Quantum Field Theory II

Lecture notes by

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ABSTRACT

The lecture notes grew out of a course given at the University of Hamburg in the winter term 2007/2008 and the summer term 2011. A first version of these lecture notes were written by Jasper Hasenkamp, Manuel Meyer, Björn Sarrazin, Michael Grefe, Jannes Heinze, Sebastian Jacobs, Christoph Piefke.

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1 Lecture 1: Path Integral in Quantum mechanics

Let start by considering a non-relativistic particle in one dimension with a Hamiltonian

$$H = \frac{p^2}{2m} + V(x) \tag{1.1}$$

We are interested in the amplitude of the particle to travel from x_a to x_b in time T. In quantum mechanics this is given by the position representation of the Schrödinger time evolution operator

$$U(x_a, x_b, T) = \langle x_b | e^{-\frac{iHT}{\hbar}} | x_a \rangle \tag{1.2}$$

Feynman showed that there is an alternative representation known as the path integral given by

$$U(x_a, x_b, T) = \int Dx(t)\; e^{\frac{iS[x(t)]}{\hbar}} \ . \tag{1.3}$$

where

$$S[x(t)] = \int_0^T dt\; L = \int_0^T dt\big(\tfrac{m}{2}v^2 - V(x)\big) \ . \tag{1.4}$$

S depends on the path x(t) from $x_a \to x_b$. Mathematical it is a <u>functional</u> in that it maps functions (or a path) to numbers

$$S: \quad x(t) \to S[x(t)] \ . \tag{1.5}$$

The integration in (1.3) is over all possible path, i.e.

$$\int Dx(t) \ = \sum_{\text{all path from } x_a \to x_b.} \tag{1.6}$$

This is an integral over the function $x(t)$ and is called a <u>functional integral</u>.

Let us prove (1.3) by discretizing time and showing that both representations of U satisfy the same differential equation with the same boundary conditions. On a discrete time grid with spacing $\Delta t = \epsilon$ one has

$$S = \int_0^T dt\big(\tfrac{m}{2}\dot{x}^2 - V(x)\big) = \sum_k \Big(\frac{m}{2}\frac{(x_{k+1}-x_k)^2}{\epsilon} - \epsilon V\big(\frac{x_{k+1}+x_k}{2}\big)\Big) \ , \tag{1.7}$$

and

$$\int Dx(t) = \frac{1}{c(\epsilon)} \int_{-\infty}^{\infty} \frac{dx_1}{c(\epsilon)} \int \frac{dx_2}{c(\epsilon)} \cdots \int \frac{dx_{N-1}}{c(\epsilon)} = \frac{1}{c(\epsilon)} \prod_{k=1}^{N-1} \int_{-\infty}^{+\infty} \frac{dx_k}{c(\epsilon)} \ . \tag{1.8}$$

Consider the last time step

$$U(x_a, x_b, T) = \int_{-\infty}^{+\infty} \frac{dx'}{c(\epsilon)}\; e^{[\frac{i}{\hbar}(\frac{m}{2}\frac{(x_b-x')^2}{\epsilon} - \epsilon V(\frac{x_b+x'}{2}))]}\; \mathcal{U}(x_a, x', T-\epsilon) \ . \tag{1.9}$$

For $\epsilon \to 0$ the first factor is rapidly oscillating and thus a non-zero contribution only arises for $x' \sim x_b$. Taylor expanding around $x' = x_b$ yields

$$\begin{aligned} U(x_a, x_b, T) = \int_{-\infty}^{+\infty} \frac{dx'}{c(\epsilon)} e^{[\frac{i}{\hbar}\frac{m}{2\epsilon}(x_b - x')^2]} (1 - \frac{i\epsilon}{\hbar} V(x_b) + \mathcal{O}(\epsilon^2)) \cdot \\ \cdot (1 + (x' - x_b)\frac{\partial}{\partial x_b} + \frac{1}{2}(x' - x_b)^2 \frac{\partial^2}{\partial x_b^2} + \ldots)\, \mathcal{U}(x_a, x_b, T - \epsilon) \end{aligned} \tag{1.10}$$

Using the Gauss integrals

$$\int d\xi\ e^{-b\xi^2} = \sqrt{\frac{\pi}{b}}\ , \qquad \int d\xi\ \xi\ e^{-b\xi^2} = 0\ , \qquad \int d\xi\ \xi^2\ e^{-b\xi^2} = \frac{1}{2b}\sqrt{\frac{\pi}{b}}\ , \tag{1.11}$$

one obtains

$$U(x_a, x_b, T) = \frac{1}{c}\sqrt{\frac{\pi}{b}} \left[1 - \frac{i\epsilon}{\hbar}V(x_b) + \frac{1}{4b}\frac{\partial^2}{\partial x_b^2} + \mathcal{O}(\epsilon^2)\right] U(x_a, x_b, T - \epsilon)\ , \tag{1.12}$$

for $b = \frac{-im}{2\hbar\epsilon}$. For $\epsilon \to 0$ we obtain

$$U(x_a, x_b, T) = \frac{1}{c}\sqrt{\frac{\pi}{b}}\, U(x_a, x_b, T)\ , \tag{1.13}$$

which determines $c = \sqrt{\frac{\pi}{b}}$. With this normalization one can rewrite (1.12) as

$$i\hbar\, \frac{U(x_a, x_b, T) - U(x_a, x_b, T - \epsilon)}{\epsilon} = \left(V(x_b) - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_b^2}\right) U(x_a, x_b, T - \epsilon)\ , \tag{1.14}$$

which, in the limit $\epsilon \to 0$, yields

$$i\hbar\, \frac{\partial U(x_a, x_b, T)}{\partial T} = H\, U(x_a, x_b, T)\ . \tag{1.15}$$

From quantum mechanics we know that this is indeed the differential equation which the time evolution operator U satisfies. All that is left to check are the boundary conditions. Again from quantum mechanics we recall that in the limit $T \to 0$ (1.2) yields

$$U(x_a, x_b, T \to 0) \to \delta(x_a - x_b)\ . \tag{1.16}$$

In the path integral representation (1.3) we have

$$\lim_{T\to 0} \int Dx(t) e^{\frac{iS[x(t)]}{\hbar}} = \lim_{\epsilon \to 0} \frac{i}{c}\, e^{[\frac{1}{\hbar}(\frac{m}{2}\frac{(x_b - x_a)^2}{\epsilon} + \mathcal{O}(\epsilon^2))]} \to \delta(x_a - x_b)\ , \tag{1.17}$$

and thus both expressions also satisfies the same boundary conditions. Thus one can conclude the equality of (1.2) and (1.3).

Remarks:

1. The path integral formalism quantizes a system by summing over all path weighted with $e^{\frac{iS[x(t)]}{\hbar}}$.

2. In the classical limit $\hbar \to 0$ the path integral is dominated by the path which extremize S, i.e. the classical path.

3. The path integral depends only on classical quantities and no operators appear.

This method can be generalized to quantum systems with Hamiltonian $H(q^i, p^i)$ and canonical variables $q^i, p^i, \ i = 1, \ldots, n$. For such systems one has again the quantum mechanical expression

$$U(q_a, q_b, T) = \langle q_b | e^{-iHT} | q_a \rangle \ , \tag{1.18}$$

with $q_a \equiv \{q^i_{in}\}$, $q_b \equiv \{q^i_{fin}\}$. For this case the path integral representation is given by[1]

$$U(q_a, q_b, T) = \prod_i \int Dq^i(t) Dp^i(t) e^{[\frac{i}{\hbar} \int_0^T dt (\sum_j p^j \dot{q}^j - H(p,q))]} \ , \tag{1.19}$$

where $p^i \neq \dot{q}^i$. For

$$H = \sum_i \frac{p_i^2}{2m} + V(q) \ , \tag{1.20}$$

the p^i integrals are Gaussian and thus can be performed resulting in

$$U(q_a, q_b, T) = \prod_i \int Dq^i \ e^{\frac{iS[q(t)]}{\hbar}} \ . \tag{1.21}$$

[1]For a proof see, for example, [2].

2 Lecture 2: Path Integral for Scalar Fields

The generalization of path integral of quantum mechanical systems to field theories replaces the path of the particle by a field configuration or in other words

$$q^i(t) \to \phi(\vec{x}, t)\ . \tag{2.1}$$

One similarly defines the amplitude (setting $\hbar = 1$)

$$\langle \phi_b(\vec{x}, T)| e^{-2iHT} |\phi_a(\vec{x}, -T)\rangle = \int D\phi\, e^{iS[\phi]}\ , \qquad S[\phi] = \int_{-T}^{T} d^4y \mathcal{L}[\phi]\ , \tag{2.2}$$

with the boundary conditions

$$\begin{aligned} t = -T: &\quad \phi = \phi_a(\vec{x})\ , \\ t = T: &\quad \phi = \phi_b(\vec{x})\ . \end{aligned} \tag{2.3}$$

The integration $\int D\phi$ in (2.2) is over all field configuration with these boundary conditions.

Compared to canonical quantization the path integral quantization has the advantage that it is manifestly Lorentz-invariant and, furthermore, perturbation theory was not necessary to define the quantum theory.

Let us consider the quantity

$$I = \int D\phi\, \phi(x_1)\phi(x_2)\, e^{iS[\phi]}\ , \tag{2.4}$$

with boundary conditions (2.3) and show that it corresponds to a correlation function. To do so let us break up I into field configurations with the boundary condition that at times x_1^0 and x_2^0 the field configurations are fixed as

$$\phi(\vec{x}, x_1^0) = \phi_1(\vec{x})\ , \qquad \phi(\vec{x}, x_2^0) = \phi_2(\vec{x})\ , \tag{2.5}$$

and then integrate over ϕ_1, ϕ_2. Or in other words the integrations splits according to

$$\int D\phi = \int D\phi_1 \int D\phi_2 \int D\phi|_{\phi(\vec{x},x_1^0)=\phi_1(\vec{x}),\phi(\vec{x},x_2^0)=\phi_2(\vec{x})} \tag{2.6}$$

Now use

$$\begin{aligned} &\int D\phi|_{\phi(x_1^0)=\phi_1(\vec{x}),\phi(x_2^0)=\phi_2(\vec{x})}\, e^{iS[\phi]} = \\ &\qquad \langle \phi_b| e^{-iH(T-x_2^0)} |\phi_2\rangle \langle \phi_2| e^{-iH(x_2^0-x_1^0)} |\phi_1\rangle \langle \phi_1| e^{-iH(x_1^0-(-T))} |\phi_a\rangle \end{aligned} \tag{2.7}$$

for $x_1^0 < x_2^0$ and with an interchanged order for $x_2^0 < x_1^0$. Furthermore define the Schrödinger operator $\hat{\phi}_S$ via

$$\hat{\phi}_S(\vec{x}_{1,2})|\phi_{1,2}\rangle = \phi_{1,2}(\vec{x}_{1,2})|\phi_{1,2}\rangle\ , \tag{2.8}$$

and use

$$\int D\phi_1 |\phi_1\rangle\langle\phi_1| = \mathbb{1} \ . \tag{2.9}$$

Inserted into (2.4) using (2.7)-(2.9) yields

$$I = \langle\phi_b|e^{-iH(T-x_2^0)}\hat{\phi}_S(\vec{x}_2)e^{-iH(x_2^0-x_1^0)}\hat{\phi}_S(\vec{x}_1)e^{-iH(x_1^0+T)}|\phi_a\rangle \tag{2.10}$$

Expressed in terms of Heisenberg operators $\hat{\phi}_H(\vec{x},x^0) := e^{iHx^0}\hat{\phi}_S(\vec{x})e^{-iHx^0}$ one obtains

$$I = \langle\phi_b|e^{-iHT}\hat{\phi}_H(x_2)\hat{\phi}_H(x_1)e^{-iHT}|\phi_a\rangle \ . \tag{2.11}$$

If one now adds the expression for $x_2^0 < x_1^0$ one arrives at

$$I = \langle\phi_b|e^{-iHT}T\{\hat{\phi}_H(x_2)\hat{\phi}_H(x_1)\}e^{-iHT}|\phi_a\rangle \ . \tag{2.12}$$

In the canonical formalism we took the limit $T \to \infty(1-\epsilon)$ to single out the vacuum state via

$$e^{-iHT}|\phi_a\rangle = \sum_n e^{-iHT}|n\rangle\langle n|\phi_a\rangle \overset{T\to\infty(1-\epsilon)}{=} |\Omega\rangle\langle\Omega|\phi_a\rangle e^{-iE_0T} \tag{2.13}$$

By appropriate normalization, the factors and phases cancel out and we arrive at

$$\langle\Omega|T\{\hat{\phi}_H(x_2)\hat{\phi}_H(x_1)\}|\Omega\rangle = \lim_{T\to\infty(1-\epsilon)} \frac{\int D\phi\, \phi(x_1)\phi(x_2)\, e^{i\int_{-T}^{T}\mathcal{L}[\phi]}}{\int D\phi\, e^{i\int_{-T}^{T}\mathcal{L}[\phi]}} \ , \tag{2.14}$$

Similarly for an n-point function one has

$$\langle\Omega|T\{\hat{\phi}_H(x_1)\ldots\hat{\phi}_H(x_n)\}|\Omega\rangle = \lim_{T\to\infty(1-\epsilon)} \frac{\int D\phi\, \phi(x_1)\ldots\phi(x_n)\, e^{i\int_{-T}^{T}\mathcal{L}[\phi]}}{\int D\phi\, e^{i\int_{-T}^{T}\mathcal{L}[\phi]}} \ . \tag{2.15}$$

Let us now explicitly evaluate the two-point function for a free theory. To do so we discretize the space-time and replace $x \to x_i$ which lives on some lattice with volume $V = L^4$. The integration measure is thus replaced by an integration at each lattice site

$$D\phi \to \prod_i d\phi(x_i) \ . \tag{2.16}$$

The Fourier transformation of $\phi(x_i)$ is given by

$$\phi(x_i) = \frac{1}{V}\sum_n e^{-ik_{n\mu}x_i^\mu}\phi(k_n) \ , \qquad \text{with} \qquad k_{n\mu} = \frac{2\pi n_\mu}{L} \ , \quad n_\mu \in \mathbb{Z}^4 \ . \tag{2.17}$$

Since ϕ is real one also has $\phi^*(k) = \phi(-k)$. Inserted into the free action of a real scalar field one obtains

$$\begin{aligned} S &= \tfrac{1}{2}\int d^4x\big(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\big) \\ &= \tfrac{1}{2V}\sum_n \Big(k_n\cdot k_n\,\phi(k_n)\,\phi(-k_n) - m^2\phi(k_n)\,\phi(-k_n)\Big) \\ &= -\tfrac{1}{V}\sum_{k_n^0>0}\big(m^2-k_n^2\big)\big((\mathrm{Re}\phi)^2 + (\mathrm{Im}\phi)^2\big) \ , \end{aligned} \tag{2.18}$$

where we used

$$\tfrac{1}{V}\int d^4x e^{-i(k_n+k_m)\cdot x} = \delta(k_n+k_m)\ . \tag{2.19}$$

We can use Reϕ and Imϕ as independent integration variables for $k_n^0 > 0$

$$\prod_i d\phi(x_i) = c\prod_{k_n^0>0} d\mathrm{Re}\phi(k_n^0)d\mathrm{Im}\phi(k_n^0), \tag{2.20}$$

where c is a functional determinant from the change of variables which drops out later on. Since (2.18) is symmetric in Reϕ and Imϕ one obtains

$$\int D\phi e^{iS} = c\prod_{k_n^0>0}\left(\int d\mathrm{Re}\phi(k_n^0)e^{-\frac{i}{V}\left(m^2-k_n^2\right)(\mathrm{Re}\phi)^2}\right)^2 = c\prod_{k_n^0>0}\frac{-i\pi V}{m^2-k_n^2}\ , \tag{2.21}$$

where we used (1.11).

Now we need to compute the numerator $N \equiv \int D\phi\phi(x_1)\phi(x_2)\, e^{iS}$ of (2.14). Using (2.17) we arrive at

$$\begin{aligned} N = c\prod_{k_n^0>0}\int d\mathrm{Re}\phi(k_n)d\mathrm{Im}\phi(k_n)e^{-\frac{i}{V}\left(m^2-k_n^2\right)((\mathrm{Re}\phi(k_n))^2+(\mathrm{Im}\phi(k_n))^2)}\cdot \\ \cdot\tfrac{1}{V^2}\sum_{m,l}e^{-i(k_m\cdot x_1+k_l\cdot x_2)}(\mathrm{Re}\phi(k_m)+i\mathrm{Im}\phi(k_m))(\mathrm{Re}\phi(k_l)+i\mathrm{Im}\phi(k_l)) \end{aligned} \tag{2.22}$$

Since N is odd under $\mathrm{Re}\phi(k_n)\to-\mathrm{Re}\phi(k_n)$ it vanishes unless $k_m = \pm k_l$. For $k_m = +k_l$ the terms proportional to $(\mathrm{Re}\phi(k_n))^2$ cancel against the terms proportional to $(\mathrm{Im}\phi(k_n))^2$ and thus N vanishes again. Only for $k_m = -k_l$ there is a non-zero contribution due to the sign from $\phi^*(k) = \phi(-k)$. Using (1.11) again one obtains

$$\begin{aligned} N = c\prod_{k_n^0>0}\int d\mathrm{Re}\phi(k_n)d\mathrm{Im}\phi(k_n)\, e^{-\frac{i}{V}\left(m^2-k_n^2\right)\left((\mathrm{Re}\phi(k_n))^2+(\mathrm{Im}\phi(k_n))^2\right)}\ \cdot \\ \cdot\tfrac{2}{V^2}\sum_m e^{-i(k_m\cdot(x_1-x_2))}(\mathrm{Re}\phi(k_m))^2 \\ = \frac{c}{V}\prod_{k_n^0>0}\frac{-i\pi V}{m^2-k_n^2}\sum_m e^{-i(k_m\cdot(x_1-x_2)}\ \frac{-i}{m^2-k_m^2}\ . \end{aligned} \tag{2.23}$$

Inserting (2.21) and (2.23) into (2.14) we arrive at

$$\langle 0|T\{\hat\phi(x_1)\hat\phi(x_2)\}|0\rangle = \frac{-i}{V}\sum_m\frac{e^{-i\left(k_m\cdot(x_1-x_2)\right)}}{m^2-k_m^2-i\epsilon} \tag{2.24}$$

Finally, in the continuum limit one replaces $V^{-1}\sum_m \to \int\frac{d^4k}{(2\pi)^4}$ and thus obtains

$$\langle 0|T\{\hat\phi(x_1)\hat\phi(x_2)\}|0\rangle = i\int\frac{d^4k}{(2\pi)^4}\frac{e^{-i\left(k\cdot(x_1-x_2)\right)}}{k^2-m^2+i\epsilon} = G_F(x_1-x_2)\ . \tag{2.25}$$

Thus by explicitly computation one can show that in the free theory the path integral representation of correlation function coincides with expressions obtained in canonical quantization.

3 Lecture 3: The Generating Functional

Defining a quantum theory via the path integral does not require the use of any perturbation theory and n-point correlations functions are defined in (2.15). However, if the theory has a perturbative regime one can expand the path integral perturbatively in the corresponding coupling and express it in terms of n-point functions of the free theory.

It is thus useful to obtain simpler formulas for such a perturbation theory and consider

$$\langle 0|T\{\phi(x_1)\dots\phi(x_n)\}|0\rangle = \frac{\int D\phi\,\phi(x_1)\dots\phi(x_n)\,e^{iS_0[\phi]}}{\int D\phi\,e^{iS_0[\phi]}}\ , \tag{3.1}$$

where $S_0[\phi]$ is the action of the free theory. One can compute (2.15) directly as we did in the last lecture or with the help of a generating functional which we want to introduce in this lecture.

Let us first define the functional derivative $\frac{\delta}{\delta J(x)}$ via[2]

$$\frac{\delta J(y)}{\delta J(x)} = \delta(x-y)\ . \tag{3.2}$$

or

$$\frac{\delta}{\delta J(x)}\int d^4y J(y)\phi(y) = \int d^4y\delta(x-y)\phi(y) = \phi(x)\ . \tag{3.3}$$

The functional derivative obeys all rules of a derivative, in particular

$$\frac{\delta}{\delta J(x)}\,e^{i\int d^4yJ(y)\phi(y)} \quad = \quad i\phi(x)e^{i\int d^4yJ(y)\phi(y)}\ , \tag{3.4}$$

which can be shown by expanding the exponential. If a space-time derivative is acting on a function one partial integrates and obtains

$$\frac{\delta}{\delta J(x)}\int d^4yV^\mu(y)\frac{\partial}{\partial y^\mu}J(y) = -\frac{\delta}{\delta J(x)}\int d^4y\big(\frac{\partial V^\mu(y)}{\partial y^\mu}\big)\ J(y) = -(\partial_\mu V^\mu)\ , \tag{3.5}$$

where it was assumed that the fields vanish a (spatial) infinity.

One defines the generating functional $Z[J]$ by

$$Z[J] := \int D\phi\, e^{i\int d^4x\left(\mathcal{L}+J(x)\phi(x)\right)}\ . \tag{3.6}$$

With this definition we can check

$$\langle\Omega|T\{\phi(x_1)\phi(x_2)\}|\Omega\rangle = \frac{1}{Z_0}(-i\frac{\delta}{\delta J(x_1)})(-i\frac{\delta}{\delta J(x_2)})Z[J]\Big|_{J=0}\ , \tag{3.7}$$

where we abbreviated $Z_0 \equiv Z[J=0]$. For n-point functions we have analogously

$$\langle\Omega|T\{\phi(x_1)\dots\phi(x_n)\}|\Omega\rangle = \frac{1}{Z_0}(-i\frac{\delta}{\delta J(x_1)})\dots(-i\frac{\delta}{\delta J(x_n)})Z[J]\Big|_{J=0}\ . \tag{3.8}$$

[2]This is the functional analog of $\frac{\partial}{\partial x_i}\sum_j x_jk_j = k_i$.

This formula is useful since $Z[J]$ can be explicitly computed for the free theory. For concreteness let us consider $\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2)$ which can be rewritten as

$$I = \int d^4x(\mathcal{L}_0+J\phi) = \int d^4x\tfrac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2) + J\phi = -\int d^4x\tfrac{1}{2}\phi(\Box+m^2)\phi - J\phi\ . \quad (3.9)$$

Changing variables with

$$\phi(x) = \phi'(x) + i\int d^4y G_F(x-y)J(y) \quad (3.10)$$

and using

$$(\Box + m^2)G_F(x-y) = i\delta^{(4)}(x-y) \quad (3.11)$$

results in

$$I = -\int d^4x\tfrac{1}{2}(\phi'(\Box+m^2)\phi') + \frac{i}{2}\int d^4xd^4yJ(y)G_F(x-y)J(x)\ . \quad (3.12)$$

The change of variables (3.10) is just a ϕ-independent shift in the path integral and thus the Jacobian is unity. As a consequence we arrive at

$$Z[J] = \int D\phi e^{i\int d^4x(\mathcal{L}_0+J\phi)} = Z_0 e^{-\frac{1}{2}\int d^4xd^4yJ(x)G_F(x-y)J(y)}\ . \quad (3.13)$$

Let us check this expression by computing the two-point function. Inserting (3.13) into (3.7) we obtain

$$\begin{aligned}\langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle &= \frac{1}{Z_0}\Big(-i\frac{\delta}{\delta J(x_1)}\Big)\Big(-i\frac{\delta}{\delta J(x_2)}\Big)Z_0 e^{-\frac{1}{2}\int d^4xd^4yJ(x)G_F(x-y)J(y)}\Big|_{J=0}\\ &= \frac{\delta}{\delta J(x_1)}\int d^4yG_F(x_2-y)J(y)\frac{Z[J]}{Z_0}\Big|_{J=0}\\ &= G_F(x_1-x_2)\ .\end{aligned} \quad (3.14)$$

Another useful quantity is

$$E[J] := i\ln Z[J] \quad (3.15)$$

Let us compute

$$\frac{\delta E[J]}{\delta J(x)} = \frac{i}{Z[J]}\frac{\delta Z[J]}{\delta J(x)} = -\frac{\int D\phi\phi(x)e^{i\int d^4x\mathcal{L}+J\phi}}{\int D\phi e^{i\int d^4x\mathcal{L}+J\phi}} \equiv -\langle\phi(x)\rangle_J \quad (3.16)$$

Furthermore

$$\begin{aligned}\frac{\delta^2 E[J]}{\delta J(x)\delta J(y)} &= \frac{i}{Z[J]}\frac{\delta^2 Z[J]}{\delta J(x)\delta J(y)} - \frac{i}{Z[J]^2}\frac{\delta Z[J]}{\delta J(y)}\frac{\delta Z[J]}{\delta J(x)}\\ &= -i\big(\langle\phi(x)\phi(y)\rangle_J - \langle\phi(x)\rangle_J\langle\phi(y)\rangle_J\big)\ ,\end{aligned} \quad (3.17)$$

where we abbreviate

$$\frac{i}{Z[J]}\frac{\delta^2 Z[J]}{\delta J(x)\delta J(y)} = i^3\frac{\int D\phi\phi(x)e^{i\int d^4x\mathcal{L}+J\phi}}{\int D\phi e^{i\int d^4x\mathcal{L}+J\phi}} \equiv -i(\langle\phi(x)\phi(y)\rangle_J\ . \quad (3.18)$$

From (3.17) we see that the second term $\langle\phi(x)\rangle_J\langle\phi(y)\rangle_J$ subtracts the disconnected pieces so that we have

$$\frac{\delta^2 E}{\delta J(x)\delta J(y)} = -i\langle\phi(x)\phi(y)\rangle_{connected} = -iG(x-y) \ , \tag{3.19}$$

where $G(x-y)$ is the full quantum propagator.

