 Supergravity In D=10,11

In D=11. The massless spectrum contains:

$$g_{(MN)},\ \ A_{[MNP]},\ \ \Psi_M, \tag{137}$$

where $M, N, P = 0, ...10$, $A_{[MNP]}$ is an antisymmetric 3-index tensor or equivalently related to a three form and Ψ_M is the gravitino. The bosonic Lagrangian is:

$$\mathcal{L}_{bosonic} = \sqrt{-g}(\mathcal{R} - \frac{1}{2}|F_4|^2) - \frac{1}{6}A_3 \wedge F_4 \wedge F_4 \tag{138}$$

where $F_4 = dA_3$ is the field strength of the three from. The claim is that $\mathcal{L}$ has the following symmetries:

- General covariance
- Local susy (upon adding appropriate fermionic terms.)
- Three form gauge invariance:

$$\begin{aligned} A_3 \rightarrow A'_3 &= A_3 + d\Lambda_2, \\ F_4 \rightarrow F'_4 &= F_4, \\ \mathcal{L} \rightarrow \mathcal{L}' &= \mathcal{L} + \frac{-1}{6}d(\Lambda_2 \wedge F_4 \wedge F_4), \end{aligned} \tag{139}$$

$$\tag{140}$$

where Λ_2 is a 2-form. The last line implies the the action S is invariant, i.e. $\delta S = \delta \int \mathcal{L} = 0$.

Let us consider type IIA supergravity in $D = 10$.The metric is:

$$g_{MN} = \begin{pmatrix} g_{\hat{M}\hat{N}} & g_{\hat{M}10} \\ g_{\hat{10}M} & g_{1010} \end{pmatrix} \tag{141}$$

with $\hat{M} = 0, ..., 9$. A_{MNP} splits into three parts:

- $A_{\hat{M}\hat{N}\hat{P}}$
- $A_{\hat{M}\hat{N}10} \equiv B_{\hat{M}\hat{N}}$
- $A_{\hat{M}1010} = 0$

For the massless spectrum we have:

$$g_{\hat{M}\hat{N}},\ B_{\hat{M}\hat{N}},\ \phi, \tag{142}$$

$$A_{\hat{M}},\ A_{\hat{M}\hat{N}\hat{P}}, \tag{143}$$

$$\Psi^{1,2}_{\hat{M}},\ \lambda^{1,2} \tag{144}$$

where ϕ is the scalar field called dilaton and (141) and (142) are called NS-NS and R-R sectors respectively. The Lagrangian reads [6]:

$$\mathcal{L}^{\mathbb{II}\mathbb{A}}_{bosonic} = e^{-\phi}(\mathcal{R} + 4\partial_M \phi \partial^M \phi - \frac{1}{2}|H_3|^2) - \frac{1}{4}|F_2|^2 + |\hat{F}_4|^2 \tag{145}$$

where $H_3 = dB_2$, $F_2 = dA_1$ and $|F_4| = dA_3$ furthermore we abbreviate:

$$\begin{aligned} |H_3|^2 &= H_{MNP}H^{MNP} \\ |F_2|^2 &= F^{MN}F_{MN} \\ |\hat{F}_4|^2 &= \hat{F}_{MNPQ}\hat{F}^{MNPQ} \\ \hat{F}_4 &= F_4 - A_1 \wedge H_3 \end{aligned}$$

The symmetries of the theory are:

- general covariance
- $N = 2$ local supersymmetry (There are two 16-dim supercharges Q',Q^2)
- Three gauge invariances with p-forms λ_p as gauge parameters:
  (i) $\delta A_3 = A_3' - A_3 = d\Lambda_2$,
  (ii) $\delta B_2 = d\Lambda_1$,
  (iii) $\delta A_1 = d\Lambda_0$, $\delta A_3 = \Lambda_0 H_3$ which implies:

$$\delta \hat{F}_4 = (d\Lambda_0)H_3 - d\Lambda_0 H_3 = 0$$

Note that the theory contains no charged fermions.

Next, we consider the case of type IIB supergravity. The massless spectrum is given by:

$$g_{MN},\ B^1_{MN}, \phi \tag{146}$$

$$l,\ B^2_{MN}, A^*_{MNPQ} \tag{147}$$

where A^*_{MNPQ} is a 4-form with a self-dual 5-form field strength, B^2_{MN} is a second 2-form, and (146) and (147) are NS-NS and R-R sectors respectively. This theory has no Lorentz invariant action but only field equations.

7.7 Supersymmetric Background In $\mathbb{R}^{1,3} \times X^6$

Let us now consider compactifications, i.e. study the theories in backgrounds of the form $\mathbb{R}^{1,3} \times X^6$. Let us additionally impose the following conditions:

[6] we omit the "$\wedge$" hence forth.

1. Super charges should exist. This implies X^6 is a manifold with G-structure (globally well defined spinor η).
2. The background does not break any Susy spontaneously.

Let's look at the susy transformation of the fermion evaluated at the background:

$$\delta_{\text{fermion}}\Big|_{\text{background}} = 0 \tag{148}$$

The variation of the gravitino's field Ψ_M is:

$$\delta\Psi_M \sim \nabla_M^{LC}\epsilon|_{\text{bg}} + \sum_p a_p\gamma.F_p\epsilon|_{\text{bg}} + ... = 0 \tag{149}$$

where ϵ,γ and a_p are the parameter of the susy transformation, the Dirac γ matricies and a number respectively.$\gamma.F_p$ denotes contractions of the form $\gamma^M\gamma^N F_{MN}$ for all p-forms present in the theory.

Now choose $\gamma.F_P|_{\text{bg}} = 0$. This means:

$$\nabla_M\epsilon_{(x,y)} = \sum_I \nabla_M\epsilon^I(x)\eta^I(y) = 0, \tag{150}$$

or equivalently:

$$\nabla_\mu\epsilon(x,y) = 0 \Rightarrow \nabla_\mu\epsilon^I(x) \tag{151}$$

$$\nabla_m\epsilon(x,y) = 0 \Rightarrow \nabla_m\eta^I = 0. \tag{152}$$

the latter implies:

$$[\nabla_m, \nabla_n]\eta^I = R_{mnkl}\gamma^{[k}\gamma^{l]}\eta^I = 0. \tag{153}$$

This says that the Hol$(\nabla_m) \subseteq su(3)$, i.e. it is reduced. A further consequence is $\mathcal{R}_{mn} = 0$, That is X^d is Ricci flat and is called *the Calabi-Yau* manifold.
Classes of examples:

- The 6-dimensional Torus T^6 which has four spinors η^I.It is flat and leads to $N = 8$ supersymmetriy in $R^{1,3}$.
- $K_3 \times T^2$ which has two spinors, Hol= $su(2)$ and leads to $N = 4$.
- The Calabi-Yau three-fold cy_3 which has one spinor, Hol= $su(3)$ and leads to $N = 2$.