Similarly one finds ($J_i \equiv J(x_i)$)

$$\begin{aligned}
\frac{\delta^3 E[J]}{\delta J_1 \delta J_2 \delta J_3} &= \frac{\delta}{\delta J_3}\left(\frac{i}{Z[J]}\frac{\delta^2 Z[J]}{\delta J_1 \delta J_1} - \frac{i}{Z^2}\frac{\delta Z}{\delta J_1}\frac{\delta Z}{\delta J_2}\right) \\
&= \frac{i}{Z}\frac{\delta^3 Z}{\delta J_1 \delta J_2 \delta J_3} - \frac{i}{Z^2}\frac{\delta Z}{\delta J_3}\frac{\delta^2 Z}{\delta J_1 \delta J_2} \\
&\quad - \frac{i}{Z^2}\left(\frac{\delta^2 Z}{\delta J_1 \delta J_3}\frac{\delta Z}{\delta J_2} + \frac{\delta Z}{\delta J_1}\frac{\delta^2 Z}{\delta J_2 \delta J_3}\right) + \frac{2i}{Z^3}\frac{\delta Z}{\delta J_3}\frac{\delta Z}{\delta J_2}\frac{\delta Z}{\delta J_1} \\
&= \langle\phi_1\phi_2\phi_3\rangle - \langle\phi_3\rangle\langle\phi_1\phi_2\rangle - \langle\phi_2\rangle\langle\phi_1\phi_3\rangle - \langle\phi_1\rangle\langle\phi_2\phi_3\rangle + 2\langle\phi_1\rangle\langle\phi_2\rangle\langle\phi_3\rangle \\
&= \langle\phi_1\phi_2\phi_3\rangle_{connected} \ .
\end{aligned} \tag{3.20}$$

In general one obtains

$$\frac{\delta^n E[J]}{\delta J_1 \ldots \delta J_n} = i^{n+1}\langle\phi_1 \ldots \phi_n\rangle_{connected} \ . \tag{3.21}$$

Thus $E[J]$ is the generating functional for connected correlation functions.

4 Lecture 4: Path Integral Quantization of the Electromagnetic Field

The classical Maxwell action for the photon field A_μ is expressed in terms of its field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and given by

$$S = -\tfrac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} = \tfrac{1}{2}\int d^4x A_\mu D^{\mu\nu} A_\nu \qquad \text{for} \qquad D^{\mu\nu} = \eta^{\mu\nu}\Box - \partial^\mu\partial^\nu \ , \tag{4.1}$$

where the second equation was obtained by partial integration. S is gauge invariant in that it obeys

$$S[A^\alpha_\mu] = S[A_\mu] \qquad \text{for} \qquad A_\mu \to A^\alpha_\mu = A_\mu + \tfrac{1}{e}\partial_\mu\alpha \tag{4.2}$$

Furthermore $S[A_\mu] = 0$ for $A_\mu = \frac{1}{e}\partial_\mu\alpha$ and thus the path integral for A_μ is ill-defined.[3] The cure of this problem is to integrate only over gauge in-equivalent field configurations. One way to implement this is the Fadeev-Popov procedure which we present in this lecture. However, we need to define <u>functional determinant</u> and <u>functional δ-functions</u>.

For a matrix M an integral representation of $\det(M)$ is proved in problem 1.1 to be

$$\sqrt{\det(M)} = (2\pi)^{N/2}\int_{-\infty}^{\infty} dx_1 \dots dx_N \, e^{-\frac{1}{2}x^T\cdot M\cdot x} \ . \tag{4.3}$$

This can be generalized to a <u>functional determinant</u> via a path integral representation

$$\sqrt{\det(M)} = \int D\phi \, e^{-\frac{1}{2}\int d^4x \phi(x) M(x)\phi(x)} \ , \tag{4.4}$$

where M is a differential operator, e.g. $M = \Box + m^2$.

For an N-dimensional δ-function one has the identity

$$\int_{-\infty}^{\infty} dx_1 \dots dx_N \, \delta^{(N)}(\vec{g}(x_1,\dots,x_N)) \det\big(\frac{\partial g_i}{\partial x_j}\big) = 1 \ , \tag{4.5}$$

which can analogously be generalized as

$$\int D\alpha \, \delta(G(A^\alpha)) \det\Big(\frac{\delta G(A^\alpha)}{\delta\alpha}\Big) = 1 \ . \tag{4.6}$$

Inserted into the path integral we obtain

$$I = \int DA \, e^{iS[A]} = \int DADα \, \delta(G(A^\alpha)) \det\Big(\frac{\delta G(A^\alpha)}{\delta\alpha}\Big) e^{iS[A]} \ . \tag{4.7}$$

Choosing

$$G(A^\alpha) = \partial^\mu A^\alpha_\mu - \omega(x) = \partial^\mu A_\mu + \tfrac{1}{e}\Box\alpha - \omega(x) \ , \tag{4.8}$$

with $\omega(x)$ being an arbitrary scalar function, we see that $\frac{\delta G(A^\alpha)}{\delta\alpha}$ is independent of A and α and thus can be moved outside the path integral. Furthermore, we can change

[3] In problem 1.3 it is shown that correspondingly the Greens function of $D^{\mu\nu}$ is ill-defined.

variables $A \to A^\alpha$. This implies $DA \to DA^\alpha$ since the change is merely a A-independent shift, and $S[A] = S[A^\alpha]$ since S is gauge invariant. Thus we arrive at

$$I = \det\Big(\frac{\delta G(A^\alpha)}{\delta\alpha}\Big) \int DA^\alpha D\alpha\, \delta(G(A^\alpha)) e^{iS[A^\alpha]} \ . \tag{4.9}$$

Since A^α is an (arbitrary) integration variable we can rename it back to A and factor out the infinite-dimensional factor $\int D\alpha$ which causes the problem, i.e.

$$I = \det\Big(\frac{\delta G(A^\alpha)}{\delta\alpha}\Big)\Big(\int D\alpha\Big) \int DA\, \delta(G(A)) e^{iS[A]} \ . \tag{4.10}$$

$\delta(G(A))$ can be expressed as a correction to S by integrating over all $\omega(x)$ with Gaussian weight centered around $\omega(x) = 0$. This yields

$$\begin{aligned} I' &= N(\xi) \int D\omega\, e^{-i\int d^4x \frac{\omega^2}{2\xi}} I \\ &= N(\xi) \det\Big(\frac{\delta G(A^\alpha)}{\delta\alpha}\Big)\Big(\int D\alpha\Big) \int DA\, e^{iS[A]} \int D\omega\, e^{-i\int d^4x \frac{\omega^2}{2\xi}} \delta\big(\partial^\mu A_\mu - \omega(x)\big) \\ &= N(\xi) \det\Big(\frac{\delta G(A^\alpha)}{\delta\alpha}\Big)\Big(\int D\alpha\Big) \int DA\, e^{iS'[A]} \ , \end{aligned} \tag{4.11}$$

where N is a normalization factor and

$$S' = S - \tfrac{1}{2\xi} \int d^4x\, (\partial^\mu A_\mu)^2 = \tfrac{1}{2} \int d^4x A_\mu D'^{\mu\nu} A_\nu \ , \tag{4.12}$$

with

$$D'^{\mu\nu} = \eta^{\mu\nu}\Box - (1 - \tfrac{1}{\xi})\partial^\mu\partial^\nu \ . \tag{4.13}$$

We see that S' has no problems with longitudinal photons since $D'^{\mu\nu}\partial_\nu\alpha \neq 0$ for $\xi \neq \infty$.

In problem 1.3 we show that for arbitrary ξ the photon propagator is given by[4]

$$G(x-y) = -i \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \big(\eta^{\mu\nu} - (1-\xi)\tfrac{k^\mu k^\nu}{k^2}\big) e^{-ik(x-y)} \ . \tag{4.14}$$

It does depend on ξ but, due to the Ward identity, the amplitudes are ξ-independent. Thus the path integral definition of correlation function is given by

$$\langle\Omega|T\{O(A)\}|\Omega\rangle \ = \ \lim_{T\to\infty(1-\epsilon)} \frac{\int DA\, O(A)\, e^{i\int_{-T}^{T}\left(\mathcal{L} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2\right)}}{\int D\phi\, e^{i\int_{-T}^{T}\left(\mathcal{L} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2\right)}} \ . \tag{4.15}$$

Another way to discover this gauge fixing term is to note that the canonical momentum π^0 vanishes for the action (4.1) and the theory cannot be properly canonically quantized. From (4.1) one computes

$$\pi^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_0 A_\mu)} = -F^{0\mu} \ , \tag{4.16}$$

[4]Note that in QFT I we used the photon propagator in the gauge $\xi = 1$ (Feynman gauge).

which implies $\pi^0 = 0$. This can be remedied by augmenting the action (4.1) by a gauge fixing term

$$-\int d^4x \Big(\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2\xi} (\partial^\mu A_\mu)^2 \Big) = \tfrac{1}{2} \int d^4x A_\mu D^{\mu\nu} A_\nu \tag{4.17}$$

for

$$D^{\mu\nu} = \eta^{\mu\nu} \Box - \partial^\mu \partial^\nu (1 - \frac{1}{\xi}) \ . \tag{4.18}$$

$D^{\mu\nu}$ is now invertible and $\pi^0 = -\xi^{-1}(\partial^\mu A_\mu)$. Therefore canonical quantization is possible.

5 Lecture 5: Path Integral for Fermions

Fermionic operators obey anti-commutation relations and therefore fermions need to be represent classical anti-commuting fields. This is achieved by Grassman numbers and Grassman valued fields. Let θ, η, χ be Grassman numbers which obey the following properties.

1. The anti-commutation relation
$$\theta\eta = -\theta\eta \ , \tag{5.1}$$
which implies $\theta^2 = 0$.

2. The product of two Grassman numbers is commuting
$$(\theta\eta)\chi = -\theta\chi\eta = \chi(\theta\eta) \ . \tag{5.2}$$

3. Grassman numbers can be added with the sum being again a Grassmann number
$$\theta + \eta = \chi \ . \tag{5.3}$$

4. Multiplication with c-numbers leave any Grassmann number anti-commuting
$$c\theta = \theta c = \chi \ . \tag{5.4}$$

5. Due to (5.1) any function of Grassman variables $f(\theta, c)$ can be at most linear in θ
$$f(\theta, c) = A(c) + B(c)\theta \ . \tag{5.5}$$

6. Differentiation obeys a graded Leibniz rule
$$\frac{\mathrm{d}}{\mathrm{d}\eta}(\theta\eta) = \frac{\mathrm{d}\theta}{\mathrm{d}\eta}\eta - \theta\frac{\mathrm{d}\eta}{\mathrm{d}\eta} = -\theta \ . \tag{5.6}$$

7. Integration (Berezin integration) is defined by
$$\int \mathrm{d}\theta\, 1 = 0, \qquad \int \mathrm{d}\theta\theta = 1 \tag{5.7}$$
such that
$$\int \mathrm{d}\theta f(\theta, c) = B(c) \ . \tag{5.8}$$
The definition (5.7) preserves a variable shift, i.e.
$$\int \mathrm{d}\theta(\theta + \eta) = \int \mathrm{d}\theta\theta - \eta\int \mathrm{d}\theta = \int \mathrm{d}\theta\theta = 1 \ . \tag{5.9}$$
Let us further choose the convention
$$\int \mathrm{d}\theta\mathrm{d}\eta\, \eta\theta = 1$$

8. Complex Grassman numbers are defined by

$$\theta := \tfrac{1}{\sqrt{2}}(\theta_1 + i\theta_2) \ , \qquad \theta^* = \tfrac{1}{\sqrt{2}}(\theta_1 - i\theta_2) \ , \tag{5.10}$$

with $\theta_{1,2}$ real. Then

$$(\theta\eta)^* := \eta^*\theta^* = -\theta^*\eta^* \ . \tag{5.11}$$

The integration over complex θ is defined as in (5.7).

Later on we need

$$\int d\theta^* d\theta e^{-\theta^* b\theta} = \int d\theta^* d\theta(1-\theta^* b\theta) = b\int d\theta^* d\theta\theta\theta^* = b \ , \qquad b\in\mathbb{C} \ . \tag{5.12}$$

Note that an ordinary Gauß-integral results in $\frac{2\pi}{b}$. We also need

$$\int d\theta^* d\theta\, \theta\theta^* e^{-\theta^* b\theta} = \int d\theta^* d\theta\theta\theta^*(1-\theta^* b\theta) = \int d\theta^* d\theta\theta\theta^* = 1 = \frac{b}{b} \ . \tag{5.13}$$

Thus adding $\theta\theta^*$ in the integrand gives an additional $\frac{1}{b}$ after integration.

If there are N Grassman variables $\theta_i, i = 1,\ldots,N$ the product can be written as

$$\prod_{i=1}^{N}\theta_i = \tfrac{1}{N!}\,\epsilon^{i_1\ldots i_N}\theta_{i_1}\ldots\theta_{i_N} \qquad \text{or} \qquad \theta_{i_1}\ldots\theta_{i_N} = \epsilon_{i_1\ldots i_N}\prod_{k=1}^{N}\theta_k \ . \tag{5.14}$$

Under a unitary transformation $\theta_i \to \theta'_i = \sum_j U_{ij}\theta_j, UU^\dagger = 1$ one derives

$$\begin{aligned}\prod_{k=1}^{N}\theta'_k &= \tfrac{1}{N!}\epsilon^{i_1\ldots i_N}U_{i_1 i'_1}\ldots U_{i_N i'_N}\theta_{i'_1}\ldots\theta_{i'_N} \\ &= \tfrac{1}{N!}\epsilon^{i_1\ldots i_N}U_{i_1 i'_1}\ldots U_{i_N i'_N}\epsilon_{i'_1}\ldots\theta_{i'_N}\prod_{k=1}^{N}\theta_k \\ &= \det U\prod_{k=1}^{N}\theta_k \ ,\end{aligned} \tag{5.15}$$

where the second equation used (5.14). Using (5.14) and (5.15) one derives (see problem 3.2)

$$\begin{aligned}\int \Big(\prod_i d\theta_i^* d\theta_i\Big)\, e^{-\sum_{k,j}\theta_k^* B_{kj}\theta_j} &= \det B \\ \int \Big(\prod_i d\theta_i^* d\theta_i\Big)\, \theta_n\theta_l^*\, e^{-\sum_{k,j}\theta_k^* B_{kj}\theta_j} &= B^{-1}_{nl}\det B \ .\end{aligned} \tag{5.16}$$

Now we can Dirac spinors $\psi_a(x)$, $a = 1,\ldots,4$ to be Grassmann valued or in other words to be classically anti-commuting fields. The path integral can be computed in analogy with bosonic fields by Fourier transforming and discretizing the measure. In this way one confirms (5.16) and computes

$$\int D\bar{\psi}D\psi e^{i\int d^4x\mathcal{L}_0[\psi,\bar{\psi}]} = c\,\det(i\partial\!\!\!/ - m) \ , \tag{5.17}$$

for $\mathcal{L}_0 = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ with c being a normalization constant. For the propagator one obtains (cf. (5.16))

$$\begin{aligned} \langle 0|T\{\psi(x_1)\bar{\psi}(x_2)\}|0\rangle &= \frac{\int D\bar{\psi}D\psi e^{i\int \mathrm{d}^4x\mathcal{L}_0[\psi,\bar{\psi}]}\psi(x_1)\bar{\psi}(x_2)}{\int D\bar{\psi}D\psi e^{i\int \mathrm{d}^4x\mathcal{L}_0[\psi,\bar{\psi}]}} \\ &= \frac{c\cdot \det(i\partial\!\!\!/ - m)}{c\cdot \det(i\partial\!\!\!/ - m)}\, S_F(x_1-x_2) = S_F(x_1-x_2)\ . \end{aligned} \tag{5.18}$$

The generating functional for fermions is defined by

$$Z[\bar{\eta},\eta] := \int D\bar{\psi}D\psi\, e^{i\int \mathrm{d}^4x\left(\mathcal{L}[\psi,\bar{\psi}]+\bar{\eta}(x)\psi(x)+\bar{\psi}(x)\eta(x)\right)}\ , \tag{5.19}$$

where $\bar{\eta}(x)$ and $\eta(x)$ are Grassman valued sources, i.e. the analog of $J(x)$ for scalar fields. For the free theory $\mathcal{L} = \mathcal{L}_0$ Z is computed in problem 3.3 to be given by

$$Z[\bar{\eta},\eta] = Z[\bar{\eta}=0,\eta=0]\, e^{-\int \mathrm{d}^4x\mathrm{d}^4y\, \bar{\eta}(x)S_F(x-y)\eta(y)}\ . \tag{5.20}$$

Defining the functional derivative of Grasmmann fields by

$$\frac{\delta\eta(x)}{\delta\eta(y)} = \delta(x-y)\ , \tag{5.21}$$

one shows

$$\langle 0|T\{\psi(x_1)\bar{\psi}(x_2)\}|0\rangle = Z[0]^{-1}\Big(-i\frac{\delta}{\delta\bar{\eta}(x_1)}\Big)\Big(+i\frac{\delta}{\delta\eta(x_2)}\Big)Z[\eta,\bar{\eta}]\Big|_{\eta=\bar{\eta}=0}\ . \tag{5.22}$$

Using (5.19) one checks (5.18) and using (5.20) one confirms that the left hand side of (5.22) also is given by $S_F(x_1-x_2)$.

6 Lecture 6: Symmetries in the Path Integral Formalism

The aim of this section is to derive the Schwinger-Dyson equation and a quantum version of the Noether theorem or in other words the Ward-identities for a generic QFT.

6.1 Schwinger-Dyson equation

Let us start with a free scalar field

$$S = \int d^4x \mathcal{L}_0 = \int d^4x \tfrac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2) = -\int d^4x \tfrac{1}{2}\phi(\Box + m^2)\phi \ . \tag{6.1}$$

and consider

$$\langle\Omega|T\{\phi(x_1)\ldots\phi(x_n)\}|\Omega\rangle \ = \ \frac{\int D\phi\, \phi_1 \ldots \phi_n\, e^{iS[\phi]}}{\int D\phi\, e^{iS[\phi]}} \ , \tag{6.2}$$

where for later use we already gave the formula for an interacting QFT and use the notation $\phi_i \equiv \phi(x_i)$.

The classical equation of motion for ϕ leaves S stationary under the arbitrary variation

$$\phi(x) \to \phi'(x) = \phi(x) + \epsilon(x) \ . \tag{6.3}$$

In the path integral the measure is invariant under this shift, i.e. $D\phi = D\phi'$. This implies

$$\int D\phi\, \phi_1 \ldots \phi_n\, e^{iS[\phi]} = \int D\phi'\, \phi'_1 \ldots \phi'_n\, e^{iS[\phi']} = \int D\phi\, \phi'_1 \ldots \phi'_n\, e^{iS[\phi']} \ , \tag{6.4}$$

where in the first step we merely changed the names of the integration variable from ϕ to ϕ' while in the second step we used the invariance of the measure under the shift (6.3). Now we expand the right hand side to first order in ϵ to arrive at

$$\begin{aligned} 0 = \int D\phi\, e^{iS[\phi]}\Big(-\tfrac{i}{2}\int d^4y\big(\epsilon(y)(\Box_y + m^2)\phi(y) + \phi(y)(\Box_y + m^2)\epsilon(y)\big)\phi_1 \ldots \phi_n \\ + \epsilon_1\phi_2 \ldots \phi_n + \phi_1\epsilon_2 \ldots \phi_n + \phi_1\phi_2 \ldots \epsilon_n\Big) \ . \end{aligned} \tag{6.5}$$

By partially integrating twice the first two terms can be combined and one obtains

$$0 = -i\int D\phi\, e^{iS[\phi]} \int d^4y \epsilon(y)\Big((\Box_y + m^2)\phi(y)\phi_1 \ldots \phi_n + i\sum_{i=1}^{n} \delta(y - x_i)\phi_1 \ldots \hat{\phi}_i \ldots \phi_n\Big) \ , \tag{6.6}$$

where $\hat{\phi}_i$ denotes the field which is omitted from the sum. Since (6.6) should hold for any ϵ e can drop the $\int d^4y\epsilon(y)$ from the equation. Let us first consider the case $n = 1$. We can move the Klein-Gordon operator out of the path integral to arrive at

$$(\Box_y + m^2)\int D\phi\, e^{iS[\phi]}\phi(y)\phi(x_1) = -i\delta(y - x_1)\, Z[0] \ , \tag{6.7}$$

or equivalently

$$(\Box_y + m^2)\langle\Omega|T\{\phi(y)\phi(x_1)|\Omega\rangle = -i\delta(y - x_1) \ . \tag{6.8}$$

This confirms once more $\Omega|T\{\phi(y)\phi(x_1)|\Omega\rangle = G_F((y-x_1)$. For n arbitrary one obtains analogously

$$(\Box_y + m^2)\langle\Omega|T\{\phi(y)\phi(x_1)\ldots\phi(x_n)|\Omega\rangle = -i\sum_{i=1}^{n}\langle\Omega|T\{\phi(x_1)\ldots\delta(y-x_i)\ldots\phi(x_n)|\Omega\rangle \ . \tag{6.9}$$

Note that the previous derivation only depended on the invariance of the measure $D\phi = D\phi'$ and thus holds for any QFT! Let us denote an arbitrary field by φ and its action by $S[\varphi] = \int d^4x\mathcal{L}[\varphi(x), \partial_\mu\varphi]$. Using the functional chain rule one obtains

$$\frac{\delta S[\varphi]}{\delta\varphi(y)} = \int d^4x\Big(\frac{\delta\mathcal{L}}{\delta\varphi(x)}\,\delta(x-y) + \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi(x)}\,\partial_\mu\delta(x-y)\Big) = \frac{\delta\mathcal{L}}{\delta\varphi(y)} - \partial_\mu\,\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi(x)} \ , \tag{6.10}$$

where we partially integrated in the second step. We see that the functional derivative of S gives the Euler-Lagrangian equations. With this fact we can redo the previous computation and first use the invariance of the measure $D\varphi = D\varphi'$ under $\varphi \to \varphi' = \varphi + \epsilon$ to obtain

$$\int D\varphi\,\varphi_1\ldots\varphi_n\,e^{iS[\varphi]} = \int D\varphi'\,\varphi'_1\ldots\varphi'_n\,e^{iS[\varphi']} = \int D\varphi\,\varphi'_1\ldots\varphi'_n\,e^{iS[\varphi']} \ , \tag{6.11}$$

Next we expand

$$S[\varphi'] = S[\varphi] + \int d^4y\epsilon(y)\frac{\delta S[\varphi]}{\delta\varphi(y)} + \mathcal{O}(\epsilon^2) \tag{6.12}$$

and insert it into (6.11). This yields

$$0 = \int D\varphi\,e^{iS[\varphi]}\Big(i\int d^4y\epsilon(y)\frac{\delta S[\varphi]}{\delta\varphi(y)}\,\varphi_1\ldots\varphi_n + \sum_{i=1}^{n}\varphi_1\ldots\epsilon_i\ldots\varphi_n\Big) \tag{6.13}$$

We drop again the $\int d^4y\epsilon(y)$ to arrive at the Schwinger-Dyson equation

$$\langle\Omega|\,\frac{\delta S[\varphi]}{\delta\varphi(y)}\,T\{\phi(x_1)\ldots\phi(x_n)|\Omega\rangle = i\sum_{i=1}^{n}\langle\Omega|T\{\phi(x_1)\ldots\delta(y-x_i)\ldots\phi(x_n)|\Omega\rangle \ . \tag{6.14}$$

It states that the classical Euler-Lagrange equations are obeyed by all n-point functions up to contact terms (the terms on the right hand side).

Note that the functional derivative is outside the time-ordering in (6.14). In the following it will be convenient to introduce a separate notation for this situation and simply write

$$\langle\frac{\delta S[\varphi]}{\delta\varphi(y)}\,\phi(x_1)\ldots\phi(x_n)\rangle = i\sum_{i=1}^{n}\langle\phi(x_1)\ldots\delta(y-x_i)\ldots\phi(x_n)\rangle \ , \tag{6.15}$$

to denote that the derivative is outside the time-ordering.

6.2 Ward identities

Let us first consider a free complex scalar field with the Lagrangian

$$\mathcal{L}_0 = \partial_\mu\phi\partial^\mu\phi^* - m^2\phi\phi^* \tag{6.16}$$

It has a symmetry

$$\phi \to \phi' = e^{i\alpha}\phi\ , \quad \alpha\in\mathbb{R}\ , \qquad \text{or infinitesimally} \quad \delta\phi = i\alpha\phi\ , \tag{6.17}$$

which leaves $\mathcal{L}_0$ invariant $\mathcal{L}_0[\phi] = \mathcal{L}_0[\phi']$. The transformation $\delta\phi = i\alpha(x)\phi$ is not a symmetry and $\mathcal{L}_0$ changes

$$\delta\mathcal{L}_0 = \big(\partial_\mu\alpha(x)\big)j^\mu(x)\ , \qquad \text{for} \qquad j^\mu(x) = i\big(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi\big)\ . \tag{6.18}$$

It is easy to check that j^μ is the Noether current of the symmetry (6.17) and therefore conserved $\partial_\mu j^\mu = 0$ if one uses the equation of motion.

We can now perform analogous steps as in the previous section with the difference that the transformation of ϕ is not arbitrary (as in (6.3)) but as in (6.18). The analog of (6.4) for a complex scalar field reads

$$\int D\phi D\phi^*\, \phi_1\ldots\phi_n\, e^{iS[\phi,\phi^*]} = \int D\phi D\phi^*\, \phi'_1\ldots\phi'_n\, e^{iS[\phi',\phi'^*]}\ , \tag{6.19}$$

where we used again the invariance of the measure. (The ϕ_i could also be ϕ_i^* or any mixture.) Expanding to first order in α we obtain

$$0 = \int D\phi D\phi^*\, e^{iS[\phi,\phi^*]}\Big(i\int d^4y(\partial_\mu\alpha(y))j^\mu(y)\phi_1\ldots\phi_n + \sum_{i=1}^n \phi_1\ldots\delta\phi_i\ldots\phi_n\Big)\ , \tag{6.20}$$

Partially integrating and dropping $\int d^4y\,\alpha(y)$ we arrive at the <u>Ward identity</u>

$$0 = \int D\phi D\phi^*\, e^{iS[\phi,\phi^*]}\Big((\partial_\mu j^\mu(y))\phi_1\ldots\phi_n + i\sum_{i=1}^n \phi_1\ldots(\pm i)\delta(y-x_i)\phi_i\ldots\phi_n\Big)\ , \tag{6.21}$$

where the sign ambiguity comes from considering $\delta\phi$ or $\delta\phi^*$. In the "bracket-notation" introduced in (6.15) it reads

$$\langle\partial_\mu j^\mu(y)\,\phi(x_1)\ldots\phi(x_n)\rangle = -i\sum_{i=1}^n\langle\phi(x_1)\ldots(\pm i)\delta(y-x_i)\phi_i\ldots\phi(x_n)\rangle\ , \tag{6.22}$$

This analysis can be generalized to an arbitrary theory characterized by $S[\varphi_a]$ with a global symmetry

$$\varphi_a \to \varphi'_a = \varphi_a + \epsilon\Delta\varphi_a\ , \tag{6.23}$$

such that

$$\mathcal{L}[\varphi_a] \to \mathcal{L}[\varphi'_a] = \mathcal{L}[\varphi_a] + \epsilon\partial_\mu\mathcal{J}^\mu\ , \qquad S[\varphi'_a] = S[\varphi_a] \tag{6.24}$$

Replacing $\epsilon\to\epsilon(x)$ the transformation of $\mathcal{L}$ is modified according to

$$\mathcal{L}[\varphi_a] \to \mathcal{L}[\varphi'_a] = \mathcal{L}[\varphi_a] + \epsilon\partial_\mu\mathcal{J}^\mu + (\partial_\mu\epsilon)\sum_a\Delta\varphi_a\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)}\ , \tag{6.25}$$

which results is the change

$$S[\varphi'_a] = \int d^4y\, \mathcal{L}[\varphi'_a] = \int d^4y \left(\mathcal{L}[\varphi_a] - \epsilon\partial_\mu j^\mu\right) = S[\varphi_a] - \int d^4y\, \epsilon\partial_\mu j^\mu \ , \tag{6.26}$$

with j^μ being the Noether current

$$j^\mu = \sum_a \Delta\varphi_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)} - \mathcal{J}^\mu \ . \tag{6.27}$$

Therefore the Ward identity (6.22) is modified as

$$\langle \partial_\mu j^\mu(y)\, \varphi_{a_1}(x_1)\ldots\varphi_{a_n}(x_n)\rangle = -i\sum_{i=1}^{n}\langle\varphi_{a_1}(x_1)\ldots\delta(y-x_i)\Delta\varphi_{a_i}(x_i)\ldots\phi_{a_n}(x_n)\rangle \ . \tag{6.28}$$

Let us check this for QED. In this case we have the Lagrangian

$$\mathcal{L} = \bar\psi(i\partial\!\!\!/ - m)\psi + A_\mu j^\mu \ , \qquad \text{for} \qquad j^\mu = e\bar\psi\gamma^\mu\psi \ . \tag{6.29}$$

It has a global symmetry

$$\begin{aligned} \psi \to \psi' &= e^{i\alpha}\psi \ , \quad \delta\psi = i\alpha\psi \ , \\ A_\mu \to A'_\mu &= A_\mu \ . \end{aligned} \tag{6.30}$$

If we replace $\alpha \to \alpha(x)$ but do not transform the A_μ we obtain the transformation law

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} - i(\partial_\mu\alpha)j^\mu \ . \tag{6.31}$$

Inserted into (6.28) for $n = 2$ we get

$$\begin{aligned} \partial_\mu\langle 0|T\{j^\mu(y)\,\psi(x_1)\bar\psi(x_2)\}|0\rangle = \delta(y-x_1)\langle 0|T\{\psi(x_1)\bar\psi(x_2)\}|0\rangle \\ - \delta(y-x_2)\langle 0|T\{\psi(x_1)\bar\psi(x_2)\}|0\rangle \ , \end{aligned} \tag{6.32}$$

or in Fourier space

$$k_\mu\langle 0|T\{j^\mu(k)\,\psi(q)\bar\psi(p)\}|0\rangle = \langle 0|T\{\psi(k-q)\bar\psi(p)\}|0\rangle - \langle 0|T\{\psi(-q)\bar\psi(p+k)\}|0\rangle \ . \tag{6.33}$$

The right hand side does not contribute to the S-matrix as can be seen from the LSZ formula. Thus we confirmed the QED Ward identity

$$k_\mu\langle 0|T\{j^\mu(k)\,\psi(q)\bar\psi(p)\}|0\rangle = 0 \ . \tag{6.34}$$

7 Lecture 7: Renormalization of ϕ^4

In QFT I we discussed the renormalization of QED. In this section we recall the concept of renormalization for the example of a ϕ^4-theory.

We introduced a quantity D which represents the superficial degree of divergence of a Feynman diagram. It counts the power of k-factors in the numerator minus the power of k-factors in the denominator

$$\int^{\Lambda} d^4k\, \frac{k^{a-4}}{k^b} \;=\; \begin{cases} \Lambda^D & \text{for} \quad D = a - b \neq 0 \\ \ln \Lambda & \text{for} \quad D = 0 \end{cases} \tag{7.1}$$

For $D \geq 0$ a Feynman diagram is naively divergent.

Before we focus on a ϕ^4-theory it is useful to consider a more general ϕ^n-theory defined by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) + \tfrac{\lambda}{n!}\, \phi^n \ . \tag{7.2}$$

The propagator is represented by a line and is in momentum space proportional to $(k^2 + m^2)^{-1}$. The interaction is represented by a vertex with n lines and proportional to λ.

The following relations hold:

$$i) \quad L = P - V + 1 \ , \tag{7.3}$$

where L is the number of loops, P counts the number of internal propagators and V the number of vertices. (7.3) holds since each propagator has a momentum integral but each vertex has a δ-function (momentum conservation) and the $+1$ expresses the overall momentum conservation.

$$ii) \quad nV = 2P + N \ , \tag{7.4}$$

where N counts the number of external lines. (7.4) holds since out of each vertex comes n lines. They can be external (N) or internal (P). The factor of two accounts for the fact that an internal line always connects two vertices while an external line does not.

$$iii) \quad D = 4L - 2P \tag{7.5}$$

since each loop contributes a factor d^4k and each internal propagator a k^{-2}. As a consequence of (7.3) and (7.4) one derives

$$D = 4(P - V + 1) - 2P = (n-4)V - N + 4 \ . \tag{7.6}$$

The last expression is useful as it shows that D is independent of P. Recall that in QED we obtained $D = 4 - N_\gamma - \frac{3}{2}N_e$ where $N_e(N_\gamma)$ counts the number of external fermion(photon) lines. Hence D solely depends on the number of external legs and thus shows that only diagrams with a small number of external legs can have a UV divergence. In our case we have the same situation for $n = 4$.

Let us consider different values of n.

$n = 2$: This is the free theory.

$n = 3$: In this case one has $D = 4 - N - V$ and a finite number of divergent diagrams. Such theories are called super-renormalizable

$n = 4$: In this case one has $D = 4 - N$ and a finite number of divergent amplitudes. Such theories are called renormalizable but the divergence occurs at all orders in perturbation theory.

$n > 4$: In this case one has $D = (n - 4)V - N + 4$ and an infinite number of divergences-
occurs. Such theories are called non-renormalizable.

These properties can also be formulated in terms of the dimension of the coupling λ which we denote by $[\lambda]$ in the following. In natural units $\hbar = c = 1$ one has

$$\text{length} = \text{mass}^{-1} = \text{time} = \text{energy}^{-1} \ . \tag{7.7}$$

It is conventional to give the dimensions of the couplings either in units of mass or length. In the following we will use mass dimensions. With this conventions one has

$$[S] = 0 \ , \quad [d^4x] = -4 \ , \quad [\mathcal{L}] = 4 \ , \quad [\partial_\mu] = 1 \ . \tag{7.8}$$

For a ϕ^4-theory we then determine from (7.2)

$$[\phi] = 1 \ , \quad [\lambda] = 4 - n \ . \tag{7.9}$$

Inserted into (7.6) we get

$$D = 4 - N - [\lambda]V \ , \tag{7.10}$$

and see that for $[\lambda] < 0$ the theory is non-renormalizable.

Before we continue let consider QED with the Lagrangian

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m)\psi + eA_\mu\bar{\psi}\gamma^\mu\psi \ . \tag{7.11}$$

In this case we readily determine

$$[\psi] = \tfrac{3}{2} \ , \quad [A_\mu] = 1 \ , \quad [e] = 0 \ . \tag{7.12}$$

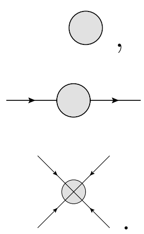
So we have a dimensionless coupling e exactly as for a ϕ^4-theory. Let us now turn to the renormalization of the latter.

For a ϕ^4-theory we have $D = 4 - N$ which is semi-positive for all diagrams with zero to four external legs. Due to the symmetry $\phi \to -\phi$ diagrams with N odd cannot occur and we are left with

$D = 0$,

$D = 2$,

$D = 4$. (7.13)



In renormalized perturbation theory one starts from the 'bare' Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m_0^2\phi^2) + \tfrac{1}{4!}\,\lambda_0\phi^4 \tag{7.14}$$

with the 'bare' parameters m_0, λ_0. One then defines renormalized field variables and renormalized couplings as

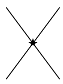
$$\phi = \sqrt{Z}\,\phi_r\ , \qquad Z = 1+\delta_Z\ , \qquad Z m_0 = m+\delta_m\ , \qquad Z\lambda_0 = \lambda+\delta_\lambda\ . \tag{7.15}$$

Inserted into (7.14) one obtains

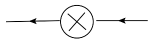
$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m_0^2\phi^2) + \tfrac{1}{4!}\,\lambda_0\phi^4 \\ &= \tfrac{1}{2}(\partial_\mu\phi_r\partial^\mu\phi_r - m^2\phi_r^2) + \tfrac{1}{4!}\,\lambda\phi_r^4 + \tfrac{1}{2}\delta_Z\partial_\mu\phi_r\partial^\mu\phi_r - \tfrac{1}{2}\delta_m\phi_r^2 + \tfrac{\delta_\lambda}{4!}\,\phi_r^4\end{aligned} \tag{7.16}$$

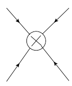
The prescription now is to do perturbation in λ (instead of λ_0) with the redefined Lagrangian (7.16) instead of (7.14). For this we need new Feynman rules. The first three terms in (7.16) lead to the Feynman rules

 $= \dfrac{1}{p^2-m^2}$,

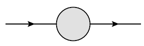
 $= -i\lambda$. (7.17)

The last three terms are called counterterms and they are denoted by

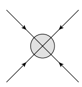
 $= i(p^2\delta_Z - \delta_m)$,

 $= -i\delta_\lambda$. (7.18)

Now one defines the split given in (7.15) by the requirement that the renormalized field ϕ_r has a propagator like a free field with the renormalized mass m being the position of the pole. In other words

 $= \dfrac{1}{p^2-m^2} \quad + \quad \text{regular at} \quad p^2 = m^2$ (7.19)

Furthermore, at $\vec{p} = 0$ one imposes

 $= -i\lambda$, (7.20)

where in these diagrams all contribution of the counterterms are included.

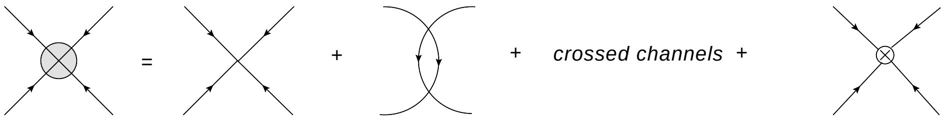
Now one follows the following procedure:

1. Compute the divergent diagrams in perturbation theory and regulate them.
2. Impose the renormalization conditions (7.19), (7.20) and in this way determine $\delta_m, \delta_\lambda, \delta_Z$.

After this procedure all amplitudes are finite and independent of the regulator. Concretely for the ϕ^4 at one-loop the following diagrams contribute

, (7.21)

and



, (7.22)

The diagrams in (7.21) contribute

$$\frac{1}{p^2-m^2} - \tfrac{i}{2}\lambda \int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2} + i(p^2\delta_Z - \delta_m) \tag{7.23}$$

which, imposing (7.19), results in (see problem 4.1)

$$\delta_Z = 0\ , \qquad \delta_m = -\lim_{d\to 4}\frac{\lambda}{2(4\pi)^{d/2}}\,\frac{\Gamma(1-\frac{d}{2})}{m^{2-d}}\ . \tag{7.24}$$

The diagrams in (7.22) contribute

$$-i\lambda + (-i\lambda)^2\big(iV(s) + iV(t) + iV(u) + \big) - i\delta_\lambda \tag{7.25}$$

for

$$s = (p_1+p_2)^2\ , \qquad t = (p_1-p_3)^2\ , \qquad s = (p_1-p_4)^2\ , \qquad s+t+u = 4m^2\ , \tag{7.26}$$

and

$$\begin{aligned} iV(p^2) :=& \tfrac{i}{2}\lambda \int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2}\,\frac{i}{(k+p)^2-m^2} \\ =& -\tfrac{i}{2}\lim_{d\to 4}\frac{1}{(4\pi)^{d/2}}\int_0^1 dx\,\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \\ =& -\tfrac{1}{32\pi^2}\lim_{\epsilon\to 0}\int_0^1 dx\,\Big(\frac{2}{\epsilon} - \ln\big(\frac{\Delta e^\gamma}{4\pi}\big) + \mathcal{O}(\epsilon)\Big)\ , \end{aligned} \tag{7.27}$$

for $\Delta = m^2 - x(1-x)p^2$. Imposing (7.20), results in

$$\delta_\lambda = -\lambda^2\big(V(4m^2) + 2V(0)\big)\ . \tag{7.28}$$

$V(p^2)$ and the final finite amplitude is computed in problem 4.2.

8 Lecture 8: Wilson's Approach to Renormalization

The Wilsonian approach to renormalization is the 'modern' view of a quantum field theory and it offers a more physical way to understand the role of UV-divergences. The basic idea is to define any QFT with a cut-off Λ and analyze the theory successively at different length or equivalently momentum scales. Concretely this is achieved by integrating over the short-distance or high-momentum fluctuations of a quantum field.

As an example let us consider a ϕ^4 theory with a momentum cut-off Λ in a Euclidean space-time, i.e. with $x^0 = ix^0$.[5] In this case the path integral reads

$$Z = \int D\phi_\Lambda \, e^{-S[\phi]} \; , \qquad D\phi_\Lambda := \prod_{|k|<\Lambda} d\phi(k) \; , \tag{8.1}$$

for

$$S[\phi] = \int d^4x \mathcal{L} \; , \qquad \mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi\partial^\mu\phi + m^2\phi^2) + \tfrac{1}{4!}\,\lambda\phi^4 \; . \tag{8.2}$$

Now one splits off the high-frequency Fourier-modes of ϕ via

$$\phi \to \phi + \hat{\phi} \; , \tag{8.3}$$

where the momenta of ϕ obey $|k| < b\Lambda, b < 1$ while those of $\hat{\phi}$ lie in the high-momentum shell $b\Lambda < |k| \leq \Lambda$. Inserted into $\mathcal{L}$ yields

$$S = S[\phi] + \hat{S}[\phi, \hat{\phi}] \; , \tag{8.4}$$

with

$$\begin{aligned} \hat{S}[\phi,\hat{\phi}] &= \int d^4x \big(\hat{\mathcal{L}}_0[\hat{\phi}] + \hat{\mathcal{L}}_{int}[\phi,\hat{\phi}]\big) \\ \hat{\mathcal{L}}_0[\hat{\phi}] &= \tfrac{1}{2}\partial_\mu\hat{\phi}\partial^\mu\hat{\phi} \\ \hat{\mathcal{L}}_{int}[\phi,\hat{\phi}] &= \tfrac{1}{2}m^2\hat{\phi}^2 + \lambda\big(\tfrac{1}{3!}\,\phi^3\hat{\phi} + \tfrac{1}{4}\phi^2\hat{\phi}^2 + \tfrac{1}{3!}\,\phi\hat{\phi}^3 + \tfrac{1}{4!}\hat{\phi}^4\big) \; . \end{aligned} \tag{8.5}$$

Note that we treat $\frac{1}{2}m^2\hat{\phi}^2$ as an interaction. Furthermore, there are no terms proportional to $\phi\hat{\phi}$ since in Fourier-space they would vanish due to (8.3) which enforces

$$\int \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4}\; \phi(k)\hat{\phi}(k')e^{i(k+k')x} \sim \delta(k+k') = 0 \; . \tag{8.6}$$

Within this setup one now performs the path integral over $\hat{\phi}$ and expresses it as a correction to $S[\phi]$

$$Z = \int D\phi_\Lambda \, e^{-S[\phi]} = \int D\phi_{b\Lambda} D\hat{\phi}_\Lambda \, e^{-S[\phi,\hat{\phi}]} = \int D\phi_{b\Lambda} e^{-S_{eff}[\phi]} \; , \tag{8.7}$$

with

$$S_{eff}[\phi] = S[\phi] + \delta S[\phi] \; , \qquad \text{where} \qquad e^{-\delta S[\phi]} \equiv \int D\hat{\phi}_\Lambda \, e^{-\hat{S}[\phi,\hat{\phi}]} \; . \tag{8.8}$$

[5]The Euclidean cut-off ensure that all momenta are below Λ.

$S_{eff}[\phi]$ can be computed perturbatively for $m \ll \Lambda, \lambda \ll 1$. One defines a $\hat\phi$ propagator by

$$\overline{\hat\phi(k)\,\hat\phi(p)} = \frac{\int D\hat\phi\,\hat\phi(k)\hat\phi(p)e^{-S_0[\phi]}}{\int D\hat\phi\, e^{-S_0[\phi]}} = \frac{(2\pi)^4}{k^2}\,\delta(k+p)\,\theta(k) \tag{8.9}$$

where

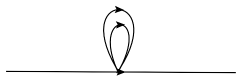
$$\theta(k) = \begin{cases} 1 & \text{if } b\Lambda \le |k| \le 1 \\ 0 & \text{otherwise} \end{cases} \quad , \tag{8.10}$$

and denotes it diagrammatically by a double line. Now one does perturbation theory where ϕ only appears on external lines, $\hat\phi$ only appears as an internal propagator. The resulting diagrams are then viewed as corrections to the couplings of $S_{eff}[\phi]$, i.e. as corrections to m and λ but also higher order terms are generated in $S_{eff}[\phi]$. This yields generically

$$\mathcal{L}_{eff} = \tfrac{1}{2}(1+\Delta Z)\partial_\mu\phi\partial^\mu\phi + \tfrac{1}{2}(m^2+\Delta m^2)\phi^2 + \tfrac{1}{4!}(\lambda+\Delta\lambda)\phi^4 + \text{higher order}\ , \tag{8.11}$$

where the higher order terms include $\phi^6, (\partial\phi)^4$, etc..