7.8 Massless Spectrum

We have p-forms A_p corresponding to the field strength $F_{p+1} = dA_p$. The field equation of the p-form is:

$$d^\dagger F_p = 0. \tag{154}$$

We define the action for this case:

$$S = \int F_{p+1} \wedge^* F_{p+1} \tag{155}$$

A_p is not well-defined due to the gauge invariance $A_p \to A_p + d\Lambda_{p-1}$. We impose a gauge choice to simplify the field equations:

$$d^\dagger A_p = 0 \tag{156}$$

Rewrite the field equation:

$$(d^\dagger d + dd^\dagger)A_p = \Delta_{10} A_p = (\Delta_4 + \Delta_6)A_p = 0 \tag{157}$$

where $d^\dagger d + dd^\dagger = \Delta$ is the Laplacian acting on p-forms. For massless modes $\Delta_6 A_p = 0$ solutions are harmonic forms :

$$dA_p = 0 = d^\dagger A_p \tag{158}$$

where $A_p = d\Lambda_{p-1}$ is a gauge transformation. Therefore:

$$A_p \in H^{(p)}(X^6) = \frac{\text{closed p-forms}}{\text{exact p-forms}}. \tag{159}$$

On a complex manifold A_p decomposes as follows:

$$A_p = \sum_{q+k=p} A_{q,k} \tag{160}$$

where $A_{q,k} = A_{i_1,\cdots,i_q,j_1,\cdots,j_k} dz^{i_1} \cdots dz^{i_q} dz^{j_1} \cdots dz^{j_k}$. Correspondingly there is a Hodge decomposition:

$$H^p = \sum_{q+k=p} H^{(q,k)}. \tag{161}$$

Let us introduce the Hodge number $h^{q,k} = \dim H^{(q,k)}$. They are conventionally arranged in *the Hodge diamond*:

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array} \tag{162}$$

For the Calabi-Yau this diagram reads:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & 0 & & h^{2,2} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{163}$$

Genericaly we have the following rules:

$$\left\{ \begin{array}{l} h^{q,k} = h^{k,q} \\ h^{q,k} = h^{\frac{d}{2}-q,\frac{d}{2}-k} \end{array} \right. \tag{164}$$

which are used in sketching the diagram. To compute the spectrum for IIA we have:
the dilaton expanded in massless modes:

$$\phi(x,y) = \phi(x) \tag{165}$$

the 2-form:

$$B_{MN}(x,y) = \left\{ \begin{array}{l} B_{\mu\nu}(x) \\ B_{\mu\nu} = 0 \\ B_{mn} = \sum_{a=1}^{h^{1,1}} b^a(x) w_{1,1}^a(y) \end{array} \right. \tag{166}$$

the 3-form:

$$A_{MNP}(x,y) = \left\{ \begin{array}{l} A_{\mu\nu\rho}(x) \\ A_{\mu mn} = \sum_a A_\mu^A(x) w_{(1,1)}^a(y) \\ A_{\mu\nu m} = 0 \\ A_{mnp} = \xi^I(x)\alpha_I(y) + \tilde{\xi}_I(x)\beta^I(x) \end{array} \right. \tag{167}$$

where $I = 0, ..., h^{1,2}$, $(\alpha_I, \beta^I) \in H^3$ and $\xi^I(x), \tilde{\xi}_I(x)$ are real scalar fields. The metric g_{MN} splits into $g_{\mu\nu}(x)$, $g_{\mu u} = 0$ and $g_{mn}(x,y)$. We expand g_{mn} around the Calabi-Yau metric:

$$g_{mn}(x,y) = g_{mn}^0(y) + \delta g_{mn}(x,y) \tag{168}$$

Inserting it in the Ricci tensor we get:

$$0 = \mathcal{R}_{mn}(g^0 + \delta g) = ... = \Delta \delta g_{mn} + R_{mn}^{kl} \delta g_{kl} \tag{169}$$

This equation is known as the Lichnerowicz equation. Going to complex coordinates, then we have:

$$\delta g_{mn} = \left\{ \begin{array}{l} \delta g_{\alpha\bar{\beta}} \\ \delta g_{\alpha\beta} \end{array} \right. \tag{170}$$

where $\alpha, \beta = 1, 2, 3$. The properties are:

- $\delta g_{\alpha\bar{\beta}}$, $\delta g_{\alpha\beta}$ satisfy Lichnerowicz equation separately.
- Both obey $(dd^\dagger + d^\dagger d)\delta g = 0$. Therefore we can write down the solution:

$$\delta g_{\alpha\bar{\beta}} = \sum_{a=1}^{h^{1,1}} v^a(x) w^a(y) \tag{171}$$

$$\delta g_{\alpha\beta} = \sum_{i=1}^{h^{1,2}} t^i(x) \Omega_{\alpha\gamma\delta} \chi^i_{\beta\bar{\gamma}\bar{\delta}} g^{0\gamma\bar{\gamma}} g^{0\delta\bar{\delta}} \tag{172}$$

where $\chi^i \in H^{1,2}$ is a harmonic $(1,2)$-form and $\Omega \in H^{3,0}$. $v^a(x)$ are $h^{1,1}$ real scalars while t^i are $h^{1,2}$ complex scalars.
Remark. There is a Kähler form $J = -ig_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^{\bar{\beta}}$ and if we expand it we get:

$$J = J^0 + \delta J^0. \tag{173}$$

We can view δJ^0 as the deformation of the Kähler form which leaves the Calabi-Yau Ricci flat.

- There is a complex structure I with $I^m_n I^n_k = -\delta^m_k$. Deformations of I should leave the Nijenhuis tensor N invariant, i.e.

$$I \to I + \delta I \tag{174}$$

with $N(I + \partial I) = 0$. On a Calabi-Yau one finds:

$$\partial I^\alpha_{\bar{\beta}} \Omega_{\alpha\beta\gamma} = \chi_{\bar{\beta}\beta\gamma} \tag{175}$$

That is the deformations of I are related? to the $(1,2)$-form and thus also to $\partial g_{\alpha\beta}$.

7.9 Final (bosonic) Spectrum

Let us summarize the final spectrum in $\mathbb{R}^{1,3}$:

gravity multiplet: $(g_{\mu\nu}, A^0_\mu)$
$h^{1,1}$vector multiplets: (A^α_μ, z^a)
$h^{1,2}$ hyper multiplets: $(t^i, \xi^I, \bar{\xi}^I, \phi, a)$

where a is the dual of $B_{\mu\nu}$.The Lagrangian reads:

$$\mathcal{L}^{N=2} = \sqrt{g}(\frac{1}{2}\mathcal{R} - Im\mathcal{N}_{AB}(z) F^A_{\mu\nu} F^{B\mu\nu} - \frac{i}{2} Re\mathcal{N}_{AB} F^A_{\mu\nu} F^B_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma}) \tag{176}$$
$$-G_{a\bar{b}}(z) \partial_\mu z^a \partial^\mu \bar{z}^{\bar{b}} - h_{uv}(q) \partial_\mu q^n \partial^\mu q^v$$

where $G_{a\bar{b}} = \partial_a \partial_{\bar{b}} K_J$ and K_J is the Kähler potential:

$$K_J = -ln \int_{cy_3} J \wedge J \wedge J \tag{177}$$

The holomorphic prepotential is $\mathcal{F} = \kappa_{abc} z^a z^b z^c$ and:

$$\kappa_{abc} = \int_{cy_3} w_{1,1}^a \wedge w_{1,1}^b \wedge w_{1,1}^c. \tag{178}$$

The last term in the Lagrangian is given by:

$$\begin{aligned} h_{uv}(q)\partial_\mu q^u \partial^\mu q^v = \partial\phi\partial\phi + e^{4\phi}(\partial a + \tilde{\xi}^I \partial_\mu \xi_I - \xi_I \partial_\mu \tilde{\xi}^I)^2 + \\ g_{i\bar{j}} \partial t^i \partial \bar{t}^{\bar{j}} + \\ e^{2\phi} Im\mathcal{M}^{IJ}(\partial\tilde{\xi}^I + \mathcal{M}_{IK}\xi^K)(\partial\tilde{\xi}_J + \mathcal{M}_{JL}\xi^L) \end{aligned} \tag{179}$$

where $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K_\Omega$ and $K_\Omega = -ln \int_{cy_3} \Omega \wedge \bar{\Omega}$. Note that no potential arises $(V = 0)$ and no charged fields are present.

7.10 IIB

The spectrum in $\mathbb{R}^{1,3}$ is:

$$g_{MN}, B^{1,2}_{MN}, A^*{}_{MNPQ}, \phi, l \tag{180}$$

Repeating the analysis of the previous section we find: $h^{1,1} + 2h^{1,2}$ real scalars arising from the deformation of the Calabi-Yau metric, $2(h^{1,1} + 1)$ real scalars arising from the two $B^{1,2}_{mn}$, $h^{1,1}$ real scalars arising from $A^*{}_{MNPQ}$ and the two additional ϕ, l. Furthermore from $A_{\mu mnp}$ arise $h^{1,2} + 1$ vector fields. So altogether we have $h^{1,2}$ vector multiplets and $h^{1,1}$ hyper multiplets.

String theory suggests that IIA symmetry in the background $\mathbb{R}^{1,3} \times cy_3$ is equivalent to IIB symmetry in $\mathbb{R}^{1,3} \times \tilde{cy}_3$ with:

$$\begin{cases} h^{1,1}(cy_3) = h^{1,2}(\tilde{cy}_3) \\ h^{1,2}(cy_3) = h^{1,1}(\tilde{cy}_3) \\ F(cy_3) = G(\tilde{cy}_3) \end{cases} \tag{181}$$

This is called mirror symmetry. $\tilde{cy}_3$ is called the mirror Calabi-Yau manifold. The conjencture of the mirror symmetry says that for every Calabi-Yau threefold cy_3 there is a mirror threefold $\tilde{cy}_3$ with reversed Hodge numbers and identical prepotentials.

7.11 Flux Compactification and Generalized Geometries

So far we have assumed $F_p|_{bg} = 0$. Now we will relax this condition and have $F_p|_{bg} \neq 0$. However, we keep the following property:

$$dF_p = 0 \tag{182}$$

$$d^\dagger F_p = 0. \tag{183}$$

We introduce the flux:

$$\int_{\gamma_p^I} F_p = e_I \neq 0 \tag{184}$$

where $\gamma_p^I \in X^6$ is a p-cycle. As a result we can write:

$$F_p = \sum_I e_I \omega_p^I \tag{185}$$

and $\omega_p^I \in H^{(p)}(X^6)$. e_I induces a potential $V(e)$ in the action. We studied the gravitino variation:

$$\delta\Psi_M(x,y) = \nabla_M \epsilon(x,y) + \sum_p a_p(\gamma.F_p)_M\ \epsilon + ... \tag{186}$$

where $M = 0, ..., 9$. Now we want to evaluate this in a supersymmetric background $\delta\Psi_M(x,y) = 0$ for $\mathbb{R}^{1,3} \times X^6$ and $\epsilon(x,y) = \sum_I \epsilon^I(x)\eta^I(y)$. We have:

$$\delta\Psi_m = \nabla_m\epsilon + ... = \sum_I \epsilon^I(x)\nabla_m\eta^I + ...,\ \ m = 1, ..., 6 \tag{187}$$

which means $\nabla_m\eta^I \neq 0$.

In addition, if we want to have a supersymmetric theory we should have a manifold X^6 with G-structure. If X^6 has $SU(3)$ structure only one globally defined spinor η exists, and one can define:

$$J_{[mn]} \sim \bar{\eta}\gamma_{[m}\gamma_{n]}\eta. \tag{188}$$

Furthermore, $I_m^\kappa = J_{mn}g^{n\kappa}$ obeys $I_m^\kappa I_\kappa^n = -\delta_m^n$. Then it follows that in general the Nijenhuis tensor $N(I)$ is non-vanishing, i.e.:

$$N(I) \neq 0. \tag{189}$$

This is a consequence of $\nabla_m\eta \neq 0$. In this case I is an almost complex structure. One can check that J is of the type $(1,1)$ with respect to I and $dJ \neq 0$ due to $\nabla_m\eta \neq 0$.

Analogously we can define a 3-form:

$$\Omega_{mnp} \sim \bar{\eta}\gamma_{[m}\gamma_n\gamma_{p]}\eta \tag{190}$$

such that it obeys:

- $d\Omega \neq 0$
- It is of the type $(3,0)$ with respect to I
- $J \wedge J \wedge J \sim \Omega \wedge \bar{\Omega},\ J \wedge \Omega = 0$

  Note that for $\nabla_m\eta = 0$ we have:

$$dJ = 0,\ N(I) = 0,\ d\Omega = 0 \tag{191}$$

therefore X^6 is cy_3 with J a Kähler form and Ω $(3,0)$ form of cy_3.