As an example let us compute

$I_1 =$ 

$$\begin{aligned} &= -\tfrac{1}{4}\lambda \int d^4x\phi^2\overline{\hat\phi\hat\phi} \\ &= -\tfrac{1}{4}\lambda \int d^4x\frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_4}{(2\pi)^4}e^{i(k_1+\ldots k_4)x}\phi(k_1)\phi(k_2)\frac{(2\pi)^4}{k_3^2}\,\delta(k_3+k_4)\,\theta(k_3)\ , \end{aligned} \tag{8.12}$$

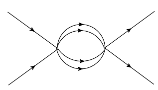
where we went to momentum space and used (8.9). Performing the integral over k_4 and x we arrive at

$$\begin{aligned} I_1 &= -\tfrac{1}{4}\lambda \int \frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_3}{(2\pi)^4}\phi(k_1)\phi(k_2)\frac{(2\pi)^4}{k_3^2}\,\delta(k_1+k_2)\,\theta(k_3) \\ &\equiv -\tfrac{1}{2}\mu \int \frac{d^4k_1}{(2\pi)^4}\,\phi(k_1)\phi(-k_1)\ , \end{aligned} \tag{8.13}$$

with

$$\mu = \tfrac{1}{2}\lambda \int \frac{d^4k_3}{(2\pi)^4}\,\frac{\theta(k_3)}{k_3^2} = \frac{\lambda}{2(2\pi)^4}\int d\Omega \int_{b\Lambda}^{\Lambda} dkk = \frac{\lambda}{32\pi^2}\,(1-b^2)\Lambda^2\ . \tag{8.14}$$

Similarly one can compute

$I_2 =$  $= -\tfrac{1}{4!}\xi \int d^4x\phi^4(x)$ (8.15)

with

$$\xi = -\frac{3\lambda^2}{16\pi^2}\,\ln\frac{1}{b}\ . \tag{8.16}$$

This procedure can be performed perturbatively for each coupling to arbitrary order computing $\Delta Z, \Delta m^2, \Delta\lambda, \ldots$. On the other hand inspecting (8.7) we see that the two path integrals are related by the transformation

$$k' = k/b \ , \qquad x' = bx \tag{8.17}$$

such that

$$k'x' = kx \ , \qquad |k| \leq b\Lambda \quad \Rightarrow \quad |k'| \leq \Lambda \ , \tag{8.18}$$

Inserted into (8.11) we obtain

$$\begin{aligned} S_{eff} &= \int d^4x' b^{-4} \Big(\tfrac{1}{2}(1+\Delta Z) b^2 \partial'_\mu \phi \partial'^\mu \phi + \tfrac{1}{2}(m^2 + \Delta m^2)\phi^2 + \tfrac{1}{4!}\,(\lambda + \Delta\lambda)\phi^4 + \ldots \Big) \\ &= \int d^4x' \Big(\tfrac{1}{2}\partial'_\mu \phi' \partial'^\mu \phi' + \tfrac{1}{2} m'^2 \phi'^2 + \tfrac{1}{4!}\,\lambda' \phi'^4 + \ldots + \tfrac{1}{n!}\,\lambda'_n \phi'^n + \ldots \Big) \end{aligned} \tag{8.19}$$

where

$$\begin{aligned} \phi' &= \sqrt{(1+\Delta Z) b^{-1}}\ \phi \ , \\ m'^2 &= (m^2 + \Delta m^2)(1+\Delta Z)^{-1} b^{-2} \ , \\ \lambda' &= (\lambda + \Delta\lambda)(1+\Delta Z)^{-2} \ , \\ \lambda'_n &= (\lambda_n + \Delta\lambda_n)(1+\Delta Z)^{-n/2} b^{n-4} \ , \\ &\ldots . \end{aligned} \tag{8.20}$$

Therefore integration over a momentum shell $b\Lambda \leq |k| \leq \Lambda$ can be viewed as a transformation of $\mathcal{L}$. Doing it again one successively integrates out the high-momentum or short-distance fluctuations. For $b \approx 1$ the transformation can be viewed as continuous and is called a renormalization group transformation.[6] Precisely due to the cut-off dependence the couplings become also scale dependent which is seen from their b dependence.

Let us focus on this point more concretely. For simplicity one starts at a point $m^2 = \lambda = \lambda_n = 0$ in coupling space and looks for the first order change. From (8.20) one then obtains

$$\begin{aligned} m'^2 &= b^{-2} \Delta m^2 \ , \\ \lambda' &= b^0 \Delta\lambda \ , \\ \lambda'_n &= b^{n-4} \Delta\lambda_n \ . \end{aligned} \tag{8.21}$$

We see that for $b < 1$ the mass m^2 grows. Growing couplings are called <u>relevant</u>. For λ on the other hand one needs further information to determine its behavior. Couplings which scale with b^0 are called a <u>marginal</u> couplings. Finally λ_n grows for $n < 4$ and is relevant, as we already said is marginal for $n = 4$ and decreases for $n > 4$. Couplings which decrease are called <u>irrelevant</u>. Obviously the scaling behavior of a coupling is related to its mass dimension in that λ_n has mass dimension $[\lambda_n] = 4 - n$ and scales with b^{n-4}. At large distance, i.e. in the infrared, only relevant and marginal couplings have to be considered. These are precisely the renormalizable couplings.

[6]This is a misnomer since mathematically there is no underlying group.

Finally let us inspect the scale dependence of the marginal coupling λ more closely. From (8.16) one finds explicitly

$$\lambda' = \lambda - \frac{3\lambda^2}{16\pi^2} \ln \frac{1}{b} \, , \tag{8.22}$$

i.e. a slow logarithmic decrease. The logarithmic correction of λ is precisely the correction computed in the previous lecture and problem 4.2.

9 Lecture 9: Callan-Symanzik Equation

In the last lecture we saw that in a Wilsonian picture, i.e. for a quantum field theory with a cut-off Λ, the couplings are scale dependent. However, a finite cut-off has the technical problem of generically violating the Ward identities and therefore it is often more convenient to send $\Lambda \to \infty$ and instead recover the scale dependence of the couplings from a modification of the renormalization conditions. In lecture 7 we discussed the on-shell renormalization scheme in that we imposed the renormalization conditions (7.19) and (7.20) on-shell at $p^2 = m^2$. However this choice is by no means mandatory and we could instead impose these conditions at an arbitrary space-like momentum $p^2 = -M^2$. This would again remove all UV-divergences but the counterterms as well as the renormalized couplings now depend on M^2. Thus also in this set-up one ends up with scale-dependent couplings.

The on-shell scheme has the additional problem that the quantities are singular in a massless theory. Since we are primarily interested in the behavior of the couplings far above the physical masses it is necessary to introduce a renormalization procedure which works for massless theories.

Let us see how this works in slightly more detail for the example of a massless ϕ^4-theory. The renormalized field is defined as in (7.15) by $\phi = \sqrt{Z}\,\phi_r$. The n-point functions $G^{(n)}(x_1, \ldots x_n, \lambda, M)$ then depend on the renormalized coupling λ and the renormalization scale M and are given by

$$G^{(n)}(x_1, \ldots x_n, \lambda, M) := \langle\Omega|T\{\phi_r(x_1)\ldots\phi_r(x_n)\}|\Omega\rangle = Z^{-n/2}\langle\Omega|T\{\phi(x_1)\ldots\phi(x_n)\}|\Omega\rangle \tag{9.1}$$

A change $M \to M + \delta M$ then induces a change in the coupling λ and the wave function renormalization Z

$$\lambda \to \lambda + \delta\lambda\ , \qquad \sqrt{Z} \to \sqrt{Z}(1 - \delta\eta)\ . \tag{9.2}$$

Since the bare $\langle\Omega|T\{\phi(x_1)\ldots\phi(x_n)\}|\Omega\rangle$ does not change one obtains from (9.1)

$$0 = \delta\left(Z^{\frac{n}{2}} G^{(n)}\right) = \delta Z^{\frac{n}{2}}\, G^{(n)} + Z^{\frac{n}{2}}\,\frac{\partial G^{(n)}}{\partial M}\,\delta M + Z^{\frac{n}{2}}\,\frac{\partial G}{\partial\lambda}\,\delta\lambda\ . \tag{9.3}$$

Using $\delta Z^{\frac{n}{2}} = -nZ^{n/2}\delta\eta$ and defining

$$\beta := \frac{M}{\delta M}\,\delta\lambda\ , \qquad \gamma := -\frac{M}{\delta M}\,\delta\eta \tag{9.4}$$

one arrives at the Callan-Symanzik-equations

$$\left[M\frac{\partial}{\partial M} + \beta\,\frac{\partial}{\partial\lambda} + n\gamma\right] G^{(n)}(x_1, \ldots x_n, \lambda, M) = 0\ . \tag{9.5}$$

The renormalized coupling λ does not depend on the UV cut-off Λ and neither does the arbitrary scale M. From their definition (9.4) we thus conclude that also β and γ do not depend on Λ. Since β and γ are dimensionless and for a massless theory there is no dimensionfull quantity other than Λ in the theory one concludes that β and γ are also independent of M and one only has $\beta = \beta(\lambda)$ and $\gamma = \gamma(\lambda)$. This is not all obvious from their definition (9.4). This in turn implies that the Callan-Symanzik-equations only contain two universal functions of λ for all $G^{(n)}$.

Let us now explicitly compute β and γ for massless ϕ^4-theory by computing $G^{(2)}(M)$ and $G^{(4)}(M)$ and inserting them into (9.5). In lecture 7 we showed in problem 4.1 and (7.23) that the one loop corrections of the propagator are p-independent. Therefore imposing the renormalization conditions at $p^2 = -M^2$ (instead of $p^2 = m^2$) does not change the result and we conclude

$$M\frac{\partial}{\partial M}\,G^{(2)} = 0, \qquad \frac{\partial}{\partial\lambda}G^{(2)} = 0\ . \tag{9.6}$$

Inserted int (9.5) then implies

$$\gamma = 0 + \mathcal{O}(\lambda^2)\ . \tag{9.7}$$

$G^{(4)}$ was computed in problem 4.2 and (7.25) to be proportional to

$$G^{(4)} = -i\lambda + (-i\lambda)^2\big(iV(s) + iV(t) + iV(u) + \big) - i\delta_\lambda \tag{9.8}$$

In lecture 7 we determine δ_λ by imposing an on-shell renormalization condition. Now we impose $G^{(4)}_{amp}|_{s=t=u=-M^2} = -i\lambda$ which implies

$$\begin{aligned}\delta_\lambda &= (-i\lambda)^2\,3V(-M^2)\\ &= \frac{3\lambda^2}{2}\lim_{d\to 4}\frac{1}{(4\pi)^{d/2}}\int_0^1 dx\,\frac{\Gamma(2-d/2)}{[x(1-x)M^2]^{2-d/2}}\\ &= \frac{3\lambda^2}{32\pi^2}\lim_{\epsilon\to 0}\Big(\frac{2}{\epsilon} - \ln M^2 + \text{finite}\Big)\ .\end{aligned} \tag{9.9}$$

Thus

$$M\frac{\partial}{\partial M}\ G^{(4)} \ = \ \frac{3i\lambda^2}{16\pi^2} \tag{9.10}$$

Inserted into (9.5) together with $\gamma = 0$ we arrive at

$$-\beta\frac{\partial}{\partial\lambda}\,G^{(4)} = \frac{3i\lambda^2}{16\pi^2}\ . \tag{9.11}$$

From (9.8) we see that at leading order $G^{(4)} = -i\lambda$ so that we conclude

$$\beta = \frac{3\lambda^2}{16\pi^2} + O(\lambda^3)\ . \tag{9.12}$$

The analysis can also be done for massless QED for $G^{(n,m)}(x_1,\dots M,e)$ where n counts the number of electron fields Ψ while m counts the number of photon fields A_μ. In this case the CS-equation reads

$$\left[M\frac{\partial}{\partial M} + \beta(e)\frac{\partial}{\partial e} + n\gamma_2 + m\gamma_3\right]\ G^{(n,m)}(x_1,\dots M,e) = 0\ , \tag{9.13}$$

where

$$\gamma_{2,3} = -\frac{M}{\delta M}\,\delta\eta_{2,3}\ , \tag{9.14}$$

with $\eta_{2,3}$ being the change in the wave-function renormalization of Ψ or A_μ respectively.

In QFT I we used the on-shell renormalization conditions. In problem 5.1 the counterterms $\delta_{2,3}$ are determined at $p^2 = -M^2$ to be

$$\begin{aligned}
\delta_2 &= -\frac{e^2}{16\pi^2}\frac{\Gamma(2-d/2)}{M^{4-d}} + \text{finite} \ , \\
\delta_3 &= -\frac{e^2}{12\pi^2}\frac{\Gamma(2-d/2)}{M^{4-d}} + \text{finite} \ .
\end{aligned} \tag{9.15}$$

In QFT I we further recorded $\delta_1 = \delta_2$. From these expressions and (9.13) one obtains (see problem 5.1)

$$\begin{aligned}
\gamma_2 &= -\tfrac{1}{2}M\partial_M\delta_2 = \frac{e^2}{16\pi^2} \ , \\
\gamma_3 &= -\tfrac{1}{2}M\partial_M\delta_3 = \frac{e^2}{12\pi^2} \ , \\
\beta &= eM\partial_M(-\delta_1 + \delta_2 + \tfrac{1}{2}\delta_3) = \frac{e^3}{12\pi^2} \ .
\end{aligned} \tag{9.16}$$

10 Lecture 10: Solution of the CS Equation

Let us consider $G^{(2)}(p)$. Dimensional analysis determines

$$G^{(2)}(p) = \frac{i}{p^2} f\left(-p^2/M^2\right) , \tag{10.1}$$

where f is an arbitrary function for now. As a consequence one has

$$\begin{aligned} M\partial_M G^{(2)} &= \frac{i}{p^2} f' \frac{2p^2}{M^2} , \\ p\partial_p G^{(2)} &= -2G^{(2)} - \frac{i}{p^2} f' \frac{2p^2}{M^2} . \end{aligned} \tag{10.2}$$

which implies

$$M\partial_M G^{(2)} = (-p\partial_p - 2)\, G^{(2)} \tag{10.3}$$

Inserted into the Callan-Symanzik equation (9.5) we obtain

$$\left(p\partial_p - \beta\frac{\partial}{\partial\lambda} + 2 - 2\gamma\right) G^{(2)}(p, \lambda, M) = 0 . \tag{10.4}$$

This is a partial differential equation of the type

$$\left[\frac{\partial}{\partial t} + v(x)\,\frac{\partial}{\partial x} - \rho(x)\right] D(x,t) = 0 , \tag{10.5}$$

where for the case at hand

$$t = \ln(p/M) , \qquad v(x) = -\beta(\lambda) , \qquad \rho(x) = 2\gamma(\lambda) - 2 . \tag{10.6}$$

In problem 5.2 we show that (10.5) is solved by

$$D(t,x) = \hat{D}\big(\overline{x}(t,x)\big)\, \exp\Big(\int_0^t dt'\, \varrho(\overline{x}(t',x))\Big) \tag{10.7}$$

with $\hat{D}$ arbitrary and

$$\frac{\partial\overline{x}(t',x)}{\partial t'} = -v(\overline{x}) , \qquad \overline{x}(0,x) = x . \tag{10.8}$$

With the identification (10.6) one thus determines

$$G^{(2)}(p,\lambda,M) = \hat{G}\left(\overline{\lambda}(p,\lambda)\right)\, \exp\Big(-\int_{p'=M}^{p'=p} d(\ln \tfrac{p'}{M})\, 2\, \left[1 - \gamma(\overline{\lambda}(p',\lambda))\right]\Big) \tag{10.9}$$

with

$$\frac{d}{d(\ln \frac{p}{M})}\, \overline{\lambda}(p,\lambda) = \beta(\overline{\lambda}) , \qquad \overline{\lambda}(M,\lambda) = \lambda . \tag{10.10}$$

$\overline{\lambda}$ is called running coupling "constant". Since (10.10) governs the flow of λ when changing the scale p it is often called the renormalization group equation. Note that the quantity $\hat{G}$ in (10.9) coincides with $G^{(2)}$ at $p = M$ but for $p \neq M$ it depends effectively on the new coupling $\overline{\lambda}$ obtained as a solution of (10.10).

Similar solutions can be found for all $G^{(n)}$.

Let us now turn again to massless ϕ^4-theory as an example. In this case we computed in (9.12)

$$\frac{d}{d \ln \frac{p}{M}} \overline{\lambda}(p, \lambda) = \beta(\overline{\lambda}) = \frac{3\overline{\lambda}^2}{16\pi^2} + O(\overline{\lambda}^3) \; . \tag{10.11}$$

This differential equation can be solved by integration which yields

$$\overline{\lambda}(p) = \frac{\lambda}{1 - \frac{3\lambda}{16\pi^2} \ln \frac{p}{M}} \; . \tag{10.12}$$

We can check that for $p = M$ indeed $\overline{\lambda} = \lambda$ holds as required by the boundary condition. Furthermore in the IR, i.e. for $p \to 0$ the theory becomes weakly coupled in that $\overline{\lambda} \to 0$. In the UV, i.e. for $p \to \infty$ the theory becomes strongly coupled in that $\overline{\lambda} \to \infty$. However, perturbation theory breaks down at the Landau pole

$$1 - \frac{3\lambda}{16\pi^2} = 0 \quad \Rightarrow \quad p = M\, e^{\frac{16\pi^2}{3\lambda}} \; . \tag{10.13}$$

From (10.10) we see that the question if a theory is weakly or strongly coupled in the UV or IR is determined by the sign of β. Therefore one can identify the following generic cases:

$$\frac{d}{d(\ln \frac{p}{M})} \overline{\lambda}(p, \lambda) \;=\; \beta(\overline{\lambda}) \begin{cases} > 0 & \text{IR-free ,} \\ = 0 & \text{scale independent (finite QFT) ,} \\ < 0 & \text{UV-free (asymptotically free) .} \end{cases} \tag{10.14}$$

Here IR-free means that $\overline{\lambda}$ decreases for decreasing p ($\overline{\lambda} \downarrow$ for $p \downarrow$). For asymptotically free theories one has instead $\overline{\lambda} \downarrow$ for $p \uparrow$.

11 Lecture 11: Non-Abelian Gauge Theories

Let us first recall the situation in an Abelian gauge theory with QED as its prominent example. Here one starts from the Lagrangian

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi \ . \tag{11.1}$$

It has a gauge symmetry

$$\psi \to \psi' = e^{i\alpha}\psi \ , \qquad \bar{\psi} \to \bar{\psi}' = e^{-i\alpha}\bar{\psi} \ , \qquad \alpha \in \mathbb{R} \ , \tag{11.2}$$

and a corresponding Noether current

$$j^\mu = \bar{\psi}\gamma^\mu\psi \ . \tag{11.3}$$

The associated gauge symmetry corresponds to the replacement $\alpha \to \alpha(x)$. Of course $\mathcal{L}$ is no longer invariant but has to be coupled to a gauge boson A_μ via the covariant derivative

$$D_\mu\psi := \partial_\mu\psi - igA_\mu\psi \ , \tag{11.4}$$

where g is called the gauge coupling constant ($g = e$ for QED). A_μ is assigned the transformation law

$$A_\mu \to A'_\mu = A_\mu + \tfrac{1}{g}\partial_\mu\alpha \ , \tag{11.5}$$

such that

$$D_\mu\psi \to (D_\mu\psi)' = \partial_\mu\psi' - igA'_\mu\psi' = e^{i\alpha}D_\mu\psi \ . \tag{11.6}$$

The Lagrangian

$$\mathcal{L}(\psi, A_\mu) = i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi + gj^\mu A_\mu \tag{11.7}$$

is gauge invariant, i.e. $\mathcal{L}(\psi', A'_\mu) = \mathcal{L}(\psi, A_\mu)$. In order to promote A_μ to a propagating field one needs to add a kinetic term (the Maxwell term)

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \ , \qquad \text{with} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{11.8}$$

$F_{\mu\nu}$ is gauge invariant, i.e. obeys $F_{\mu\nu} \to F'_{\mu\nu} = F_{\mu\nu}$.

This story can be generalized to n Dirac spinors $\psi_i, i = 1, \ldots, n$ with the Lagrangian

$$\mathcal{L} = \sum_{i=1}^{n}(i\bar{\psi}_i\partial\!\!\!/\psi_i - m\bar{\psi}_i\psi_i) \ . \tag{11.9}$$

$\mathcal{L}$ has the unitary symmetry

$$\psi_i \to \psi'_i = \sum_j U_{ij}\psi_j \ , \qquad \bar{\psi}_i \to \bar{\psi}'_i = \sum_j \bar{\psi}_j U^\dagger_{ji} \ , \tag{11.10}$$

with $UU^\dagger = 1$ since $\sum_i \bar{\psi}_i\psi_i \to \sum_i \bar{\psi}'_i\psi'_i = \sum_{ijk} \bar{\psi}_i U^\dagger_{ij} U_{jk}\psi_k = \sum_i \bar{\psi}_i\psi_i$.

The associated gauge symmetry is obtained by the replacement $U \to U(x)$. $\mathcal{L}$ is no longer invariant but has to be coupled to a matrix gauge boson $(A_\mu)_{ij}$ via the covariant derivative

$$D_\mu\psi_i := \partial_\mu\psi_i - ig\sum_j (A_\mu)_{ij}\psi_j \ , \tag{11.11}$$

such that

$$D_\mu\psi_i \to (D_\mu\psi)'_i = \partial_\mu\psi'_i - ig\sum_j (A_\mu)'_{ij}\psi'_j = \sum_j U_{ij}D_\mu\psi_j \ . \qquad (11.12)$$

This determines the transformation law of A_μ to be (in matrix notation)

$$A_\mu \to A'_\mu = UA_\mu U^\dagger - \tfrac{i}{g}(\partial_\mu U)U^\dagger \ , \qquad (11.13)$$

which can be checked by inserting (11.13) and (11.10) into (11.12).

The field strength for $A_{\mu ij}$ is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \ . \qquad (11.14)$$

One checks the transformation law

$$F_{\mu\nu} \to F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu - ig[A'_\mu, A'_\nu] = UF_{\mu\nu}U^\dagger \ . \qquad (11.15)$$

As a consequence

$$\mathrm{tr}F'_{\mu\nu}F^{\mu\nu\prime} = \mathrm{tr}(UF_{\mu\nu}U^\dagger UF^{\mu\nu}U^\dagger) = \mathrm{tr}F_{\mu\nu}F^{\mu\nu} \ . \qquad (11.16)$$

Therefore the gauge invariant Lagrangian ($\mathcal{L}(\psi', A'_\mu) = \mathcal{L}(\psi, A_\mu)$) for this non-Abelian gauge theory (or Yang-Mills theory) is given by

$$\mathcal{L} = \sum_{i=1}^{n}(i\bar{\psi}_i\gamma^\mu D_\mu\psi_i - m\bar{\psi}_i\psi_i) - \kappa\mathrm{tr}F_{\mu\nu}F^{\mu\nu} \ , \qquad (11.17)$$

where the normalization κ will be determined later.

It is often easier to consider infinitesimal transformations

$$U = \mathbb{1} + \hat{H} + \mathcal{O}(\hat{H}^2) \ , \qquad U^{-1} = \mathbb{1} - \hat{H} + \mathcal{O}(\hat{H}^2) \ , \qquad (11.18)$$

such that for unitary U

$$U^{-1} = U^\dagger \quad \Rightarrow \quad \hat{H} = -\hat{H}^\dagger \ , \qquad (11.19)$$

i.e. $\hat{H}$ is anti-Hermitian. Therefore one conventionally defines $\hat{H} = iH$ with $H = H^\dagger$. It is convenient to separate the parameters of a transformation (e.g. rotation angles) from the basis of hermitian matrices and define

$$\hat{H}_{ij} = i\sum_a \alpha^a t^a_{ij} \ , \qquad \alpha^a \in \mathbb{R} \ , \qquad t^a = (t^a)^\dagger \ , \qquad (11.20)$$

where t^a denotes a basis of all hermitian $n \times n$ matrices. (They are called generators.) There are n^2 linearly independent such matrices and thus $a = 1, \ldots, n^2$. If additionally $\det U = 1$ the t^a are traceless and there are only $n^2 - 1$ generators. In the next lecture we show that the t^a satisfy

$$[t^a, t^b] = i\sum_c f^{abc}t^c \ . \qquad (11.21)$$

The f^{abc} are called structure constants of the Lie-algebra.

12 Lecture 12: A little Group Theory

A group G consists of a set of elements $g_1, \ldots, g_n \in G$ with a multiplication "$\cdot$" such that

(1)	$g_i \cdot g_j \in G$	(closure of the multiplication)
(2)	$g_i \cdot (g_j \cdot g_k) = (g_i \cdot g_j) \cdot g_k$	(associativity)
(3)	$g_i \cdot \mathbb{1} = \mathbb{1} \cdot g_i = g_i$	(existence of the identity)
(4)	$g_i \cdot g_i^{-1} = \mathbb{1}$	(existence of an inverse)

An additional option is

$(\star)\quad g_i \cdot g_j = g_j \cdot g_i$

in which case the group is called Abelian.

In quantum field theory the concept of a group is an indispensable tool since symmetry transformations such as rotations, Lorentz-transformations and gauge transformations are mathematically described by a group. However, in this application the group elements depend on the rotation angles, the boost parameters or the gauge parameters. Such groups are called Lie-groups which have the property that their elements g depend continuously on a finite number of parameters $\alpha_a, a = 1, \ldots, d$, i.e. $g = g(\alpha_1, \ldots, \alpha_d)$. The group multiplication now reads

$$g(\alpha_1, \ldots, \alpha_d) \cdot g(\beta_1, \ldots, \beta_d) = g(\gamma_1, \ldots, \gamma_d) \ , \tag{12.1}$$

where

$$\gamma_a = \gamma_a(\alpha_1, \ldots, \alpha_d, \beta_1, \ldots, \beta_d) \tag{12.2}$$

is a differentiable function of α_a and β_a.