7.12 Compute Kaluza-Klein Reduction

The conditions $[\nabla_m, \nabla_n]\eta \neq 0$ implies $\int_{X^6} R^{(6)} \neq 0$ generates a potential V which implies some modes become massive. To proceed we should distinguish two cases:
(i) $m \approx m_{KK}$ (ignore all massive modes)
(ii)$m \ll m_{KK}$ (keep the light modes)
For case (ii) one expands in bases of light fields:

$$J = J^0 + \sum_a v^a \omega^a \tag{192}$$

where w^a is the solution of $(\Delta^6 + m^2)w = 0$. Additionally we assume there is no extra light gravitino and one only has two massless gravitinos (i.e. $N = 2$). Geometrically this means:

$$\begin{cases} d(J \wedge J) = 0 \\ (d\Omega)^{3,1} = 0 \end{cases} \tag{193}$$

Supergravity implies $M = M^{\nu}_{S\kappa} \times M^{h}_{Q\kappa}$. The result in IIA is $M = M_{\delta J} \times M_{\delta\Omega}$ where $M_{\delta J} \in M^{\nu}_{S\kappa}$ and $M_{\delta\Omega} \in M^{h}_{Q\kappa}$. In IIB we have:

$$M = M_{\delta\Omega} \times M_{\delta J}$$

7.13 Compute The Potential

The Killing verctor is defined by:

$$D_\mu q^u = \partial_\mu q^u - V^A_\mu k^u_A(q) \tag{194}$$

where $k^u_A(q)$ is a function of the scalars q^u in the hyper multiplets, $u = 1, .., 4n_h$ and $A = 0, ..., n_\nu$. Then:

$$k^u_A k^x_{uv} = -\partial_v P^x_A + \epsilon^{xyz} \omega^y_v P^z_A = -D_v P^x_A \tag{195}$$

where $x = 1, 2, 3$. We get the following result for IIA:

$$P^1 + iP^2 = e^{\frac{1}{2}K_\Omega + \phi} \int_{X^6} J_c \wedge d\Omega \tag{196}$$

$$P^3 = -e^{2\phi} \int e^{J_c} \wedge \sum_P F_P \tag{197}$$

where ϕ is dilaton. And for IIB:

$$P^1 + iP^2 = e^{\frac{1}{2}K_J + \phi} \int_{X^6} J_c \wedge d\Omega \tag{198}$$

$$P^3 = e^{2\phi} \int \Omega \wedge (H_3 + F_3) \tag{199}$$

Remarks:

- P^x mirror symmetries
- Can also be computed for $SU(3) \times SU(3)$

8 String theory

8.1 Basic concepts[7]

String theory can be defined as the quantum theory of extended objects. The basic idea of string theory is to replace point-like objects by closed or open strings.

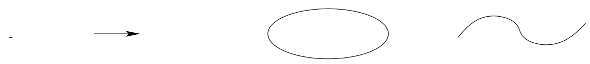

Figure 2: Open and closed string

l_s denotes the extension of the string, and thus is the fundamental length scale.

Let us look at a *closed string* moving in D-dimensional Minkowskian background $M^{1,D-1}$. It sweeps out a 2-dimensional world sheet $\Sigma(\sigma,\tau)$, where σ denotes the coordinate on the string and τ the time direction.

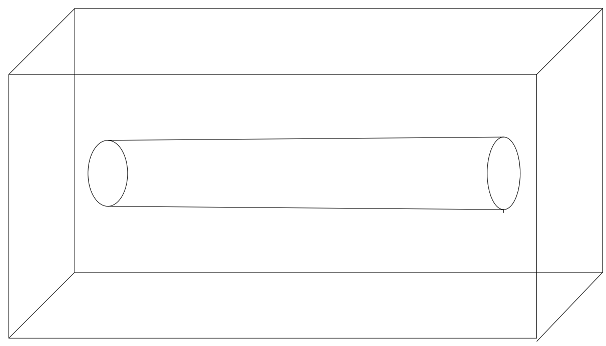

Figure 3: World sheet of a closed string

The motion of the string is described by 2 dimensional quantum field theory. Therefore one looks at the Polyakov action:

$$S = -\frac{1}{4\pi\alpha'} \int_\Sigma d\sigma d\tau \sqrt{det\, h}\, h^{\beta\gamma}(\sigma,\tau)\, \partial_\beta X^M \partial_\gamma X^N \eta_{MN} + \int_\Sigma R^{(2)} \qquad (200)$$

Here α' denotes the string tension with $\sqrt{\alpha'} = l_s$, h is the metric on the world sheet Σ, η_{MN} the Minkowski metric on $M^{1,D-1}$ with $\eta = diag(-1,1,\dots,1)$ and $X^M : \Sigma \to M^{1,D-1}$, $M = 0,\dots,D-1$ are fields on the world sheet in

[7] [17], [16]

2 dimensional QFT which can be seen as coordinates of the target space of Σ. The term $\int_\Sigma R^{(2)}$ is proportional to the Euler number $\chi(\Sigma)$ of the world sheet and therefore does not affect the dynamics of the 2-dimensional QFT.

The symmetries of this action are:

1. Lorentz transformations $SO(1, D-1)$ acting on X^M as $X^M \to X'^M = \Lambda^M_N X^N$, $\Lambda \in SO(1, D-1)$
2. Translations of X^M: $X^M \to X'^M = X^M + a^M$

   (1. and 2.: Poincare group in target space)
3. Reparametrization of Σ
4. Weyl-invariance with respect to h: $h_{\alpha\beta} \to \Lambda(\sigma, \tau) h_{\alpha\beta}$

Due to the last two properties, $h_{\alpha\beta}$ can be chosen locally flat (i.e. there are no dynamical degrees of freedom in $h_{\alpha\beta}$) and as a consequence of the Weyl-invariance one can take $h^{\alpha\beta}T_{\alpha\beta} = 0$ where $T_{\alpha\beta} \sim \frac{\partial S}{\partial h_{\alpha\beta}}$. Therefore the 2 dimensional QFT is conformal and hence it is described by 2 dimensional CFT.