An infinitesimal element of a Lie-group can be parametrized as[7]

$$g(\alpha) = \mathbb{1} + i \sum_{a=1}^{d} \alpha_a t^a + \tfrac{1}{2} \sum_{a,b=1}^{d} \alpha_a \alpha_b T^{ab} + \mathcal{O}\left(\alpha^3\right) \ , \tag{12.3}$$

where T^{ab} can be chosen symmetric $T^{ab} = T^{ba}$. The group multiplication can then be expressed as relations among the t^a.

In order to derive this relation let us first note that $g(0) = \mathbb{1}$ so that

$$\gamma_a(\alpha_1, \ldots, \alpha_d, 0, \ldots, 0) = \gamma_a(0, \ldots, 0, \alpha_1, \ldots, \alpha_d) = \alpha_a \ . \tag{12.4}$$

This implies

$$\gamma_a(\alpha, \beta) = \alpha_a + \beta_a + \sum_{b,c=1}^{d} C_a^{bc} \alpha_b \beta_c + \mathcal{O}\left((\alpha, \beta)^3\right) \ , \tag{12.5}$$

or in other words no terms proportional to α^2 or β^2 can appear as they would not satisfy

[7] This does not capture global issues.

(12.4). We can compute explicitly

$$
\begin{aligned}
g(\alpha)\cdot g(\beta) &= \mathbb{1} + i\sum_a(\alpha_a+\beta_a)t^a + \tfrac{1}{2}\sum_{a,b}\left((\alpha_a\alpha_b+\beta_a\beta_b)T^{ab} - 2\alpha_a\beta_b t^a t^b\right) + \mathcal{O}\left(\alpha^3\right) \\
&= g(\gamma) = \mathbb{1} + i\sum_a \gamma_a t^a + \tfrac{1}{2}\sum_{a,b}\gamma_a\gamma_b T^{ab} + \mathcal{O}\left(\gamma^3\right) \\
&= \mathbb{1} + i\sum_a(\alpha_a+\beta_a)t^a \\
&\qquad + \sum_{b,c} C_a^{bc}\alpha_b\beta_c t^a + \tfrac{1}{2}\sum_{ab}(\alpha_a+\beta_a)(\alpha_b+\beta_b)T^{ab} + \dots\ ,
\end{aligned}
\tag{12.6}
$$

where we used (12.5) in the last step. Decomposing $t^a t^b = \frac{1}{2}\{t^a, t^b\} + \frac{1}{2}[t^a, t^b]$ and comparing the second order terms we arrive at

$$
[t^a, t^b] = i\sum_c (C_c^{ba} - C_c^{ab})t^c\ , \tag{12.7}
$$

together with a slightly more complicated (and uninteresting) equation for $\{t^a, t^b\}$. Defining the structure constants $f^{abc} := C_c^{ba} - C_c^{ab} = -f^{bac}$ eq. (12.7) turns into the Lie-algebra of the group G

$$
\left[t^a, t^b\right] = i\sum_{c=1}^{d} f^{abc}t^c\ . \tag{12.8}
$$

The t^a are called the generators of the algebra while d is called the dimension of the algebra. It is left to show that at higher orders no further constraints appear. A proof of this fact can be found, for example, in [6]. Before we continue let us note that (12.8) says that the group multiplication is characterized by a finite-dimensional algebra independently of any parameter.

The structure constants f^{abc} satisfy the Jacobi-identity which holds for any commutator

$$
\begin{aligned}
0 &= \left[t^a, \left[t^b, t^c\right]\right] + \left[t^b, \left[t^c, t^a\right]\right] + \left[t^c, \left[t^a, t^b\right]\right] \\
&= \sum_{e=1}^{d}\left(if^{bce}\left[t^a, t^e\right] + if^{cae}\left[t^b, t^e\right] + if^{abe}\left[t^c, t^e\right]\right)\ ,
\end{aligned}
\tag{12.9}
$$

where in the second equation we used (12.8). Using it again implies

$$
\sum_{e=1}^{d}\left(f^{bce}f^{aef} + f^{cae}f^{bef} + f^{abe}f^{cef}\right) = 0\ . \tag{12.10}
$$

Let us now give some examples of Lie groups. Let us start with the group $GL(n)$. This is the group of $n\times n$ matrices M acting on $\mathbb{R}^n$ with $\det M \neq 0$. These matrices satisfy the group axioms. As a second example consider the group $U(n)$ which is the group of unitary $n\times n$-matrices $U, UU^\dagger = U^\dagger U = \mathbb{1}$ again with $\det U \neq 0$ and acting on $\mathbb{C}^n$. These matrices also satisfy the group axioms. Let ξ, θ be complex vectors i.e. $\xi, \theta \in \mathbb{C}^n$. Under a unitary transformation one has

$$
\theta \to \theta' = U\theta\ , \qquad \xi \ \to \xi' = U\xi\ , \tag{12.11}
$$

or with indices

$$\theta_i \to \theta_i' = \sum_{j=1}^{n} U_{ij}\theta_j \ , \qquad \xi_i \to \xi_i' = \sum_{j=1}^{n} U_{ij}\xi_j \ , \tag{12.12}$$

The inner product $\sum_{i=1}^{n} \theta_i^* \xi_i$ is invariant under this transformation since

$$\theta^{*T} \to \theta^{*T\prime} = (U\theta)^{*T} = \theta^{*T} U^\dagger \ . \tag{12.13}$$

In the previous lecture we already saw that an infinitesimal group element can be written as

$$U = \mathbb{1} + i\sum_{a=1}^{d} \alpha_a t^a + \mathcal{O}\left(\alpha^2\right) , \tag{12.14}$$

with generators which are hermitian, i.e. $t^a = t^{a\dagger}$. There are n^2 such generators and thus the dimension of the Lie-Algebra is $d = n^2$.[8]

$\underline{SU(n)}$ denotes the group of unitary $n \times n$-matrices U with the additional property $\det U = 1$. In the previous lecture we saw that in this case the generators are also traceless $\operatorname{Tr} t^a = 0$. Such transformation are rotations in $\mathbb{C}^n$ and $d = n^2 - 1$.

$\underline{O(n)}$ is the group of orthogonal $n \times n$-matrices O with $OO^T = O^T O = \mathbb{1}, \det O \neq 0$. Let ξ, θ now be real vectors i.e. $\xi, \theta \in \mathbb{R}^n$ with transformations

$$\theta \to \theta' = O\theta \ , \qquad \xi \ \ \to \xi' = O\xi \ , \tag{12.15}$$

The inner product $\theta^T \xi$ is left invariant by these transformations and thus the transformations correspond to rotations and/or reflections of $\mathbb{R}^n$. The infinitesimal element reads again $O = \mathbb{1} + i\sum_{a=1}^{d} \alpha_a t^a + \mathcal{O}\left(\alpha^2\right)$ but now $O^T O = \mathbb{1}$ requires the generators to be antisymmetric $t^a = -t^{aT}$. The dimension of the Lie-algebra therefore is $d = \frac{1}{2}n(n-1)$. Imposing additionally $\det O = 1$ excludes reflections and this (proper) rotation group is called $\underline{SO(n)}$.

The group $\underline{SO(n,m)}$ consists of matrices Λ which satisfy

$$\Lambda^T \eta \Lambda = \eta \ , \tag{12.16}$$

for $\eta = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$ with signature (n, m). Such transformation leave the inner product $\xi^T \eta \theta$ invariant. In this notation the Lorentz-group corresponds to $SO(1, 3)$.

[8] Note that the phase rotations $U = e^{i\alpha}$ form the group $U(1)$.

13 Lecture 13: Representations of Lie Groups

Under a representation of a group G one understands the map

$$g_i \to M_i\left(g_i\right) \quad \text{with} \quad M_i\left(g_i\right)\cdot M_j\left(g_j\right) = M_k\left(g_k\right) \quad \text{for} \quad g_i \cdot g_j = g_k \tag{13.1}$$

or for Lie-groups

$$g\left(\alpha\right) \to M\left(\alpha\right) \quad \text{with} \quad M\left(\alpha\right)\cdot M\left(\beta\right) = M\left(\gamma\right) \quad \text{for} \quad g\left(\alpha\right)\cdot g\left(\beta\right) = g\left(\gamma\right)\ , \tag{13.2}$$

where M is a linear operator (a matrix or a differential operator) acting on an d_r-dimensional vector space V. d_r is the dimension of the representation. If all M's act within a subspace of V the representation is called reducible and otherwise irreducible.

The groups discussed in the previous lectures were defined by a particular representation. This is called the defining or fundamental representation.

A prominent example in physics is the group $SU(2)$ which is generated by $d = n^2 - 1 = 4 - 1 = 3$ hermitian generators. These are precisely the Pauli matrices $t^a = \frac{1}{2}\sigma^a$. They satisfy

$$\left[t^a, t^b\right] = i \sum_c \epsilon^{abc} t^c\ . \tag{13.3}$$

Comparison with (12.8) implies $f^{abc} = \epsilon^{abc}$.

However, all groups can have higher-dimensional representations. For $SU(2)$ these are the spin-j representations which are $(2j+1)$-dimensional and also satisfy (13.3). In order to notate the different representations we sometimes use t^a_r to state that the generators t^a are taken in the representation r. In particle physics all known gauge groups are among the $SU(n)$ and so we focus on this case henceforth.

The matrices $(t^a)_{bc} = i f^{bac}$ form a representation since they satisfy (12.8) due to the Jacobi-identity (12.10). (This is shown in problem 7.2.) This representation is called the adjoint representation and it has dimension $d_r = d \equiv d(G)$.

The complex conjugate representation is generated by $t^a_{\bar{r}} = -(t^a_r)^T$ (which is shown in problem 7.3). A real representation satisfies $t^a_{\bar{r}} = S t^a_r s^{-1}, \forall a$.

Let us define

$$D^{ab}_r := \mathrm{Tr}(t^a_r t^b_r)\ , \qquad a, b = 1, \ldots, d = n^2 - 1\ . \tag{13.4}$$

Due to the cyclicity of the trace $D^{ab}_r = D^{ba}_r$ and furthermore

$$(D^{ab}_r)^\dagger = \mathrm{Tr}(t^{b\dagger}_r t^{a\dagger}_r) = D^{ab}_r\ , \tag{13.5}$$

where we used $t^a_r = t^{a\dagger}_r$ and again the cyclicity of the trace. Thus D^{ab}_r is a hermitian and symmetric matrix and thus real. It can be diagonalized and furthermore the normalization of the t^a can be chosen such that

$$\mathrm{Tr}\left(t^a_r t^b_r\right) = c(r)\,\delta^{ab}\ . \tag{13.6}$$

$c(r)$ is called the index of the representation. In the fundamental n-dimensional representation of $SU(n)$ one chooses $c(n) = \frac{1}{2}$.

Another property of the f^{abc} can be learned from considering

$$\begin{aligned}\mathrm{Tr}\left([t_r^a, t_r^b] t_r^c\right) &= i f^{abd}\, \mathrm{Tr}\left(t_r^d t_r^a\right) = i f^{abc} c(r) \\ &= \mathrm{Tr}\left(t_r^a t_r^b t_r^c - t_r^b t_r^a t_r^c\right) = \mathrm{Tr}\left(t_r^a [t_r^b, t_r^c]\right) = i f^{bca} c(r)\ ,\end{aligned} \tag{13.7}$$

where we used (13.6) and the cyclicity of the trace in the second line. Thus we have the symmetry properties

$$f^{abc} = -f^{bac} = f^{bca}\ , \tag{13.8}$$

or in other words the structure constants are totally antisymmetric for $SU(n)$.

The quadratic Casimir operator is defined as

$$T_{ij}^2 := \sum_{a=1}^{d(G)} \sum_{k=1}^{d(r)} t_{ik}^a t_{kj}^a. \tag{13.9}$$

It commutes with all generators

$$\left[t^b, T^2\right] = 0\ , \quad \forall b\ . \tag{13.10}$$

Thus T^2 must be proportional to the unit matrix and one can write

$$T_{ij}^2 = c_2(r)\, \delta_{ij}\ . \tag{13.11}$$

Taking the trace of T^2 and D^{ab} one derives (problem 7.2) the relation

$$c(r) \cdot d(G) = c_2(r) \cdot d(r)\ . \tag{13.12}$$

Let us close this lecture with a brief discussion of the Lorentz group in this formalism. It corresponds to the group $SO(1,3)$ with the defining representation obeying (12.16). For infinitesimal Lorentz transformations one expands

$$\Lambda_\nu^\mu = \delta_\nu^\mu + i \sum_a \omega_a (t^a)_\nu^\mu + \mathcal{O}(\omega^2)\ . \tag{13.13}$$

Inserted into (12.16) one obtains

$$t^{aT} \eta = -\eta t^a\ , \tag{13.14}$$

or on other words the generators with both indices lowered are antisymmetric. There are six such generators and it is customary to assembles them in an antisymmetric matrix $J^{\rho\sigma} = -J^{\sigma\rho}$. Similarly the six Lorentz parameters are expressed in terms of the antisymmetric matrix $\omega_{\rho\sigma} = -\omega_{\sigma\rho}$ such that $\sum_a \omega_a t^a = \sum_{\sigma\rho} \omega_{\sigma\rho} J^{\sigma\rho}$. The $J^{\rho\sigma}$ satisfy the $SO(1,3)$ algebra

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\big(\eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho} + \eta^{\mu\sigma} J^{\nu\rho}\big) \tag{13.15}$$

All $SO(n,m)$ groups have also spinor representations. Starting from the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{13.16}$$

one can construct the operator

$$S^{\mu\nu} := \tfrac{i}{4} [\gamma^\mu, \gamma^\nu] \tag{13.17}$$

which satisfies (13.15). $S^{\mu\nu}$ is the generator of the spinor representations of $SO(n,m)$.

14 Lecture 14: Feynman Rules in Non-Abelian Gauge Theories

In lecture 11 we introduced non-Abelian gauge theories and gave the Lagrangian in (11.17) which is invariant under the transformations (11.10) and (11.13). Using the group-theoretical considerations of the previous lecture it is convenient to also determine the infinitesimal transformation laws of non-Abelian gauge theories. That is we expand U as in (12.14) and insert it into (11.10) to obtain

$$\delta\psi = \psi' - \psi = i\sum_{a=1}^{d} \alpha^a t^a \psi \ , \tag{14.1}$$

or with indices

$$\delta\psi_i = i\sum_{a=1}^{d}\sum_{j=1}^{n} \alpha^a t^a_{ij} \psi_j \ . \tag{14.2}$$

For $SU(n)$ one has $d = n^2 - 1$. $\bar\psi$ transforms as

$$\delta\bar\psi = \bar\psi' - \bar\psi = \bar\psi U^\dagger - \bar\psi = -i\sum_a \alpha^a \bar\psi\, t^a \ . \tag{14.3}$$

For the gauge bosons we find from (11.13)

$$\delta A_\mu = A'_\mu - A_\mu = i\sum_a \alpha^a [t^a, A_\mu] + \tfrac{1}{g}\sum_a (\partial_\mu \alpha^a)\, t^a \tag{14.4}$$

From the last term we see that A_μ must be an element of the Lie-algebra and we can expand A_μ in a basis of generators

$$A_\mu = \sum_a A^a_\mu t^a. \tag{14.5}$$

Inserted into (14.4) we arrive at

$$\delta A_\mu = -i\sum_a \alpha^a A^b_\mu [t^a, t^b] + \tfrac{1}{g}\sum_a (\partial_\mu \alpha^a)\, t^a \ . \tag{14.6}$$

Using (12.8) and (14.5) we can alternatively write

$$\delta A^a_\mu = -\sum_{bc} \alpha^b A^c_\mu f^{bca} + \tfrac{1}{g}\partial_\mu \alpha^a \equiv \tfrac{1}{g} D_\mu \alpha^a \tag{14.7}$$

The covariant derivative (11.11) now reads

$$D_\mu \psi = \partial_\mu \psi - igA_\mu \psi = \partial_\mu \psi - ig\sum_a A^a_\mu t^a \psi_j \tag{14.8}$$

while the field strength is given by (see problem 7.1)

$$\begin{aligned} F_{\mu\nu} =& \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] = \sum_a F^a_{\mu\nu} t^a \\ F^a_{\mu\nu} =& \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\sum_{bc} f^{abc} A^b_\mu A^c_\nu \end{aligned} \tag{14.9}$$

In the Lagrangian (11.17) we left the normalization κ undetermined. Inserting (14.5) and using (13.6) we see that for $\kappa = -\frac{1}{4c(r)}$ we obtain a properly normalized kinetic term of the gauge bosons. Thus (11.17) turns into

$$\mathcal{L} = \sum_i \bar{\psi}_i (i\gamma^\mu D_\mu - m)\psi_i - \tfrac{1}{4}\sum_a F^a_{\mu\nu}F^{a\mu\nu} \ . \tag{14.10}$$

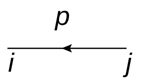
In order to derive the Feynman rules we split $\mathcal{L}$ according to

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}} \ , \tag{14.11}$$

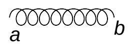
where

$$\begin{aligned} \mathcal{L}_0 &= \bar{\psi}_i(\gamma^\mu\partial_\mu - m)\psi_i + \tfrac{1}{2}A^a_\mu\left(\eta^{\mu\nu} - \partial^\mu\partial^\nu\right)A^a_\nu \ , \\ \mathcal{L}_{\text{int}} &= gA^a_\mu j^{a\mu} - gf^{abc}\partial_\mu A^a_\nu A^{b\mu}A^{c\nu} - \tfrac{1}{4}g^2 f^{cab}A^a_\mu A^b_\nu f^{cde}A^{d\mu}A^{e\nu} \ , \end{aligned} \tag{14.12}$$

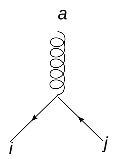
and $j^\mu = \bar{\psi}\gamma^\mu t^a\psi$. $\mathcal{L}_0$ contains the kinetic terms which are also present in the Abelian limit. Together they results in the following Feynman rules [2]



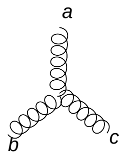
$$\sim \quad \frac{i}{\not{p} - m}\delta_{ij}$$



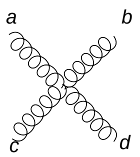
$$\sim \quad -i\Big(\frac{\eta_{\mu\nu}}{k^2} - (1-\xi)\frac{k_\mu k_\nu}{k^4}\Big)$$



$$\sim \quad ig\gamma^\mu t^a$$



$$\sim \quad gf^{abc}[g^{\mu\mu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\nu}(q-k)^\nu]$$



$$\begin{aligned} \sim \quad & -ig[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\mu\sigma}g^{\nu\rho}) \\ & + f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})] \end{aligned} \tag{14.13}$$

Note that the momenta are incoming in the second diagram.

In order to derive the gauge boson propagator recall that the kinetic term for A^a_μ in (14.12) is not invertible and as a consequence a gauge fixing is necessary. To do so we repeat the Fadeev-Popov procedure which we used in lecture 4 for the Abelian case. As we will see momentarily for a non-Abelian gauge bosons a slight complication occurs.

Let us again start from the path integral

$$I = \int DA\, e^{iS[A]} \ , \tag{14.14}$$

where $S[A]$ contains all A-dependent terms in (14.12). We again insert the identity in the form

$$\int D\alpha\, \delta(G(A^\alpha)) \det\Big(\frac{\delta G(A^\alpha)}{\delta\alpha}\Big) = 1 \ , \tag{14.15}$$

where

$$A_\mu^{a\alpha} = A_\mu^a + \tfrac{1}{g} D_\mu \alpha^a \ , \qquad \text{with} \qquad D_\mu \alpha^a = \tfrac{1}{g}\partial_\mu \alpha^a - f^{abc}\alpha^b A_\mu^c \tag{14.16}$$

Choosing

$$G(A^\alpha) = \partial^\mu A_\mu^{a\alpha} - \omega(x) \tag{14.17}$$

with $\omega(x)$ being an arbitrary scalar function, we see that $\frac{\delta G(A^\alpha)}{\delta\alpha}$ in the non-Abelian case depends on A and is given by

$$\frac{\delta G(A^\alpha)}{\delta\alpha} = \tfrac{1}{g}\partial^\mu D_\mu(A) \tag{14.18}$$

Inserted into the path integral we obtain

$$I = \int DAD\alpha\, \delta(G(A^\alpha)) \det\Big(\frac{\delta G(A^\alpha)}{\delta\alpha}\Big) e^{iS[A]} \ . \tag{14.19}$$

but now the determined cannot be moved outside the integral. Changing variables

$$A_\mu \to A_\mu^\alpha = UA_\mu U^\dagger - U\partial_\mu U^\dagger \ , \tag{14.20}$$

and using $S[A] = S[A^\alpha]$ (since it is gauge invariant) and $DA = DA^\alpha$ since the change is a unitary transformation plus an A-independent shift we arrive at

$$I = \int DA^\alpha D\alpha\, \delta(G(A^\alpha)) \det\Big(\frac{\delta G(A^\alpha)}{\delta\alpha}\Big) e^{iS[A^\alpha]} \ . \tag{14.21}$$

Since A^α is an (arbitrary) integration variable we can rename it back to A and factor out the infinite-dimensional factor $\int D\alpha$ which causes the problem, i.e.

$$I = \Big(\int D\alpha\Big) \int DA\, \delta(G(A)) \det\Big(\frac{\delta G(A^\alpha)}{\delta\alpha}\Big) e^{iS[A]} \ . \tag{14.22}$$

The determinant in I can be represented as a path integral over anti-commuting bosonic ghost fields c^a. Recall from (5.17) that a functional determined can be represented by a fermionic path integral

$$\int D\bar\psi D\psi e^{i\int d^4x \bar\psi \hat{O}\psi} = c\, \det(\hat{O}) \ . \tag{14.23}$$

However $\frac{\delta G(A^\alpha)}{\delta\alpha}$ does not act on spinors and therefore we need bosonic Grassmann fields to represent the determinant

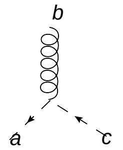
$$\det\Big(\frac{\delta G(A^\alpha)}{\delta\alpha}\Big) = \det\Big(\tfrac{1}{g}\partial^\mu D_\mu(A)\Big) = \int D\bar{c} Dc e^{i\int d^4x \mathcal{L}_{\rm ghost}[c,\bar{c},A]} \ , \tag{14.24}$$

where

$$\mathcal{L}_{\rm ghost}[c,\bar{c},A] = \bar{c}(-\partial^\mu D_\mu)c = \bar{c}^a(-\Box\delta^{ac} - g\partial^\mu f^{abc}A_\mu^b)c^c \ . \tag{14.25}$$

Since the fields $c, \bar{c}$ have the "wrong" spin-statistic they are called ghost fields. They are not physical but rather their introduction is a trick to represent $\det(\frac{\delta G(A^\alpha)}{\delta \alpha})$. Now we can evaluate this determinant by the additional Feynman rules

$$\sim \quad \frac{i\delta^{ab}}{p^2}$$



$$\sim \quad -gf^{abc}p^\mu \tag{14.26}$$

Finally $\delta(G(A))$ can again be expressed as a correction to S by integrating over all $\omega(x)$ with Gaussian weight centered around $\omega(x) = 0$. This yields

$$\begin{aligned} I' &= N(\xi) \int D\omega \, e^{-i\int d^4x \frac{\omega^2}{2\xi}} \; I \\ &= N(\xi) \Big(\int D\alpha \Big) \int DADcD\bar{c} \, e^{iS'[c,\bar{c},A]} \end{aligned} \tag{14.27}$$

where N is a normalization factor and

$$S' = \int d^4x \left(\sum_i \bar{\psi}_i (i\gamma^\mu D_\mu - m)\psi_i - \tfrac{1}{4} \sum_a F^a_{\mu\nu} F^{a\mu\nu} - \tfrac{1}{2\xi} (\partial^\mu A_\mu)^2 + \bar{c}(-\partial^\mu D_\mu) c \right) , \tag{14.28}$$

where we now also included the fermions again. As a consequence the gauge boson propagator reads again in momentum space

$$-\frac{i}{k^2} \big(\eta_{\mu\nu} - (1-\xi) \frac{k_\mu k_\nu}{k^2} \big) \, \delta^{ab} \ , \tag{14.29}$$

as already anticipated in (14.13)

15 Lecture 15: BRST Quantization

The gauged fixed action (14.28) has no gauge symmetry but instead a new global symmetry the so called Becchi-Rouet-Stora-Tyutin (BRST) symmetry. This symmetry can be seen by considering the auxiliary Lagrangian

$$\mathcal{L}_B = \sum_i \bar{\psi}_i(i\gamma^\mu D_\mu - m)\psi_i - \sum_a \Big(\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \sum_b \bar{c}^a(\partial^\mu D_\mu)_{ab}c^b - \tfrac{\xi}{2}B^aB^a - B^a\partial^\mu A^a_\mu\Big) \ . \tag{15.1}$$

Since there is no kinetic terms for B^a its Euler-Lagrangian equation is purely algebraic

$$\frac{\partial\mathcal{L}}{\partial B^a} = \xi B^a + \partial^\mu A^a_\mu = 0 \ , \tag{15.2}$$

and thus can be used to express B^a in terms of A_μ

$$B^a = -\tfrac{1}{\xi}\partial^\mu A^a_\mu \ . \tag{15.3}$$

Inserted into the $\mathcal{L}_B$ of (15.1) we get back the original gauged fixed $\mathcal{L}$ of (14.28).

$\mathcal{L}_B$ is invariant under the following BRST-transformations

$$\begin{aligned}
\delta_\epsilon A^a_\mu &= \epsilon \sum_c D^{ac}_\mu c^c \equiv \epsilon\partial_\mu c^a + g\sum_{bc} f^{abc}A^b_\mu c^c \ , \\
\delta_\epsilon \psi_i &= ig\epsilon \sum_{aj} c^a(t^a)_{ij}\psi_j \ , \\
\delta_\epsilon c^a &= -\tfrac{1}{2}g\epsilon\sum_{bc} f^{abc}c^bc^c \ , \\
\delta_\epsilon \bar{c}^a &= \epsilon B^a \ , \\
\delta_\epsilon B^a &= 0 \ ,
\end{aligned} \tag{15.4}$$

where ϵ is the Grassmann parameter of the BRST transformation and satisfies $\epsilon^2 = 0$. To show it we first observe that the gauge transformations (14.3) and (14.4) and the BRST-transformation are related by

$$\delta_\alpha A^a_\mu|_{\alpha^a = g\epsilon c^a} = \delta_\epsilon A^a_\mu \ , \qquad \delta_\alpha \psi_i|_{\alpha^a = g\epsilon c^a} = \delta_\epsilon\psi_i \ . \tag{15.5}$$

This immediately implies that the first two terms in (15.1) are invariant. The term proportional to ξ is trivially invariant so that we need to compute

$$\begin{aligned}
&\delta_\epsilon\Big(\sum_{ab}\bar{c}^a(\partial^\mu D_\mu)_{ab}c^b - \sum_a B^a\partial^\mu A^a_\mu\Big) \\
&\qquad = \sum_{ab}(\delta_\epsilon\bar{c}^a)(\partial^\mu D_{\mu ab}c^b) + \sum_{ab}\bar{c}^a\partial^\mu\delta_\epsilon(D_{\mu ab}c^b) - \sum_a B^a\partial^\mu\delta_\epsilon A^a_\mu \ .
\end{aligned} \tag{15.6}$$

Using (15.4) we see that the first and the last term cancel and we are left with

$$\begin{aligned}\sum_a \bar{c}^a\partial^\mu\big(\partial_\mu\delta_\epsilon c^a + g\sum_{bc} f^{abc}(\delta_\epsilon A^b_\mu)c^c + g\sum_{bc} f^{abc}A^b_\mu\delta_\epsilon c^c\big)\\ = \sum_a \bar{c}^a\partial^\mu\Big(g\epsilon\sum_{bc} f^{abc}\big(-\tfrac{1}{2}(\partial_\mu c^b c^c) + (\partial_\mu c^b)c^c\big)\\ + g^2\epsilon\sum_{bcef}\big(f^{abc}(f^{bef}A^e_\mu c^f c^c - \tfrac{1}{2}f^{cef}A^b_\mu c^e c^f)\big)\Big) .\end{aligned} \tag{15.7}$$

Using

$$\sum_{bc} f^{abc}(\partial_\mu c^b)c^c = \tfrac{1}{2}\sum_{bc} f^{abc}\partial_\mu(c^b c^c) , \tag{15.8}$$

which holds due to the antisymmetry of f^{abc} and the Grassmann nature of c^a, and

$$\sum_{bcef} f^{abc}f^{bef}c^f c^c = \tfrac{1}{2}\sum_{bcef} c^f c^c(f^{abc}f^{bef} - f^{abf}f^{bec}) , \tag{15.9}$$

one sees that also the remaining terms cancel. Thus $\mathcal{L}_B$ has a global BRST symmetry for any ξ as a remnant of the gauge symmetry.

One commonly defines the BRST operator Q by the action

$$\begin{aligned} QA^a_\mu &= \partial_\mu c^a + gf^{abc}A^b_\mu c^c ,\\ Q\psi &= igc^aT^a\psi ,\\ Qc^a &= -\tfrac{1}{2}gf^{abc}c^b c^c ,\\ Q\bar{c}^a &= B^a ,\\ QB^a &= 0 , \end{aligned} \tag{15.10}$$

In problem 8.1 it is shown that Q is nil-potent, i.e. satisfies

$$Q^2 = 0 , \tag{15.11}$$

and furthermore the action (15.1) can be written as

$$\mathcal{L}_B = \sum_i \bar{\psi}_i(i\gamma^\mu D_\mu - m)\psi_i - \tfrac{1}{4}\sum_a F^a_{\mu\nu}F^{a\mu\nu} + Q\sum_a\big(\bar{c}^a\partial^\mu A^a_\mu + \tfrac{\xi}{2}\bar{c}^a B^a\big) , \tag{15.12}$$

which is another way to see its invariance under BRST-transformations. One can also show

$$[Q,H] = 0 , \qquad Q = Q^\dagger . \tag{15.13}$$

The space of states $\mathcal{H}$ decomposes into three subspaces: $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_0$, where

- $\mathcal{H}_1$: contains states $|\psi_1\rangle$ which are not annihilated by Q, i.e. $Q|\psi_1\rangle \neq 0$,
- $\mathcal{H}_2$: contains states $|\psi_2\rangle$ for which $|\psi_2\rangle = Q|\psi_*\rangle$ holds. (15.11) then implies $Q|\psi_2\rangle = 0$,
- $\mathcal{H}_0$: contains states $|\psi_0\rangle$ which obey $Q|\psi_0\rangle = 0$ but $|\psi_0\rangle \neq Q|\psi_*\rangle$.

Note that $\mathcal{H}_0$ coincides with the cohomology of Q which is defined as

$$\mathrm{coh}(Q) := \frac{\mathrm{Ker}Q}{\mathrm{Im}Q} \ . \tag{15.14}$$

From (15.10) we that in the free theory ($g \to 0$)

$$QA_\mu^a = \partial_\mu c^a \ , \qquad Q\bar{c}^a = B^a \ , \qquad Q\psi = Qc^a = QB^a = 0 \ , \tag{15.15}$$

which implies that for the asymptotic states we have

$$A_\mu^{\mathrm{long}} \in \mathcal{H}_1 \ , \quad A_\mu^{\mathrm{trans}} \in \mathcal{H}_0 \ , \quad \psi \in \mathcal{H}_0 \ , \quad c^a \in \mathcal{H}_2 \ , \quad \bar{c}^a \in \mathcal{H}_1 \ , \quad B^a \in \mathcal{H}_2 \ . \tag{15.16}$$

We see that the physical degrees of freedom $A_\mu^{\mathrm{trans}}, \psi$ are in $\mathcal{H}_0$ while the unphysical degrees of freedom $A_\mu^{\mathrm{long}}, c^a, \bar{c}^a, B^a$ are in $\mathcal{H}_1$ or $\mathcal{H}_2$ respectively. Therefore in the BRST quantization procedure $\mathcal{H}_0$, or in other words the cohomology of Q, is identified with the physical Hilbert space. Thus, all asymptotic states of the theory have to be in $\mathcal{H}_0$.

It remains to show that the time evolution does not change this picture. Since Q commutes with H we have

$$Q|\psi_0, t\rangle = Qe^{iHt}|\psi_0\rangle = e^{iHt}Q|\psi_0\rangle = 0 \ . \tag{15.17}$$

Thus $|\psi_0, t\rangle \in \mathcal{H}_2 \oplus \mathcal{H}_0$. However, $\langle\psi_2|\psi_0, t\rangle = \langle\psi_*|Q|\psi_0, t\rangle = 0$ and thus $|\psi_0, t\rangle$ has no component in $\mathcal{H}_2$ or in other words $|\psi_0, t\rangle \in \mathcal{H}_0$.

This can be repeated including the interaction so that indeed the physical states are the states of $\mathcal{H}_0$, i.e. they are in the cohomology of Q.

16 Lecture 16: Renormalized perturbation theory for Non-Abelian Gauge Theories

Let us start with the bare Lagrangian determined in (14.28)

$$\mathcal{L} = -\tfrac{1}{4}F^a_{0\mu\nu}F_0^{a\mu\nu} - \tfrac{1}{2\xi}\left(\partial^\mu A^a_{0\mu}\right)\left(\partial^\nu A^a_{0\nu}\right) + \bar{\psi}_0\left(i\not{D} - m_0\right)\psi_0 + \bar{c}^a_0\left(-\partial^\mu D^{ac}_\mu\right)c^a_0 \ , \tag{16.1}$$

and define the renormalized fields by

$$\psi_0 = \sqrt{Z_2}\,\psi \ , \qquad A^a_{0\mu} = \sqrt{Z_3}\,A^a_\mu \ , \qquad c^a_0 = \sqrt{Z^c_2}\,c^a \ , \qquad \bar{c}^a_0 = \sqrt{Z^c_2}\,\bar{c}^a \ , \tag{16.2}$$

with

$$Z_2 = 1 + \delta_2 \ , \qquad Z_3 = 1 + \delta_3 \ , \qquad Z^c_2 = 1 + \delta^c_2 \ . \tag{16.3}$$

The renormalized couplings are then defined by

$$\begin{aligned} &Z_2 m_0 = m + \delta_m \ , \qquad g_0 Z_2\sqrt{Z_3} = g\left(1 + \delta_1\right) \ , \qquad g_0 Z_3^{3/2} = g\left(1 + \delta_1^{3g}\right) \ , \\ &g_0^2 Z_3^2 = g^2\left(1 + \delta_1^{4g}\right) \ , \qquad g_0 Z^c_2\sqrt{Z_3} = g\left(1 + \delta^c_1\right) \ . \end{aligned} \tag{16.4}$$

We thus have eight counterterms $(\delta_m, \delta_{1,2,3}, \delta^c_{1,2}, \delta_1^{3g,4g})$ for only five physical quantities (A, ψ, c, g, m). Indeed one can show that at one-loop (16.3) and (16.4) imply the three relations

$$\delta_1 - \delta_2 = \delta_1^{3g} - \delta_3 = \delta^c_1 - \delta^c_2 = \tfrac{1}{2}(\delta_1^{4g} - \delta_3) \ . \tag{16.5}$$

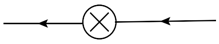
Inserting (16.2), (16.3) and (16.4) into (16.1) we obtain

$$\mathcal{L} = \mathcal{L}(\psi, A, c) + \mathcal{L}_{CT} \ , \tag{16.6}$$

where

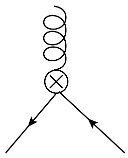
$$\begin{aligned} \mathcal{L}_{CT} = &-\tfrac{1}{4}\delta_3\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 + \bar{\psi}\left(i\delta_2\not{\partial} - \delta_m\right)\psi - \delta^c_2\bar{c}^a\Box c^a + g\delta_1 A^a_\mu j^{a\mu} \\ &- g\delta_1^{3g} f^{abc}\left(\partial_\mu A^a_\nu\right)A^{b\mu}A^{c\nu} - \tfrac{1}{4}g^2\delta_1^{4g}\left(f^{eab}A^a_\mu A^b_\nu\right)^2 - g\delta^c_1\bar{c}^a f^{abc}\partial^\mu A^b_\mu c^c. \end{aligned} \tag{16.7}$$

These counterterms lead to the additional Feynman rules

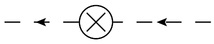


$$\sim \quad i(\not{p}\delta_2 - \delta_m)$$

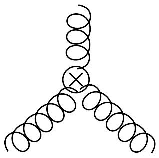
$$\sim \quad -i\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)\delta^{ab}\delta_3$$



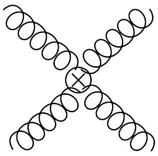
$$\sim \quad i g t^a \gamma^\mu \delta_1$$



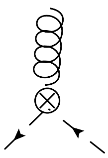
$$\sim \quad \delta_1^c$$



$$\sim \quad \delta_1^{3g}$$



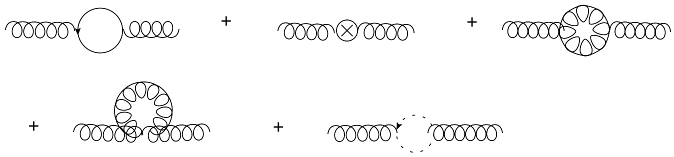
$$\sim \quad \delta_1^{4g}$$



$$\sim \quad \delta_1^c\ .$$

(16.8)

In the following we give the results for $\delta_{1,2,3}$ which we need for the computation of the β-function. For further details see [2]. The one-loop corrections to the gauge boson propagator includes the diagrams



. (16.9)

The first two diagrams are as in QED and they give

$$\Pi^{\mu\nu ab} = i\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\delta^{ab}\Pi(q^2) \tag{16.10}$$

with

$$\begin{aligned}\Pi(q^2) = &-\tfrac{g^2}{(4\pi)^2}\tfrac{4}{3}c(r)n_f\,\Gamma\left(2-\tfrac{d}{2}\right) + \text{finite}\\ &-\delta_3\ ,\end{aligned} \tag{16.11}$$

where we allowed for the possibility of n_f fermions in representation r. The last three diagrams in (16.9) only arise in non-Abelian gauge theories and they give

$$\Pi(q^2) = \tfrac{g^2}{(4\pi)^2}\tfrac{5}{3}\,c_2(G)\Gamma\left(2-\tfrac{d}{2}\right) + \text{finite}\ . \tag{16.12}$$

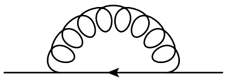
Imposing the renormalization condition

$$\Pi\left(q^2=-M^2\right)=0 \tag{16.13}$$

yields

$$\delta_3=\tfrac{g^2}{(4\pi)^2}\left[\tfrac{5}{3}c_2(G)-\tfrac{4}{3}n_f c(r)\right]\frac{\Gamma\left(2-\frac{d}{2}\right)}{\left(M^2\right)^{2-d/2}}\;. \tag{16.14}$$

The fermion self energy Σ_2 has the two contributions



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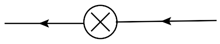


(16.15)

which are computed in problem 8.3 to give (for massless fermions)

$$\tfrac{ig^2}{(4\pi)^2}\not{p}c_2(r)\Gamma\left(2-\tfrac{d}{2}\right)+\text{finite}+i\not{p}\delta_2\;. \tag{16.16}$$

Imposing the renormalization conditions

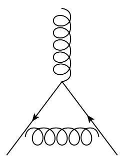
$$\Sigma\left(\not{p}=-M\right)=0\;,\qquad \frac{d}{d\not{p}}\Sigma\left(\not{p}=-M\right)=0 \tag{16.17}$$

yields[9]

$$\delta_2=-\tfrac{g^2}{(4\pi)^2}\,c_2(r)\,\frac{\Gamma\left(2-\frac{d}{2}\right)}{\left(M^2\right)^{2-d/2}} \tag{16.18}$$

Finally we consider the vertex corrections

$-ig\Gamma^\mu t^a \quad \sim$



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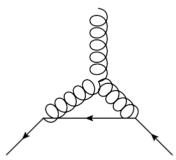

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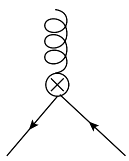


(16.19)

In problem 8.3 one computes

$$\begin{aligned}-ig\Gamma^\mu t^a =&\tfrac{ig^3}{(4\pi)^2}t^a\gamma^\mu\Gamma\left(2-\tfrac{d}{2}\right)\left(c_2(r)-\tfrac{1}{2}c_2(G)+\tfrac{3}{2}c_2(G)\right)+\text{finite}\\ &+ig\gamma^\mu t^a\delta_1\;,\end{aligned} \tag{16.20}$$

where the combination $c_2(r)-\frac{1}{2}c_2(G)$ arises from the first graph while $\frac{3}{2}c_2(G)$ results from the second graph. Imposing

$$-ig\Gamma^\mu\left(p'-p=-M\right)=-ig\gamma^\mu \tag{16.21}$$

yields

$$\delta_1=-\tfrac{g^2}{(4\pi)^2}\frac{\Gamma\left(2-\frac{d}{2}\right)}{\left(M^2\right)^{2-d/2}}\left(c_2(r)+c_2(G)\right) \tag{16.22}$$

[9]The first condition fixes δ_m.

The computation of the $\beta-$ and γ-functions proceeds as in QED since only the coefficients changed. Using

$$\frac{\Gamma\left(2-\frac{d}{2}\right)}{\left(M^2\right)^{2-d/2}} = \frac{2}{\epsilon} - \ln\left(M^2\right) + \ldots \ , \tag{16.23}$$

we obtain

$$\begin{aligned} \gamma_2 &= \frac{1}{2} M \partial_M \delta_2 = \tfrac{g^2}{(4\pi)^2} c_2(r) \ , \\ \gamma_3 &= \frac{1}{2} M \partial_M \delta_3 = \tfrac{g^2}{(4\pi)^2} \left(\tfrac{4}{3} n_f c(r) - \tfrac{5}{3} c_2(G)\right) \ , \end{aligned} \tag{16.24}$$

and

$$\beta(g) = gM \frac{\partial}{\partial M}\left(-\delta_1 + \delta_2 + \tfrac{1}{2}\delta_3\right) = -\tfrac{g^3}{(4\pi)^2}\left(\tfrac{11}{3} c_2(G) - \tfrac{4}{3} n_f\, c(r)\right) . \tag{16.25}$$

The physical significance of this results is that β can easily be negative leading to UV free (asymptotically free) theories. For example for $SU(n)$ one has $c(n) = \frac{1}{2}, c_2(G) = n$ resulting in

$$\beta(g) = -\tfrac{g^3}{(4\pi)^2}\left(\tfrac{11}{3} n - \tfrac{2}{3} n_f\right) \ , \tag{16.26}$$

which is negative for $11n - 2n_f > 0$. A prominent example of such a theory is QCD which we turn to in the next lecture.

17 Lecture 17: Quantum Chromo Dynamics (QCD)

17.1 Basic definitions and properties

The basic idea of QCD is that the strong interactions are mediated by an $SU(3)$ gauge theory. Thus there are $3^2-1 = 8$ gauge bosons of $SU(3)$ called gluons and we denote them by $G^a_\mu, a = 1, ..., 8$. The constituents of baryons and mesons are fermions called quarks and they transform in the fundamental three-dimensional representation of $SU(3)$. They are denoted by $q_i, i = 1, 2, 3$ and each of the $q_i(x)$ is a Dirac spinor. The $SU(3)$ quantum number is often called colour and in this nomenclature the q_i form a colour-triplet with i being the colour-index. Experimentally one observes six of these colour triplets and so together we denoted them by $q^I_i, I = 1, \ldots, n_f = 6$. Here one often says that there are six flavours of quarks and thus the index I is called the flavor index. Furthermore they are grouped in three families according to their electric charge.

family	quarks	charge	quark	charge
1	u (up)	2/3	d (down)	$-1/3$
2	c (charm)	2/3	s (strange)	$-1/3$
3	t (top)	2/3	b (bottom)	$-1/3$

The QCD Lagrangian reads

$$\mathcal{L}_{QCD} = -\tfrac{1}{4}\sum_{a=1}^{8} G^a_{\mu\nu} G^{a\mu\nu} - \sum_{I=1}^{6}\sum_{i=1}^{3}(i\bar{q}^I_i \not{D} q^I_i - m_{IJ}\bar{q}^I_i q^J_i) \ , \tag{17.1}$$

where the field strength is defined canonically as

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} G^b_\mu G^c_\nu \ , \tag{17.2}$$

and the covariant derivatives are

$$D_\mu q^I_i = \partial_\mu q^I_i - i g_s \sum_a \sum_j G^a_\mu t^a_{ij} q^I_j \ . \tag{17.3}$$

The t^a_{ij} are the generators of $SU(3)$, f^{abc} the corresponding structure constants and g_s is the strong coupling constant. The gauge transformation of the quarks therefore is

$$\delta q^I_i = i \sum_{a=1}^{8}\sum_{j=1}^{3} \alpha^a(x)\, t^a_{ij} q^I_j \ . \tag{17.4}$$

Before we continue let us pause and note that all six quarks are electrically charges and thus couple to the photon γ. Including the electromagnetic interactions the gauge group is $SU(3) \times U(1)_{\text{em}}$ and one has to add the Maxwell term to the Lagrangian of (17.1) with the covariant derivatives modified according to

$$D_\mu q^I_i = \partial_\mu q^I_i - i g_s \sum_a \sum_j G^a_\mu t^a_{ij} q^I_j - i Q_f e \gamma_\mu q^I_i \ , \tag{17.5}$$

where Q_f indicates the fraction of electric charge they carry, i.e. $Q_f = 2/3$ for u, c, t and $Q_f = -1/3$ for d, s, b. Accordingly the transformation law reads

$$\delta q_i^I = i \sum_{a=1}^{8} \sum_{j=1}^{3} \alpha^a(x)\, t_{ij}^a q_j^I + i\alpha_{\rm em}(x) q_i^I \ . \tag{17.6}$$

The β-function was determined in (16.26) to be

$$\beta(g) = -\frac{b_0 g^3}{(4\pi)^2} \ , \qquad \text{with} \qquad b_0 = \tfrac{11}{3} c_2(G) - \tfrac{4}{3} n_f c(r) \ , \tag{17.7}$$

which for QCD evaluates to

$$b_0 = \tfrac{11}{3}\, c_2(8) - \tfrac{4}{3}\, n_f\, c(3) = \tfrac{11}{3}\, 3 - \tfrac{4}{3}\, 6\, \tfrac{1}{2} = 7 > 0 \ , \tag{17.8}$$

where we used $c_2(8) = 3, c(3) = \frac{1}{2}, n_f = 6$. Thus QCD is an asymptotically free theory which is weakly coupled at high energies and strongly coupled at low energies. At low energies one observes experimentally only colour singlet states. This is called confinement which, however, has not been proved in QCD yet. These colour singlets are bound states of the quarks. More precisely one has

$$\begin{aligned} \text{mesons}: \quad & M^{IJ} = \sum_i \bar{q}_i^I q_i^I \ , \\ \text{baryons}: \quad & B^{IJK} = \sum_{ijk} \epsilon_{ijk} q_i^I q_j^J q_k^K \ . \end{aligned} \tag{17.9}$$

One checks that both combinations are $SU(3)$ singlets.