The equation of motion for X^M is the 2-dim Klein-Gordon-equation:

$$\Box^{(2)} X^M = 0 \tag{201}$$

The solution of this equation can be written in the following way:

$$X^M(\sigma, \tau) = X^M_L(\tau + \sigma) + X^M_R(\tau - \sigma) \tag{202}$$

where X^M_L is called left mover and X^M_R right mover. Moreover, X^M has to fulfill a periodicity condition (we are still considering the closed string) in the string coordinate σ:

$$X^M(\sigma) = X^M(\sigma + 2\pi R) \tag{203}$$

where R denotes the radius of the closed string. Hence we have a discrete spectrum of eigenvalues.

Using canonical quantization one imposes the following commutation relations:

$$\left[X^M(\sigma), P^N(\sigma')\right] = i\delta(\sigma - \sigma')\eta^{MN} \tag{204}$$

$$\left[X^M, X^N\right] = 0 = \left[P^N, P^M\right] \tag{205}$$

Here $P^M = \frac{\partial \mathcal{L}}{\partial \dot{X}_M}$ and X^M are operators acting on some Hilbert space $\mathcal{H}$. To get $\mathcal{H}$ positive semi-definite, it is necessary to set either $D = 26$, or, if the 2 dimensional CFT is supersymmetric, $D = 10$. $D = 10$ is also necessary to have fermions in the space-time.

From Poincare-invariance on can take the time translation symmetry to define the Hamiltonian H in $M^{1,9}$ (as the generator of time translation) and the Lorentz rotation symmetry to get the rotation operator J also in $M^{1,9}$ (as the generators of the rotation group). By computing the eigenvalues of H and J in each mode, one gets m^2 and s, where m is the mass and s is the spin of the according mode. From this computation one gets:

- $m^2 < 0$: tachyons, only appear in non-supersymmetric case
- $m^2 = 0$:
  $s = 2 \quad \rightarrow \quad$ graviton
  $s = \frac{3}{2} \quad \rightarrow \quad$ gravitino
  $s = 1 \quad \rightarrow \quad$ gauge bosons
  $s = \frac{1}{2} \quad \rightarrow \quad$ quarks, leptons
  $s = 0 \quad \rightarrow \quad$ scalars, Higgs
- $m^2 > 0$: there are infinitely many massive excitations: $m^2 = nm_s$, $n \in \mathbb{N}$, $m_s^2 = \frac{1}{l_s^2} = \frac{1}{\alpha'}$, where the associated spin is given by $s\left(m\right) = \frac{m^2}{m_s^2}$ (Regge-behaviour). m_s is a characteristic scale of string theory, the tension of the string, related to its length l_s.

The identification of the $s = 2$ excitation with Einstein's graviton leads to the following identification:

$$m_s \sim m_{PL} \sim 10^{19} GeV \text{ and } l_s \sim l_{PL} \sim 10^{-35} m$$

Therefore the massive modes are in general not observable. The difference between pointlike particles and strings becomes relevant only at high energies, where string theory softens the UV behaviour.

Interactions:
For interactions, one looks at two strings which merge to one, or at one string which splits into two.
For low energies, these pictures correspond to Feynman-diagrams in QFT, and they are determined by the coupling constant $g_s = e^{-\langle\phi\rangle}$, where ϕ is the scalar field.
If one wants to look at real interactions, one takes the quantization as for Feynman-diagrams by summing over all world sheet topologies of two particles going into the interaction and two particles leaving it to compute the scattering amplitude $\mathcal{A}$. The scattering amplitude obeys the following expansion:

$$\mathcal{A} = \sum_{n=0}^{\infty} A^{(n)} g_s^{2+2n} \tag{206}$$

A comparison between QFT and string theory shows, that in QFT one starts with an action $S = \int \mathcal{L}$ and then gets to Feynman-diagrams, whereas in string theory one has somesthing corresponding to the Feynman-diagrams, but one didn't start from an action functional, where no analogue is known until now.

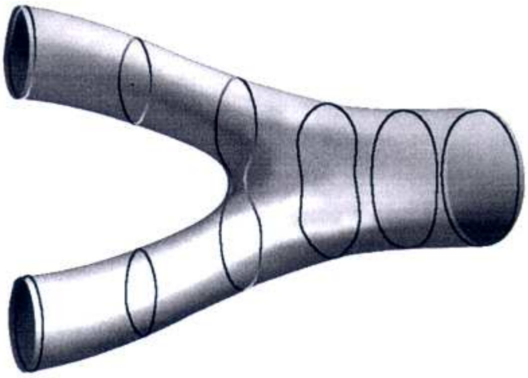

Figure 4: Two merging closed strings

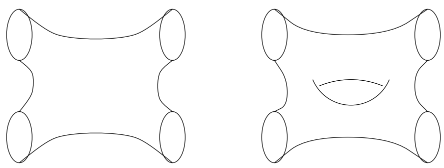

Figure 5: Interactions

Remarks:
One can show:

- for scattering with $E << E_s$, where $E_s = m_s c^2$: $\mathcal{A}^{string} \to \mathcal{A}^{QFT}_{GR}$
- $\mathcal{A}$ is finite for $E >> E_s$, therefore the theory is a candidate for *perturbative* quantum gravity

Including non-perturbative correction, one expects $\mathcal{A} = \sum_{n=0}^{\infty} A^{(n)} g_s^{2+2n} + \mathcal{O}\left(e^{-g_s^{-2}}\right)$, which is convergent for $g_s << 1$ (perturbative region).

8.2 Different string theories[8]

For differen choices of the background $M^{1,9}$, one gets the following different theories:

[8]for the following see [18]

1. $M^{1,9} = \mathbb{R}^{1,9}$: In this case, there are 5 different string theories:

$$\left.\begin{array}{l}\bullet \text{II A} \\ \bullet \text{II B}\end{array}\right\} \text{left-right symmetric on } \Sigma \qquad \overset{E<<E_{PL}}{\longrightarrow} \quad \begin{array}{l}\text{II A supergravity} \\ \text{II B supergravity}\end{array}$$

$$\bullet \text{ I gauge group } G = SO(32) \qquad \overset{E<<E_{PL}}{\longrightarrow} \quad \text{I supergravity}$$

$$\bullet \begin{array}{l}\text{heterotic} \\ \text{string}\end{array} \left\{\begin{array}{l} G = SO(32) \\ G = E_8 \times E_8 \end{array}\right\} \begin{array}{l}\text{left-right} \\ \text{antisymmetric}\end{array} \quad \overset{E<<E_{PL}}{\longrightarrow} \quad \text{I supergravity}$$

2. $M^{1,9} = \mathbb{R}^{1,D-1} \times Y^{10-D}$: Here, Y is a compact manifold with volume $V >> l_s^{10-D}$. In this case there is a family of backgrounds and hence there is some freedom of choice:

$$\left.\begin{array}{l}\bullet \text{ discrete choice of } Y \text{ (discrete degeneracy)} \\ \bullet \text{ choice of moduli (continuous degeneracy)}\end{array}\right\} \begin{array}{l}\text{vacuum degeneracy of} \\ \text{string theory (landscape} \\ \text{of string theory)}\end{array}$$

   That suggests that even if string theory is unique, the degeneracy of the ground states is not.