At high energies one can use perturbation theory in g_s as developed in this course. For low energies other methods such as lattice gauge theories are necessary. To estimate the scale $\Lambda_{\rm QCD}$ where QCD becomes strongly coupled one consider the solution of the CS equation which was determined in problem 6.4 to be

$$\bar{g}^{-2}(P) = \bar{g}^{-2}(M) + \tfrac{b_0}{(8\pi)^2}\, \ln(P/M) \tag{17.10}$$

or for $\alpha_s := \frac{g_s^2}{4\pi}$

$$\bar{\alpha}_s(P) = \frac{\bar{\alpha}_s(M)}{1 + \frac{b_0}{(2\pi)}\, \bar{\alpha}_s(M)\, \ln(P/M)} \ . \tag{17.11}$$

One now estimates $\Lambda_{\rm QCD}$ by the condition

$$\bar{\alpha}_s^{-1}(M = \Lambda_{\rm QCD}) = 0 \ . \tag{17.12}$$

Inserted into (17.10) one determines

$$\Lambda_{\rm QCD} = P e^{-\frac{2\pi}{b_0} \bar{\alpha}_s^{-1}(P)} \ . \tag{17.13}$$

In problem 9.3 it is shown that $\Lambda_{\rm QCD}$ is a renormalization group invariant scale in that it satisfies

$$\frac{d\Lambda_{\rm QCD}}{dP} = 0 \ . \tag{17.14}$$

Numerically one finds

$$\Lambda_{QCD} \approx 200 MeV \qquad \text{using} \qquad \bar{\alpha}_s(1GeV) = 0.4 \ . \tag{17.15}$$

17.2 Experimental observations

Even though perturbation is not applicable at low energies QCD is a well tested theory. The electromagnetic production of hadrons via the process $e^+e^- \to$ hadrons is very similar to the in QFT I computed process $e^+e^- \to \mu^+\mu^-$. There we found the cross section

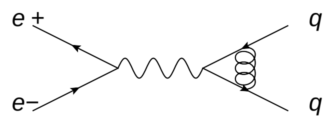
$$\sigma_{\rm tot}(e^+e^- \to \mu^+\mu^-) \approx \frac{4\pi\alpha_{\rm em}}{3E^2_{rmCM}} \equiv \sigma_0 \tag{17.16}$$

valid in the limit $E_{\rm CM} \gg m_\mu$. Here the difference is the different electric charge of the quarks and the multiplicity due to the colour quantum number. Thus one obtains

$$\sigma_{tot}(e^+e^- \to q\overline{q}) = 3\sigma_0 \sum_f Q_f^2 \ . \tag{17.17}$$

This is indeed observed including the jumps when additional quark flavor can be energetically excited.

The leading one-loop QCD correction arises from the diagram

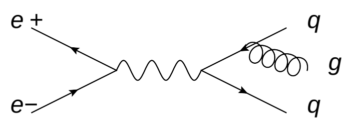


(17.18)

which leads to

$$\sigma_{tot} = 3\sigma_0 \sum_f Q_f^2 \left(1 + \frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2)\right) \ . \tag{17.19}$$

A further confirmation of QCD was the observation of three-jet events in the 70ies at DESY. They arise form diagrams such as



. (17.20)

The crucial experiment, however was the <u>SLAC-MIT</u> experiment in 1968 where $20GeV$ electrons were shot on a hydrogen-target. This in some sense repeated Rutherford's experiments in that a substructure of the proton became visible. At high momentum transfer the scattering process was seen to be on point-like constituents which were called <u>partons</u> at the time and later on identified with the quarks. Again there is a closely connected QED process $e^-\mu^- \to e^-\mu^-$ and the modification only comes from the fact that the quarks are bound in the proton. This is parametrized by the <u>parton distribution function</u> (PDF) $f_i(x)$ which gives the probability of finding the parton i inside the proton with x parameterizing the fraction of the proton momentum carried by the quark. With this parametrization one finds

$$\frac{d\sigma}{d\cos\theta} = \sum_i f_i(x) Q_i^2 \left(\frac{d\sigma}{d\cos\theta}\right)_{\rm QED} , \tag{17.21}$$

which is indeed observed with $10 - 20\%$ accuracy. (For further details see [2].)

18 Lecture 18: Spontaneous Symmetry Breaking & Goldstone's Theorem

The simplest theory which displays spontaneous symmetry breaking is a real scalar field with Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\ , \qquad \text{where} \qquad V = -\tfrac{1}{2}\mu^2\phi^2 + \tfrac{1}{4}\lambda\phi^4\ , \quad \mu^2, \lambda > 0\ . \tag{18.1}$$

Due to the "wrong sign" of the quadratic term μ is not the mass of the field. The minimum of $V(\phi)$ is found from

$$\frac{\partial V}{\partial\phi} = \phi(-\mu^2 + \lambda\phi^2) = 0 \tag{18.2}$$

to be at $\phi = \pm\sqrt{\mu^2/\lambda}$ while $\phi = 0$ is a local maximum.

The Lagrangian (18.1) has a discrete symmetry $\phi \to -\phi$ which leaves $\mathcal{L}$ unchanged, i.e. $\mathcal{L}(\phi) = \mathcal{L}(-\phi)$. However the minimum does not have this symmetry. Instead it spontaneously breaks it. This is the prototype example of spontaneous symmetry breaking:

The theory has a global symmetry which is not shared by its ground state.

For a continuous symmetry this phenomenon can be observed for a complex scalar field $\phi = \frac{1}{\sqrt{2}}(\phi_1 + \imath\phi_2)$ with Lagrangian

$$\mathcal{L} = \partial_\mu\phi\partial^\mu\phi^* - V(\phi,\phi^*)\ , \qquad \text{where} \qquad V = -\mu^2\phi\phi^* + \tfrac{1}{2}\lambda(\phi\phi^*)^2\ , \quad \mu^2, \lambda > 0\ . \tag{18.3}$$

This $\mathcal{L}$ has a global $U(1)$ symmetry $\phi \to \phi' = e^{\imath\alpha}\phi, \alpha \in \mathbb{R}$ in that $\mathcal{L}(\phi', \phi^{*\prime}) = \mathcal{L}(\phi, \phi^*)$. In this case the minimum is found at

$$\frac{\partial V}{\partial\phi} = \phi^*(-\mu^2 + \lambda\phi\phi^*) = 0 \tag{18.4}$$

which implies

$$\phi\phi^* = \frac{\mu^2}{\lambda}\ . \tag{18.5}$$

This is the equation of a circle and thus the minimum is a one-dimensional field space. Any values on that circle breaks the symmetry spontaneously. (At $\phi = 0 = \phi^*$ we have again a local maximum.) It is convenient to expand $\phi = \frac{1}{\sqrt{2}}(v + h(x) + \imath\sigma(x))$ with

$$\tfrac{1}{2}v^2 = \frac{\mu^2}{\lambda}\ , \qquad \text{such that} \qquad \phi|_{\text{min}} = \tfrac{1}{\sqrt{2}}v\ , \qquad h|_{\text{min}} = \sigma|_{\text{min}} = 0\ . \tag{18.6}$$

Inserted into V one obtains from (18.3)

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu h\partial^\mu h + \tfrac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - V(h,\sigma) \tag{18.7}$$

with

$$V(h,\sigma) = -\tfrac{1}{2}\mu^2((v+h)^2 + \sigma^2) + \tfrac{1}{8}\lambda((v+h)^2 + \sigma^2)^2 = V(v) + \mu^2h^2 + \text{cubic}\ , \tag{18.8}$$

where we used $\frac{v^2}{2} = \frac{\mu^2}{\lambda}$ in the second step. Thus we see that $\sigma(x)$ is massless while $h(x)$ has mass $m^2 = 2\mu^2$. $\sigma(x)$ is called Goldstone boson.

As a next generalization consider N complex scalar fields ϕ^i with Lagrangian

$$\mathcal{L} = \sum_{i=1}^{N} \partial_\mu \phi^i \partial^\mu \phi^{i*} - V(\phi^i, \phi^{i*}) \ , \qquad V = -\mu^2 \sum_i \phi^i \phi^{i*} + \tfrac{\lambda}{2} \big(\sum_i \phi^i \phi^{i*} \big)^2 \ . \tag{18.9}$$

$\mathcal{L}$ has a global $U(N)$ symmetry

$$\phi^i \to \phi'^i = \sum_j U^{ij} \phi^j \ , \qquad \phi^{i*} \to \phi'^{i*} = \sum_j \phi^{j*} U^{\dagger ji} \ , \qquad UU^\dagger = \mathbb{1} \ . \tag{18.10}$$

The minimum of V is found by solving

$$\frac{\partial V}{\partial \phi^i} = \phi^{i*}(-\mu^2 + \lambda \sum_j \phi^j \phi^{j*}) = 0 \ , \qquad \frac{\partial V}{\partial \phi^{i*}} = \phi^{i}(-\mu^2 + \lambda \sum_j \phi^j \phi^{j*}) = 0 \ , \tag{18.11}$$

to be on the $2N$-dimensional sphere

$$\sum_j \phi^j \phi^{j*} \ = \ \frac{\mu^2}{\lambda} \ . \tag{18.12}$$

One can parametrize the field space by

$$\phi^1 = \tfrac{1}{\sqrt{2}} \big(v + h(x) + \imath \sigma(x) \big), \phi^2, ..., \phi^N \ , \tag{18.13}$$

such that

$$\frac{v^2}{2} = \frac{\mu^2}{\lambda} \ , \qquad \phi^1|_{\rm min} = \tfrac{1}{\sqrt{2}} v \ , \qquad h|_{\rm min} = \sigma|_{\rm min} = \phi^2|_{\rm min} = \ldots = \phi^N|_{\rm min} = 0 \ . \tag{18.14}$$

We see that the minimum breaks the $U(N)$ to a residual $U(N-1)$ acting on $\phi_2, \ldots, \phi_N$. Inserting (18.13) into V given in (18.9) one computes

$$V = V(v) + \mu^2 h^2 + \text{cubic} \ . \tag{18.15}$$

We thus see that one scalar fields h is massive (with mass $m^2 = \mu^2$) while the $2N-1$ scalars $\sigma, \phi_2, \ldots, \phi_N$ are massless. These are again the Goldstone bosons.

Generically the number of massless scalar fields in a theory with a spontaneously broken global symmetry is determined by the Goldstone-theorem. It says:

For every spontaneously broken continuous symmetry there exists a real massless scalar field called Goldstone boson.

Let us prove the theorem. Consider N real scalars $\phi^i, i = 1, \ldots, N$ with Lagrangian

$$\mathcal{L} = \tfrac{1}{2} \sum_{i=1}^{N} \partial_\mu \phi^i \partial^\mu \phi^i - V(\phi^i) \ . \tag{18.16}$$

The kinetic term has a global $O(N)$ symmetry. The potential V we leave arbitrary and only assume that it has a global symmetry $G \subset O(N)$ with transformation law

$$\phi^i \to \phi^{i\prime} = \phi^i + \delta \phi^i \ , \qquad \text{with} \qquad \delta \phi^i = \imath \sum_{a=1}^{dim(G)} \alpha^a (t^a)^{ij} \phi^j \ , \tag{18.17}$$

where the t^a are the generators of G and $\alpha \in \mathbb{R}$ is the global transformation parameter. Since V is assumed to be invariant under G we have

$$V(\phi^{i\prime}) = V(\phi^i + \delta\phi^i) = V(\phi^i) + \sum_i \frac{\partial V}{\partial \phi^i}\delta\phi^i + \ldots \;= V(\phi^i) \;, \tag{18.18}$$

which implies at first order

$$\sum_i \frac{\partial V}{\partial \phi^i}\,\delta\phi^i = 0 \;. \tag{18.19}$$

Differentiating this equation w.r.t. ϕ^j yields

$$\sum_i \left(\frac{\partial^2 V}{\partial\phi^j \partial\phi^i}\delta\phi^i + \frac{\partial V}{\partial \phi^i}\frac{\partial}{\partial\phi^j}\delta\phi^i \right) = 0 \;. \tag{18.20}$$

Evaluated at the minimum where $\frac{\partial V}{\partial \phi^i} = 0$ holds we obtain

$$\sum_i m_{ij}^2 \delta\phi^i\big|_{\min} = 0 \;, \tag{18.21}$$

where we defined the $N \times N$ mass matrix

$$m_{ij}^2 \;\equiv\; \frac{\partial^2 V}{\partial\phi^i\partial\phi^j}\Big|_{\min} \;. \tag{18.22}$$

Since m_{ij}^2 is symmetric it can be diagonalized to $m_{ij}^2 = m_i^2\delta_{ij}$. Inserted into (18.21) we arrive at

$$\sum_i m_i^2\,\delta\phi^i\big|_{\min} = 0 \;. \tag{18.23}$$

One defines the unbroken generators t_u of G to be those which leave the minimum invariant, i.e. which obey

$$(t_u)^{ij}\phi^j|_{\min} = 0 \;, \qquad \text{implying} \qquad \delta_u\phi^i\big|_{\min} = 0 \;. \tag{18.24}$$

The broken generators t_b of G on the other hand are those which transform the ground state, i.e. which obey

$$(t_b)^{ij}\phi^j|_{\min} \neq 0 \;, \qquad \text{implying} \qquad \delta_b\phi^i\big|_{\min} \neq 0 \;. \tag{18.25}$$

Inspecting (18.23) we see that it is automatically satisfied for all unbroken generators while for the broken generators $m_i^2 = 0$ has to hold.

In order to confirm the physical meaning of the mass matrix m_{ij}^2 let us Taylor expand the potential around $\phi^i = \phi^i|_{\min} + \Delta\phi^i$

$$\begin{aligned} V(\phi^i|_{\min} + \Delta\phi^i) &= V(\phi^i|_{\min}) + \tfrac{1}{2}\sum_{ij} m_{ij}^2 \Delta\phi^i\Delta\phi^j + \ldots \\ &= V(\phi^i|_{\min}) + \tfrac{1}{2}\sum_{I} m_i^2 \Delta\phi^i\Delta\phi^i + \ldots \;. \end{aligned} \tag{18.26}$$

We see that m_i^2 are the mass parameters of the fields ϕ^i when they are expanded around their background values. Therefore we have completed the proof of Goldstone's theorem in that we showed that for each broken generator there exists a massless scalar field.

To conclude let us check the consistency of the $U(N)$ example with Lagrangian (18.9). In this case we found that the $U(N)$ is spontaneously broken to $U(N-1)$ and thus the number of broken generators is

$$N^2 - (N-1)^2 = 2N - 1 \ , \tag{18.27}$$

which indeed coincides with the number of massless scalars we found. Another example is given in problem 9.4.

19 Lecture 19: Higgs Mechanism

In the previous lecture we considered spontaneous symmetry breaking for a global symmetry and proved that a massless scalar field, the Goldstone boson, is necessarily present. In this lecture we generalize the analysis in that we consider instead local gauge symmetries. As we will see this results in the possibility of massive gauge bosons where the longitudinal degree of freedom is precisely the Goldstone boson.

Let us start with an Abelian example and consider the Lagrangian

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + D^\mu \phi^* D_\mu \phi - V(\phi, \phi^*) \ , \tag{19.1}$$

where ϕ is a complex charged scalar field. The covariant derivative and the potential read

$$D_\mu \phi = \partial_\mu \phi - igA_\mu \phi \ , \qquad V = -\mu^2 \phi\phi^* + \tfrac{1}{2}\lambda(\phi\phi^*)^2 \ , \qquad \mu^2, \lambda > 0 \ . \tag{19.2}$$

$\mathcal{L}$ is invariant under the gauge symmetry

$$\phi \to \phi' = e^{i\alpha(x)} \phi \ , \qquad A_\mu \to A'_\mu = A_\mu + \tfrac{1}{g}\partial_\mu \alpha \ . \tag{19.3}$$

The minimum of V is found at

$$\phi^*|_{\min} = \phi|_{\min} = \tfrac{1}{\sqrt{2}} v = \sqrt{\frac{\mu^2}{\lambda}} \ . \tag{19.4}$$

Here we use a different parametrization than in the previous lecture in that we change variables $(\phi, \phi^*) \to (h, \beta)$ according to

$$\phi(x) = \tfrac{1}{\sqrt{2}}\big(v + h(x)\big) e^{i\beta(x)} \ , \qquad \phi^*(x) = \tfrac{1}{\sqrt{2}}\big(v + h(x)\big) e^{-i\beta(x)} \ . \tag{19.5}$$

$h(x)$ will be identified as the Higgs boson while $\beta(x)$ will be the Goldstone boson. Inserted into V we obtain

$$V(h) = -\mu^2 (v+h)^2 + \tfrac{1}{8}\lambda(v+h)^4 = V(v) + \mu^2 h^2 = \tfrac{1}{2}\lambda v h^3 + \tfrac{1}{8}\lambda h^4 \ . \tag{19.6}$$

We see that the Goldstone boson β drops out completely from the potential. Inserting (19.5) into the kinetic term of ϕ yields

$$D^\mu \phi^* D_\mu \phi = \tfrac{1}{2}\Big(\partial^\mu h + i(gA^\mu - \partial^\mu \beta)(v+h)\Big)\Big(\partial_\mu h - i(gA_\mu - \partial_\mu \beta)(v+h)\Big) \ . \tag{19.7}$$

We see that β can be removed by the gauge transformation

$$A_\mu \to A'_\mu = A_\mu + \tfrac{1}{g}\partial_\mu \beta \ , \tag{19.8}$$

which leaves $F_{\mu\nu}$ invariant. This gauge is called the <u>unitary</u> gauge and it corresponds to a field basis where the Goldstone boson β is removed from the entire Lagrangian in that $\mathcal{L}$ of (19.1) now reads

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2}\partial^\mu h \partial_\mu \phi - V(h) + \tfrac{1}{2} m_A^2 A_\mu A^\mu + (g^2 v h + \tfrac{1}{2} g^2 h^2) A_\mu A^\mu \ . \tag{19.9}$$

We see that the gauge boson has a mass term given by $m_A^2 = g^2 v^2$ while the last term is an $A_\mu - h$ interaction term. Let us count the degrees of freedom. In the unbroken phase both A_μ and ϕ each have two real physical degrees of freedom. In the broken phase the massive A_μ has $2s+1=3$ degrees of freedom while there is only one real scalar h left. Also from the gauge transformation (19.8) we see that β plays the role of the longitudinal degree of freedom of a massive A_μ.

Let us generalize this situation to an $SU(n)$ gauge symmetry with complex scalar fields transforming in the fundamental **n**-dimensional representation of $SU(n)$. The Lagrangian reads

$$\mathcal{L} = -\tfrac{1}{4}\sum_{a=1}^{n^2-1} F^a_{\mu\nu}F^{a\mu\nu} + \sum_{i=1}^{n} D^\mu\phi^{*i}D_\mu\phi^i - V(\phi^i,\phi^{*i}) \ , \tag{19.10}$$

where the covariant derivative and the potential read

$$\begin{aligned} D_\mu\phi^i &= \partial_\mu\phi^i - igA^a_\mu t^{aij}\phi^j \ , \qquad D_\mu\phi^{*i} = \partial_\mu\phi^i + igA^a_\mu t^{*aij}\phi^j \ , \\ V &= -\mu^2\phi^{*i}\phi^i + \tfrac{1}{2}\lambda(\phi^{*i}\phi^i)^2 \ , \qquad \mu^2,\lambda>0 \ . \end{aligned} \tag{19.11}$$

$\mathcal{L}$ is invariant under the gauge symmetry

$$\phi^i \to \phi i\prime = U^{ij}\phi^j \ , \qquad A_\mu \to A'_\mu = UA_\mu U^\dagger - \tfrac{i}{g}U\partial_\mu U^\dagger \ . \tag{19.12}$$

The minimum of V is found at

$$\phi^{*i}|_{\min} = \phi^i|_{\min} = \frac{v^i}{\sqrt{2}} = \sqrt{\frac{\mu^2}{\lambda}} \ . \tag{19.13}$$

Expanding ϕ^i around its background or vacuum expectation value v^i as $\phi^i = \frac{1}{\sqrt{2}}v^i + \ldots$ we can directly compute the gauge boson mass term to be

$$\mathcal{L}_m = \tfrac{1}{2}m^2_{ab}A^a_\mu A^{b\mu} \ . \tag{19.14}$$

The mass matrix is given by

$$m^2_{ab} = g^2\, t^{*aij}v^jt^{bik}v^k = g^2\, v^jt^{aji}t^{bik}v^k = g^2\,\tfrac{1}{2}v^j\{t^a,t^b\}^{jk}v^k \ , \tag{19.15}$$

where we used that the generators are hermitian and that the mass matrix is symmetric. From (19.15) we see immediately that it vanishes for the unbroken generators which satisfy $t^{bik}v^k = 0$. In the previous lecture we found that there are $2N-1$ broken generators for which $t^{bik}v^k \neq 0$ and thus m^2_{ab} has $2N-1$ non-zero eigenvalues or in other words $2N-1$ massive gauge bosons. With this information we can parametrize the unitary gauge as

$$\phi^i \ = \ \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \ldots \\ 0 \\ v+h(x) \end{pmatrix} U(x) \ , \qquad U = e^{i\sum_{a=1}^{2N-1}\beta^a(x)\,t^a_{\mathrm{br}}} \ , \tag{19.16}$$

where t^a_{br} denote the broken generators and the $2N-1$ Goldstone bosons β^a are real. One can check that U is unitary and thus can be removed by an appropriate gauge transformation. In this gauge the potential is again given by (19.6).

Let us close this lecture with two explicit examples. First consider an $SU(2)$ theory with one Higgs doublet ϕ^i. In this case the generators are the Pauli matrices, i.e. $t^a = \frac{1}{2}\sigma^a$ and we parametrize ϕ^i as in (19.16). Inserted into (19.15) we obtain

$$m_{ab}^2 = \tfrac{1}{8}g^2\, v^j \sigma^{aji}\sigma^{bik}v^k = \tfrac{1}{8}g^2 v^2 \delta^{ab} \ . \tag{19.17}$$

We see that all 3 gauge bosons are massive and the $SU(2)$ is completely broken.

Finally the electro-weak sector of the standard model is based on the gauge group $G = SU(2) \times U(1)_Y$ broken to $U(1)_{em}$. $U(1)_Y$ denotes the hyper charge while $U(1)_{em}$ denotes QED. The Higgs-doublet is charged under both factors and thus the Lagrangian reads

$$\mathcal{L} = -\tfrac{1}{4}\sum_{a=1}^{3} F_{\mu\nu}^a F^{a\mu\nu} - \tfrac{1}{4}B_{\mu\nu}B^{\mu\nu} + \sum_{i=1}^{2} D^\mu \phi^{*i} D_\mu \phi^i - V(\phi^i, \phi^{*i}) \ , \tag{19.18}$$

where $F_{\mu\nu}^a$ is the field strength of the $SU(2)$ factor while $B_{\mu\nu}$ is the field strength of $U(1)_Y$. The potential is again given by (19.2) and the covariant derivative read

$$D_\mu \phi^i = \partial_\mu \phi^i - \tfrac{i}{2} g A_\mu^a \sigma^{aij}\phi^j - iyg' B_\mu \phi^i \ , \tag{19.19}$$

where $y = 1/2$ is the hyper charge of ϕ. $\mathcal{L}$ is invariant under the gauge symmetry

$$\delta\phi^i = \tfrac{i}{2}\alpha^a \sigma^{aij}\phi^j + \tfrac{i}{2}\alpha_y\, \phi^i \ . \tag{19.20}$$

Parameterizing ϕ^i by

$$\phi^i = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} + \ldots \tag{19.21}$$

we find

$$D^\mu \phi^{*i} D_\mu \phi^i = \tfrac{1}{8}(0, v)\Big(gA^{a\mu}\sigma^a + 2yg'B_\mu \mathbb{1}\Big)\Big(gA^{a\mu}\sigma^a + 2yg'B_\mu \mathbb{1}\Big)\begin{pmatrix} 0 \\ v \end{pmatrix} + \ldots \ . \tag{19.22}$$

Using

$$(0, v)\sigma^a \begin{pmatrix} 0 \\ v \end{pmatrix} = -v^2 \delta^{a3} \ , \qquad (0, v)\sigma^a \sigma^b \begin{pmatrix} 0 \\ v \end{pmatrix} = v^2 \delta^{ab} \ , \tag{19.23}$$

we obtain

$$D^\mu \phi^{*i} D_\mu \phi^i = \tfrac{v^2}{8}\Big(g^2(A^{1\mu}A_\mu^1 + A^{2\mu}A_\mu^2) + (gA_\mu^3 - 2yg'B_\mu)(gA^{3\mu} - 2yg'B^\mu)\Big) + \ldots \ . \tag{19.24}$$

Written as a mass matrix as in (19.14) this expression corresponds to

$$m_{ab}^2 = \frac{v^2}{8}\begin{pmatrix} g^2 & & & \\ & g^2 & & \\ & & g^2 & -gg' \\ & & -gg' & g'^2 \end{pmatrix} \ . \tag{19.25}$$

Now one defines

$$\begin{aligned} W_\mu^\pm &:= \tfrac{1}{\sqrt{2}}(A^1\mu \pm iA^2\mu) \ , \\ \begin{pmatrix} Z_\mu^0 \\ \gamma_\mu \end{pmatrix} &:= O \begin{pmatrix} A_\mu^3 \\ B_\mu \end{pmatrix} \ , \qquad \text{where} \qquad O \equiv \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \ , \end{aligned} \tag{19.26}$$

with

$$\cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}\ , \qquad \sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}\ . \tag{19.27}$$

θ_W is called the weak or Weinberg angle. Inserted into (19.24) yields

$$D^\mu\phi^{*i}D_\mu\phi^i = m_W^2 W^{+\mu}W_\mu^- + \tfrac{1}{2}m_Z^2 Z_\mu^0 Z^{\mu 0} + \dots\ . \tag{19.28}$$

where

$$m_W = \tfrac{1}{2}gv\ , \qquad m_Z = \tfrac{1}{2}\sqrt{g^2+g'^2}\, v = \frac{m_W}{\cos\theta_W}\ . \tag{19.29}$$

We see that three massive gauge bosons $W_\mu^\pm, Z_\mu^0$ arise while the photon γ_μ stays massless. Thus we indeed observe the spontaneous symmetry breaking $SU(2)\times U(1)_Y \to U(1)_{em}$.