3. $M^{1,9} = \mathbb{R}^{1,D-1} \times$ "SCFT": Here the SCFT is an "abstract SCFT" (with $c = \bar{c} = \frac{3}{2}(10 - D)$), i.e. there is no known geometrical interpretation. Developing a geometric notion of this case is part of the research in the area of quantum geometry. The volume V in this case is $V \approx l_s^{10-D}$.

8.3 Perturbative dualities

Perturbative dualities are dualities that hold at each order in g_s and hence are visible for $g_s \to 0$ (weak coupling region).

1. T-duality:
   Background: $\mathbb{R}^{1,8} \times \mathbb{S}^1$, coordinates: $X^M(\sigma, \tau) = (X^\mu, X^9)$ where $M = 0, \ldots, 9,\ \mu = 0, \ldots, 8$. As we are still looking at a closed string, we have the following condition for the string-coordinate: $X^9\,(\sigma = 2\pi, \tau) = X^9\,(\sigma = 0, \tau) + 2\pi l R$ Thus we get the spectrum of excitations: $m_{n,l}^2 = \frac{n^2}{R^2} + \frac{l^2 R^2}{\alpha'^2}$, where $n, l \in \mathbb{Z}$ and the first term denotes the Kaluza-Klein-modes, whereas the second term corresponds to the winding-modes, which are absent for pointlike particles. By this formula one computes that $m_{n,l}^2 \leftrightarrow m_{l,n}^2$ for $R \leftrightarrow \frac{\alpha'}{R}$. This is the so called T-duality. In type II theories one finds $IIA\,(R) = IIB\left(\frac{\alpha'}{R}\right)$.

   Remarks:

   - The T-duality does not exist for point particles

- It's a symmetry of the full string theory (not only the spectrum)
- It suggests a minimal length: $R = \sqrt{\alpha'}$
- For the background $\mathbb{R}^{1,D-1} \times T^{10-D}$ the T-duality group is $\Gamma_T = SO\left(10-D, 10-D, \mathbb{Z}\right)$

2. Mirror symmetry:
Mirror symmetry says that IIA in background $\mathbb{R}^{1,3} \times CY_3$ equals IIB in $\mathbb{R}^{1,3} \times \tilde{CY}_3$, where $\tilde{CY}_3$ is the according mirror Calabi-Yau manifold, i.e. $h^{1,1}\left(CY_3\right) = h^{1,2}\left(\tilde{CY}_3\right)$ and $h^{1,2}\left(CY_3\right) = h^{1,1}\left(\tilde{CY}_3\right)$. This symmetry should also hold for the entire theory.
For the type IIA and IIB theories, we have the following geometries:

$$M = \begin{cases} M_J^{(1,1)}(z) \times M_\Omega^{(1,2)}(t) \subset M_J^{(1,1)}(z) \times M_{QK}(t, \xi, \tilde{\xi}, \phi, a) \text{ for } IIA \\ M_J^{(1,1)}(\tilde{z}) \times M_\Omega^{(1,2)}(\tilde{t}) \subset M_{QK}(\tilde{z}, \xi, \tilde{\xi}, \tilde{\phi}, \tilde{a}) \times M_\Omega^{(1,2)}(\tilde{t}) \text{ for } IIB \end{cases}$$

Here $M_J^{(1,1)}$ and $M_\Omega^{(1,2)}$ are special Kähler manifolds, whereas M_{QK} is quaternionic Kähler and corresponds to the hypermultiplet. Therefore, the mirror map should identify $M_J^{(1,1)}(CY_3)$ with $M_\Omega^{(1,2)}(\tilde{CY}_3)$ and $M_\Omega^{(1,2)}(CY_3)$ with $M_J^{(1,1)}(\tilde{CY}_3)$. On the level of prepotentials, this means $F_J(CY_3) = F_\Omega(\tilde{CY}_3)$ and $F_\Omega(CY_3) = F_J(\tilde{CY}_3)$.

We check this in terms of the third derivatives $\partial_{z^a}\partial_{z^b}\partial_{z^c}F_J = F_{abc}$ and $\partial_{z^i}\partial_{z^j}\partial_{z^k}F_\Omega = F_{ijk}$ (Yukawa couplings). We have:

$$K_J = -ln \int J \wedge J \wedge J = -ln(F + \bar{F} - (z^a - \bar{z}^a)(F_a - \bar{F}_a)) \quad (207)$$

$$K_\Omega = -ln \int \Omega \wedge \Omega \quad (208)$$

And moreover:

$$F_{ijk} = \int \Omega \wedge \partial_i \partial_j \partial_k \Omega(t) \text{ and} \quad (209)$$

$$F_{abc}^{class} = \kappa_{abc} z^a z^b z^c \quad (210)$$

where $\kappa_{abc} = \int \omega_a^{1,1} \wedge \omega_b^{1,1} \wedge \omega_c^{1,1}$ the classical interaction numbers and $F_{abc} = F_{abc}^{class} + F_{abc}^{int}$ where F_{abc}^{int} are the worldsheet instanton corrections.

To compute these quantities, one can write F_{abc} in the following form:

$$F_{abc} = \langle O_a O_b O_c \rangle = \int [DX]\, O_a O_b O_c d^{-S} \quad (211)$$

where O_a, O_b, O_c are operators that we evaluate, and $S = S_{clas} + S_{inst}$, $S_{inst} = 2\pi i \sum_{a=1}^{n+1} m_a z^a$.

From a physical point of view, we are looking at holomorphic world sheet instantons ($\frac{\partial X^M}{\partial \bar{\sigma}} = 0$, $\sigma = \sigma + i\tau$, $\bar{\sigma} = \sigma - i\tau$), from a mathematical point of view, these are rational curves of genus 0, i.e. holomorphic maps from $\mathbb{P}_1$ to CY_3. One finds ([19]):

$$F_{abc}^{inst}(z) = \sum_{\vec{n}} N_{\vec{n}} n_a n_b n_c \frac{e^{2\pi i \sum_d n_d z^d}}{1 - e^{2\pi i \sum_d n_d z^d}} \tag{212}$$

Thus there exists a mirror map $t(z)$ relating $F_{abc}^{inst}(z)$ and $F_{ijk}(t)$.

If one draws these dualities into the diagram with the different theories, one gets the following picture:

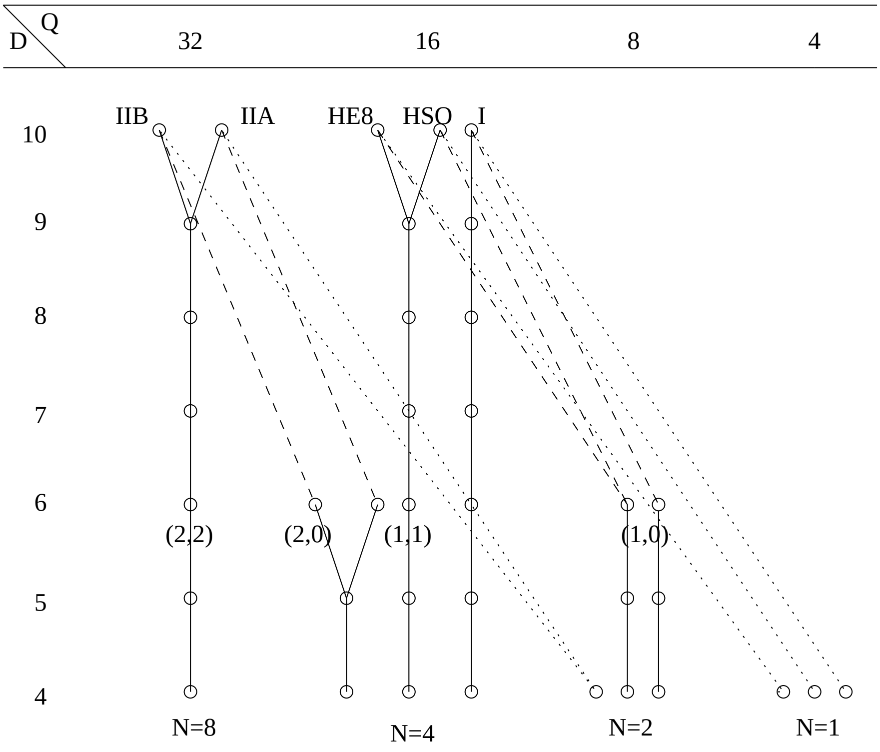

Figure 6: String theories and dualities

8.4 Non-perturbative dualities

Non-perturbative dualities involve the string coupling constant g_s.

Basic idea:
The strong coupling region of a given string theory should be described by "another" weakly coupled theory.
There are the following logical possibilites how this could be done:

1. by another string theory (S-duality)
2. by the same string theory (U-duality)
3. by another theory (M-theory)

But this is difficult to prove without a non-perturbative formulation of string theory. Therefore everything is conjectured and not yet proven, but there is "overwhelming evidence" instead. This means it has been checked for "special" couplings/ states, where non-perturbative corrections are known (exact) and thus can be extrapolated into the strong coupling region. Possible such states/ couplings are:

- BPS-states (i.e. $M = Z$)
- holomorphic couplings (e.g. prepotential $F(z)$)

D-branes:
As already mentioned, strings can be open or closed. Until now, we only considered the closed string. When looking at the open string $X^M(\sigma,\tau)$, $M = 0,\dots,9$, one can introduce boundary conditions. If these boundary conditions are imposed on $\partial X^\mu(\sigma_0)$, they are called *von Neumann* conditions. If they are imposed on $X^m(\sigma_0)$, they are *Dirichlet* conditions, and hence the D in the word "D-brane". The Dirichlet conditions define dynamical objects of string theory. Due to the attached open strings degrees of freedom, there are D-branes of different dimensions:

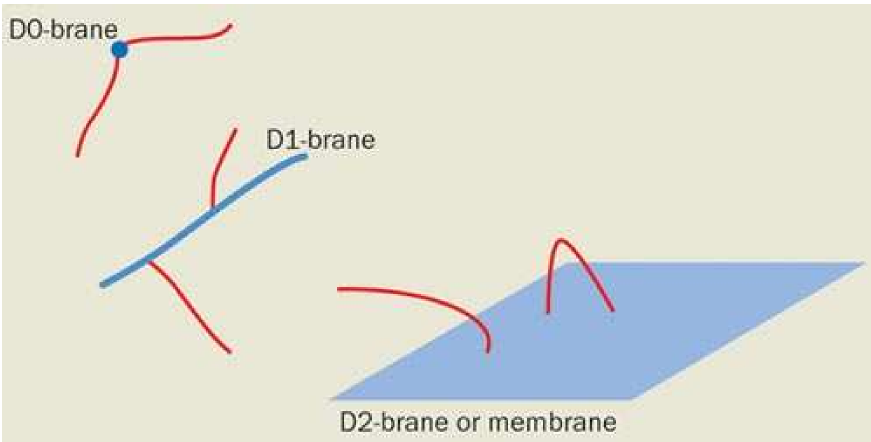

Figure 7: D-branes

The light modes correspond to the supersymmetric abelian vector multiplets. One can also look at a stack of N D-branes, where the open string can end either on the same or on different branes. This leads to an $U(N)$ gauge boson. As the mass/ tension are proportional to $\frac{1}{g_s}$, the D-branes are non-perturbative states of string theory, which is therefore not only a theory of strings, but also describes higher dimensional objects.

1. S-duality:
   Here, the strong coupling of a theory A is mapped to the weak coupling of a theory B, i.e. A and B are S-dual. This means $g_A \sim \frac{1}{g_B}$.
   Examples:

   - $Het\ SO(32)(g_s) \equiv type\ I(\frac{1}{g_s})$
     For the perturbative states, a string corresponds to a $D1$-string.

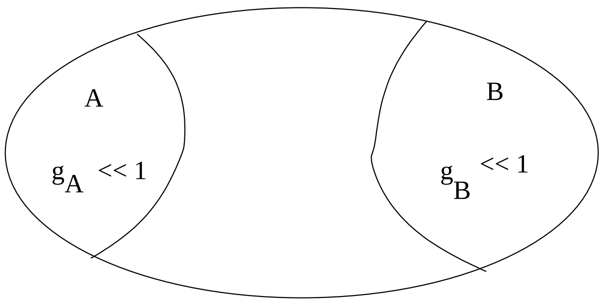

Figure 8: Duality

The non-perturbative states are more difficult and not yet checked. The supergravities coincide, i.e. $\mathcal{L}^{Het}_{supergravity}(g_s) = \mathcal{L}^{I}_{supergravity}(\frac{1}{g_s})$.

- $Het(\mathbb{R}^{1,5} \times T^4) \equiv IIA(\mathbb{R}^{1,5} \times K3)$

A closely related duality between two theories A and B is the case, if the string coupling is mapped to another geometric quantity: $g_A \leftrightarrow R_B$ and $R_A \leftrightarrow g_B$.
Examples:

- $IIA(\mathbb{R}^{1,5} \times CY_3) \equiv Het(\mathbb{R}^{1,5} \times K3)$
- $I(\mathbb{R}^{1,5} \times K3) \equiv Het(\mathbb{R}^{1,5} \times K3)$

2. U-duality:
In this case, the strong coupling limit is determined by the same theory (self duality).
Examples:

   - IIB in $\mathbb{R}^{1,9}$
   One can complexify the coupling constant $\tau = l + ie^{-\phi}$, where l is the RR scalar, ϕ corresponds to the dilaton and $e^{-\phi}|_{background} = \frac{1}{g_s}$. $SL(2,\mathbb{Z})$ acts on τ via $\tau \rightarrow \frac{a\tau+b}{c\tau+d}$, $a,b,c,d \in \mathbb{Z}$, $ad-bc = 1$. $\tau \rightarrow -\frac{1}{\tau}$ is the strong-weak coupling symmetry and implies $e^{-\phi} \rightarrow e^{\phi}$. One only has to look at a fundamental domain of possible values of τ, as the other ones are mapped into this domain by $SL(2,\mathbb{Z})$.

   - Het/T^6 also has $SL(2,\mathbb{Z})$.

3. M-theory:
Here, the strong coupling region is mapped to a new, different theory (not a string theory). Witten called this theory M-theory.
Examples:

- IIA in $\mathbb{R}^{1,9}$:
  IIA supergravity in $D = 10$ can be obtained from $D = 11$ supergravity by compactification on $\mathbb{S}^1$, where $R_{11} \sim g_A^{\frac{2}{3}}$. This means $\lim_{g_A \to \infty} IIA \to 11D-$theory.
  For the Kaluza-Klein states holds:

  $$m \sim \frac{1}{R_{11}} \sim \frac{1}{g_A^{\frac{2}{3}}} \to \begin{cases} \infty \text{ for } g_A \to 0 \\ 0 \text{ for } g_A \to \infty \end{cases}$$

  Therefore one can view them as non-perturbative states. These are BPS-states and can be identified with $D-0$-particles. Since there is no string theory with $D = 11$ this theory must be something new: M-theory
  Other strong coupling limits:
  - $M/I \leftarrow Het\, E_8 \times E_8$
  - $M/K3 \leftarrow Het/T^3$
  - $M/(T^5/\mathbb{Z}) \leftarrow IIB/K3$
  - $M/CY_3 \leftarrow Het/(K3 \times \mathbb{S}^1)$

  One therefore gets the following picture:

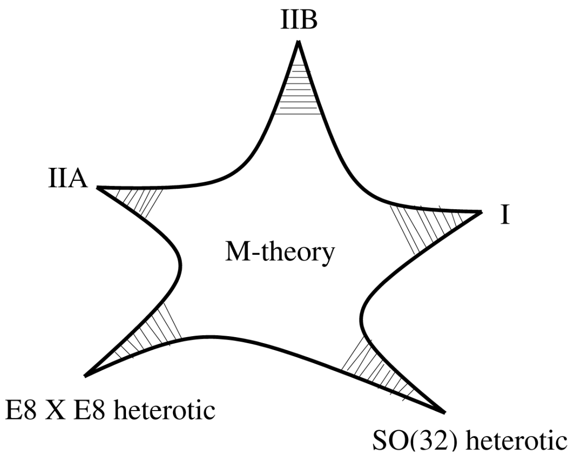

Figure 9: M-theory

Altogether, one can also draw these dualities into the diagram of the different string theories:

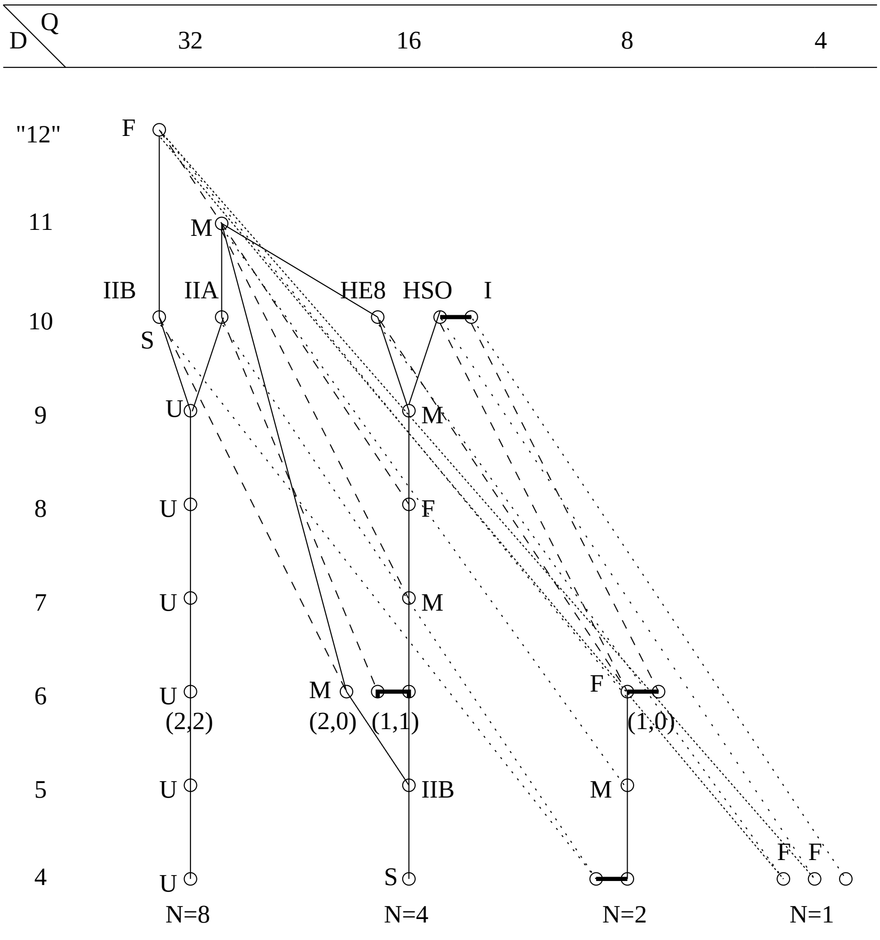

Figure 10: String theories and dualities

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