20 Lecture 20: Chiral gauge theories and the Glashow-Salam-Weinberg theory

20.1 Chiral gauge theories

Let us first recall from QFT I that any four-component Dirac spinor ψ_D can be decomposed into two two-component Weyl spinors $\psi_{L,R}$ via

$$\psi_D = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} , \qquad \overline{\psi} \ = \psi^\dagger \gamma^0 = \left(\overline{\psi}_R, \overline{\psi}_L\right) , \tag{20.1}$$

where

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} , \qquad \sigma^\mu \ = (\mathbf{1}, \vec{\sigma}) , \qquad \overline{\sigma}^\mu = (\mathbf{1}, -\vec{\sigma}) . \tag{20.2}$$

The fermionic Lagrangian decomposes accordingly

$$\mathcal{L} = \overline{\psi}_D (i\gamma^\mu D_\mu + m)\psi_D = i\overline{\psi}_L \sigma^\mu D_\mu \psi_L + i\overline{\psi}_R \overline{\sigma}^\mu D_\mu \psi_R + m\left(\overline{\psi}_L \psi_R + \overline{\psi}_R \psi_L\right) . \tag{20.3}$$

Gauge theories are called chiral gauge theories whenever ψ_L and ψ_R transform in different representations of the gauge group G.

The simplest example are chiral $U(1)$ theories with the transformations

$$\psi_L \to \psi_L' = e^{iy_L\alpha(x)}\psi_L , \qquad \psi_R \to \psi_R' = e^{iy_R\alpha(x)}\psi_R . \tag{20.4}$$

Note that they transform with the same local function $\alpha(x)$ but in general with different charges $y_L \neq y_R$. (For $y_L = y_R$ the $U(1)$ theory is called vector-like.) The corresponding covariant derivatives now read

$$D_\mu \psi_{L,R} = \partial_\mu \psi_{L,R} - igy_{L,R} A_\mu \psi_{L,R} . \tag{20.5}$$

From (20.3) we see that for $y_L \neq y_R$ the Lagrangian is only invariant for $m = 0$ while for $y_L = y_R$ an invariant mass term $m \neq 0$ is possible. Or in other words, chiral gauge theories forbid fermionic mass terms.

The non-Abelian generalization has the same $\mathcal{L}$ as in (20.3) but now with covariant derivatives

$$D_\mu \psi_L = \partial_\mu \psi_L - igA_\mu^a t_{r_L}^a \psi_L , \qquad D_\mu \psi_R = \partial_\mu \psi_R - igA_\mu^a t_{r_R}^a \psi_R , \tag{20.6}$$

and transformation laws

$$\delta\psi_L = i\alpha^a t_{r_L}^a \psi_L , \qquad \delta\psi_r = i\alpha^a t_{r_R}^a \psi_R . \tag{20.7}$$

In a chiral gauge theory the representations $\mathbf{r_L}$ and $\mathbf{r_R}$ are different and therefore no gauge invariant mass term is possible. However in spontaneously broken chiral gauge theories a fermionic mass can be generated by the Higgs mechanism. Let us see how this works.

We start with the chiral $U(1)$ gauge theory and add to (20.3) a Yukawa coupling of the form

$$\mathcal{L}_{\text{Yuk}} = \lambda(\phi\bar{\psi}_L\psi_R + \phi^*\bar{\psi}_R\psi_L) . \tag{20.8}$$

The combination $\bar{\psi}_L \psi_R$ is Lorentz invariant and $\mathcal{L}_{\text{Yuk}}$ is also gauge invariant provided we assign for ϕ the transformation law

$$\phi \to \phi' = e^{i(y_L - y_R)\alpha(x)}\phi \ . \tag{20.9}$$

Using the parametrization (19.5) we obtain

$$\mathcal{L}_{\text{Yuk}} = m(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L) + \dots \ , \qquad \text{where} \qquad m = \tfrac{1}{\sqrt{2}}\lambda v \ . \tag{20.10}$$

Thus we see that the Higgs mechanism generates both gauge boson masses and fermion masses in chiral gauge theories.

In non-Abelian gauge theories one adds the same Yukawa interaction (20.8) with ϕ in a representation such that the product $\phi\bar{\psi}_L\psi_R$ is a singlet. As example we now discuss the GSW model.

20.2 GSW model

The GSW (Glashow-Salam-Weinberg) model (with one family) is a chiral gauge theory with gauge group $G = SU(2) \times U(1)_Y$ spontaneously broken to $U(1)_{em}$. The spectrum is summarized in table 20.2.

Spectrum		SU(2)	Y	Q
1) Gauge bosons				
	$A_\mu^{a=1,2,3}$	$\mathbf{3}$	0	$0, \pm 1$
	B_μ	$\mathbf{1}$	0	0
2) Weyl fermions				
	$E_L^{i=1,2} = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L$	$\mathbf{2}$	$-\frac{1}{2}$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$
	e_R^-	$\mathbf{1}$	-1	-1
3) Higgs boson				
	$\phi^i = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$	$\mathbf{2}$	$\frac{1}{2}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Table 20.0: Spectrum of GSW model

The Lagrangian is given by

$$\begin{aligned} \mathcal{L} = &- \tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \tfrac{1}{4}B_{\mu\nu}B^{\mu\nu} + D^\mu\phi^{*i}D_\mu\phi^i - V(\phi^i, \phi^{*i}) \\ &+ i\bar{E}^i_L\sigma^\mu D_\mu E^i_L + i\bar{e}_r\sigma^\mu D_\mu e_r + \lambda(\phi^i\bar{E}^i_L e_R + \phi^{*i}\bar{e}_R E^i_L) \ , \end{aligned} \tag{20.11}$$

where $F^a_{\mu\nu}$ is the field strength of the $SU(2)$ factor while $B_{\mu\nu}$ is the field strength of $U(1)_Y$. The potential is again given by (19.11) and the covariant derivative of ϕ in (19.19). The covariant derivatives of the fermions reads

$$\begin{aligned} D_\mu E^i_L &= \partial_\mu E^i_L - \tfrac{i}{2}gA^a_\mu\sigma^a_{ij}E^j_L - ig'y(E_L)B_\mu E^i_L \ , \\ D_\mu e_r &= \partial_\mu e_R - ig'y(e_R)B_\mu e_R \ , \end{aligned} \tag{20.12}$$

where $y(E_L) = -\frac{1}{2}$ and $y(e_R) = -1$.

The spontaneous symmetry breaking in the Higgs sector of this theory we already discussed in the previous lecture. Thus we can insert the redefinitions (19.26)–(19.29) and furthermore define

$$\begin{aligned} e :=& \frac{gg'}{\sqrt{g^2+g'^2}} = g\sin\theta_W = g'\cos\theta_W \ , \\ Q :=& \tfrac{1}{2}\sigma^3 + y\mathbb{1}_{2\times 2} = \begin{pmatrix} \frac{1}{2}+y & 0 \\ 0 & -\frac{1}{2}+y \end{pmatrix} \ . \end{aligned} \tag{20.13}$$

This yields

$$\begin{aligned} D_\mu E_L &= \partial_\mu E_L - \tfrac{i}{\sqrt{2}} g\left(W_\mu^+\sigma^+ + W_\mu^-\right)E_L - \frac{ig}{\cos\theta_W} Z_\mu^0\left(\tfrac{1}{2}\sigma^3 - \sin^2\theta_W Q\right)E_L - ie\gamma_\mu Q E_L \ , \\ D_\mu e_R &= \partial_\mu e_R + \frac{ig}{\cos\theta_W} Z_\mu^0 \sin^2\theta_W e_R + ie\gamma_\mu e_R \ , \end{aligned} \tag{20.14}$$

where we defined $\sigma^\pm = \frac{1}{2}(\sigma^1 \pm i\sigma^2)$.

From the Yukawa interaction we read off the fermionic mass term. Inserting (19.21) we obtain

$$\mathcal{L}_m = m_e(\bar{e}_L e_r + \bar{e}_R e_L) \ , \qquad \text{for} \quad m_e = \tfrac{1}{\sqrt{2}}\lambda v \ . \tag{20.15}$$

Thus the electron receives a Dirac mass terms while the neutrino ν_e is massless. In problem 10.3 we show that introducing a right-handed neutrino ν_R which is a singlet under the entire $SU(2) \times U(1)_Y$ with a Majorana mass term also generates a mass for the neutrino.

21 Lecture 21: GSW II

In the previous lecture we discussed the GSW model with one family of leptons. However three families are experimentally observed and they can be accommodated by adding

$$
\begin{aligned}
E_L^{iI} &= \begin{pmatrix}\nu_e\\ e^-\end{pmatrix}_L, \begin{pmatrix}\nu_\mu\\ \mu^-\end{pmatrix}_L, \begin{pmatrix}\nu_\tau\\ \tau^-\end{pmatrix}_L , \qquad I=1,2,3\\
e_R^I &= e_R, \mu_R, \tau_R \ ,
\end{aligned}
\tag{21.1}
$$

that is the index $I = 1, 2, 3$ counts the different families. The quantum numbers for each family are identical and as given in table 20.2. The Lagrangian is as in (20.11) with an additional sum over I and modified Yukawa interactions

$$
\begin{aligned}
\mathcal{L} =& -\tfrac14 F^a_{\mu\nu}F^{a\mu\nu} - \tfrac14 B_{\mu\nu}B^{\mu\nu} + D^\mu\phi^{*i}D_\mu\phi^i - V(\phi^i,\phi^{*i})\\
&+ i\sum_{I=1}^3 \bar{E}_L^{iI}\sigma^\mu D_\mu E_L^{iI} + i\sum_{I=1}^3 \bar{e}_r^I\sigma^\mu D_\mu e_r^I + \sum_{IJ}\lambda_{IJ}(\phi^i\bar{E}_L^{iI}e_R^J + \phi^{*i}\bar{e}_R^J E_L^{iI}) \ .
\end{aligned}
\tag{21.2}
$$

The covariant derivatives are as in (20.14) as they are identical for each family. The Yukawa couplings are now 3×3 matrices in family space and as a consequence the spontaneous symmetry breaking yields the mass matrices

$$
m_{IJ} = \tfrac{v}{\sqrt{2}}\lambda_{IJ} \ , \tag{21.3}
$$

with eigenvalues which are identified with m_e, m_μ, m_τ.

It is convenient to rewrite the Lagrangian (21.2) and explicitly display the interaction of the gauge bosons with the fermionic currents.

$$
\begin{aligned}
\mathcal{L} =& -\tfrac14 F^a_{\mu\nu}F^{a\mu\nu} - \tfrac14 B_{\mu\nu}B^{\mu\nu} + D^\mu\phi^{*i}D_\mu\phi^i - V(\phi^i,\phi^{*i})\\
&+ i\sum_{I=1}^3 \bar{E}_L^{iI}\sigma^\mu \partial_\mu E_L^{iI} + i\sum_{I=1}^3 \bar{e}_r^I\sigma^\mu \partial_\mu e_r^I + \sum_{IJ}\lambda_{IJ}(\phi^i\bar{E}_L^{iI}e_R^J + \phi^{*i}\bar{e}_R^J E_L^{iI})\\
&+ g\left(W_\mu^+ J_W^{-\,\mu} + W_\mu^- J_W^{+\,\mu} + Z_\mu^0 J_Z^\mu\right) + eA_\mu J^\mu_{\text{em}} \ ,
\end{aligned}
\tag{21.4}
$$

where

$$
\begin{aligned}
J_W^{-\,\mu} &= \tfrac{1}{\sqrt2}\bar\nu_L^I\sigma^\mu e_L^I \ , \qquad J_W^{+\,\mu} = \tfrac{1}{\sqrt2}\bar e_L^I\sigma^\mu \nu_L^I \ , \qquad J^\mu_{\text{em}} = -(\bar e_L^I\sigma^\mu e_L^I + \bar e_R^I\sigma^\mu e_R^I) \ ,\\
J_Z^\mu &= \frac{1}{\cos\Theta_w}\left[\tfrac12\bar\nu_L\sigma^\mu\nu_L + \left(-\tfrac12+\sin^2\theta_w\right)\bar e_L\sigma^\mu e_L + \sin^2\theta_w\bar e_R\sigma^\mu e_R\right].
\end{aligned}
\tag{21.5}
$$

In this form we easily see that QED processes like $e^-e^-\to\mu^-\mu^-$ or $e^-e^+\to\mu^-\mu^+$ receive (measured) corrections from a Z^0 exchange. Furthermore the charged current interactions lead to new (observed) processes such as electron-neutrino scattering $e^-\nu\to e^-\nu$.

From the form (21.4) we can also easily derive the relation with Fermi's theory of the weak interactions. At low energies $p \ll m_{W^\pm,Z^0}$ one neglects the kinetic term of the heavy gauge bosons $W^\pm, Z^0$ such that the field equation become algebraic

$$
\begin{aligned}
\frac{\delta\mathcal{L}}{\delta W_\mu^\pm} &= gJ^{\mp\,\mu} - m_W^2 W^{\mp\,\mu} = 0 \qquad \Rightarrow \qquad W^{\pm\,\mu} = \frac{g}{m_W^2}J^\pm \ ,\\
\frac{\delta\mathcal{L}}{\delta Z_\mu^0} &= gJ_Z^\mu - m_Z^2 Z^{0\,\mu} = 0 \qquad \Rightarrow \qquad Z^{0\,\mu} = \frac{g}{m_Z^2}J_Z^\mu \ .
\end{aligned}
\tag{21.6}
$$

Inserted back into $\mathcal{L}$ one obtains

$$\mathcal{L}_{\mathrm{F}} = \tfrac{8}{\sqrt{2}}\, G_F J^+_\mu J^{-\,\mu} + \tfrac{g^2}{m_Z^2}\, J_{Z\,\mu} J_Z^\mu \ , \tag{21.7}$$

where $G_F = \frac{8}{\sqrt{2}} \frac{g^2}{m_W^2}$ is the Fermi constant. The first term in (21.7) is a non-renormalizable charged current interaction of Fermi's theory. The second term is a neutral current interaction predicted by the GSW-theory. It was indeed observed at CERN in 1973.

Let us close this section with a summary of the prediction of the GSW-model:

- Neutral current interactions which were observed at CERN in 1973.
- Existence of heavy gauge bosons $W^\pm_\mu, Z^0_\mu$ with a mass relation

$$\rho \equiv \frac{m_W}{m_Z \cos\theta_w} = 1 \ . \tag{21.8}$$

  They were observed at CERN in 1979.
- Existence of at least one Higgs boson with an undetermined mass m_H. This is not yet observed.

22 Lecture 22 & 23: The Standard Model

22.1 Spectrum and Lagrangian

The Standard Model (SM) combines QCD with the GSW-theory. The gauge group therefore is $G = SU(3)\times SU(2)\times U(1)_Y$ which is spontaneously broken by the Higgs mechanism to $SU(3)\times U(1)_{\rm em}$. The particle spectrum is given in table 22.1. The indices $I = 1,2,3$ denotes the three families of the SM and $\hat{i} = 1,2,3$ the colour index.

Spectrum		$SU(3)$	$SU(2)$	Y	Q
1) Gauge bosons					
	$G_\mu^{\hat{a}=1,\dots,8}$	$\mathbf{8}$	$\mathbf{1}$	0	0
	$A_\mu^{a=1,2,3}$	$\mathbf{1}$	$\mathbf{3}$	0	$0,\pm1$
	B_μ	$\mathbf{1}$	$\mathbf{1}$	0	0
2) Weyl fermions					
	$E_L^{iI} = \begin{pmatrix}\nu_e^I\\ e^{-I}\end{pmatrix}_L$	$\mathbf{1}$	$\mathbf{2}$	$-\frac{1}{2}$	$\begin{pmatrix}0\\-1\end{pmatrix}$
	e_R^{-I}	$\mathbf{1}$	$\mathbf{1}$	-1	-1
	ν_R^I	$\mathbf{1}$	$\mathbf{1}$	0	0
	$Q_L^{\hat{i}iI} = \begin{pmatrix}u^{\hat{i}I}\\ d^{\hat{i}I}\end{pmatrix}_L$	$\mathbf{3}$	$\mathbf{2}$	$\frac{1}{6}$	$\begin{pmatrix}\frac{2}{3}\\-\frac{1}{3}\end{pmatrix}$
	$u_R^{\hat{i}I}$	$\mathbf{3}$	$\mathbf{1}$	$\frac{2}{3}$	$\frac{2}{3}$
	$d_R^{\hat{i}I}$	$\mathbf{3}$	$\mathbf{1}$	$-\frac{1}{3}$	$-\frac{1}{3}$
3) Higgs boson					
	$\phi^i = \begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}$	$\mathbf{1}$	$\mathbf{2}$	$\frac{1}{2}$	$\begin{pmatrix}1\\0\end{pmatrix}$

Table 22.0: Spectrum of GSW model

The Lagrangian is given by

$$\begin{aligned}\mathcal{L} = &-\tfrac14 G_{\mu\nu}^{\hat{a}}G^{\hat{a}\mu\nu} - \tfrac14 F_{\mu\nu}^{a}F^{a\mu\nu} - \tfrac14 B_{\mu\nu}B^{\mu\nu} + i\sum_f \bar{f}\sigma^\mu D_\mu f + D^\mu\phi^{*i}D_\mu\phi^i - V(\phi^i,\phi^{*i})\\ &+ (\lambda_l)_{IJ}\phi^i\bar{E}_L^{iI}e_R^J + (\lambda_\nu)_{IJ}\epsilon_{ij}\phi^{*i}\bar{E}_L^{jI}\nu_R^J + (\lambda_d)_{IJ}\phi^i\bar{Q}_L^{iI}d_R^J + (\lambda_u)_{IJ}\epsilon_{ij}\phi^{*i}\bar{Q}_L^{jI}u_R^J + h.c.\ ,\end{aligned} \tag{22.1}$$

where f denotes all Weyl fermions of table 22.1 with covariant derivatives

$$\begin{aligned}D_\mu &= \partial_\mu - ig_s G_\mu^{\hat{a}}t^{\hat{a}} - \tfrac{i}{2}gA_\mu^a\sigma_{ij}^a - ig'y(E_L)B_\mu\\ &= \hat{D}_\mu - \tfrac{i}{\sqrt2}g\left(W_\mu^+\sigma^+ + W_\mu^-\sigma^-\right)E_L - \frac{ig}{\cos\theta_W}Z_\mu^0\left(\tfrac12\sigma^3 - \sin^2\theta_W Q\right)E_L - ie\gamma_\mu QE_L\ ,\end{aligned} \tag{22.2}$$

and we abbreviated $\hat{D}_\mu = \partial_\mu - ig_sG_\mu^{\hat{a}}t^{\hat{a}}$. Once again we rewrite

$$i\sum_f \bar{f}\sigma^\mu D_\mu f = i\sum_f \bar{f}\sigma^\mu\hat{D}_\mu f + g\left(W_\mu^+J_W^{-\mu} + W_\mu^-J_W^{+\mu} + Z_\mu^0J_Z^\mu\right) + eA_\mu J_{\rm em}^\mu\ , \tag{22.3}$$

now with

$$J_W^{-\mu} = \tfrac{1}{\sqrt{2}}\left(\overline{\nu}_L^I \sigma^\mu e_L^I + \overline{u}_L^I \sigma^\mu d_L^I\right) \ , \qquad J_W^{+\mu} = \tfrac{1}{\sqrt{2}}\left(\overline{e}_L^I \sigma^\mu \nu_L^I + \overline{d}_L^I \sigma^\mu u_L^I\right) \ ,$$

$$J_{\rm em}^\mu = -\overline{e}_D^I \gamma^\mu e_D^I + \tfrac{2}{3}\overline{u}_D^I \gamma^\mu u_D^I - \tfrac{1}{3}\overline{d}_D^I \gamma^\mu d_D^I,$$

$$\begin{aligned} J_Z^\mu = \frac{1}{\cos\Theta_w}\big[&\tfrac{1}{2}\overline{\nu}_L^I \sigma^\mu \nu_L^I - \left(\tfrac{1}{2} - \sin^2\theta_w\right)\overline{e}_L^I \sigma^\mu e_L^I + \sin^2\theta_w \overline{e}_R^I \sigma^\mu e_R^I + \left(\tfrac{1}{2} - \tfrac{2}{3}\sin^2\theta_w\right)\overline{u}_L^I \sigma^\mu u_L^I \\ &+ \left(-\tfrac{2}{3}\sin^2\theta_w\right)\overline{u}_R^I \sigma^\mu u_R^I + \left(-\tfrac{1}{2} + \tfrac{1}{3}\sin^2\theta_w\right)\overline{d}_L^I \sigma^\mu d_L^I + \left(\tfrac{1}{3}\sin^2\theta_w\right)\overline{d}_R^I \sigma^\mu d_R^I\big]. \end{aligned} \tag{22.4}$$

e_D^I, u_D^I, d_D^I denote Dirac spinors.

22.2 Fermion masses and CKM-mixing

The spontaneous symmetry breaking of $SU(2) \times U(1)_Y \to U(1)_{\rm em}$ again generates the heavy $W_\mu^\pm, Z_\mu^0$ with masses given in (19.29). In addition in analogy with (20.15) the following fermion mass terms are induced

$$\mathcal{L}_m = (m_e)_{IJ}\bar{e}_L^I e_R^J + (m_\nu)_{IJ}\bar{\nu}_L^I \nu_R^J + (m_d)_{IJ}\bar{d}_L^I d_R^J + (m_u)_{IJ}\bar{u}_L^I u_R^J \ , \tag{22.5}$$

where

$$\begin{aligned} (m_e)_{IJ} &= \tfrac{v}{\sqrt{2}}(\lambda_e)_{IJ} \ , \qquad (m_\nu)_{IJ} = \tfrac{v}{\sqrt{2}}(\lambda_\nu)_{IJ} \ , \\ (m_d)_{IJ} &= \tfrac{v}{\sqrt{2}}(\lambda_d)_{IJ} \ , \qquad (m_u)_{IJ} = \tfrac{v}{\sqrt{2}}(\lambda_u)_{IJ} \ . \end{aligned} \tag{22.6}$$

We see that generically the fermion mass matrices can be non-diagonal. This field basis is called the weak basis. Using the polar decomposition theorem one can always go instead to a field basis where the fermion mass matrices are diagonal.

The polar decomposition theorem states that any non-degenerate complex matrix M can always be written as

$$M = HW \ , \tag{22.7}$$

where H is hermitian and W is unitary. This implies that any M can be diagonalized by a bi-unitary transformation of the form

$$U_1 M U_2^\dagger = M_D \ , \tag{22.8}$$

where $U_{1,2}$ are in general different unitary matrices and M_D is a diagonal matrix.

For concreteness let us focus on the quark-sector and perform a rotation in family space of the form

$$\begin{aligned} \bar{u}_L^I &= \bar{u}_L^{\prime K} S_u^{\dagger KI} \ , \qquad u_R^J = T_u^{JM} u_R^{\prime M} \ , \\ \bar{d}_L^I &= \bar{U}_L^{\prime K} S_d^{\dagger KI} \ , \qquad d_R^J = T_d^{JM} d_R^{\prime M} \ , \end{aligned} \tag{22.9}$$

where $S_{u,d}, T_{u,d}$ are all unitary. Inserting (22.9) into (22.5) and using the polar decomposition theorem we can choose $S_{u,d}, T_{u,d}$ such that the mass terms given in (22.5) are diagonal with the physical masses being the diagonal entries. Of course we also need to insert this transformation in the rest of the Lagrangian (22.1). By inspection we see immediately that the kinetic terms are unchanged and from (22.4) we also infer that the neutral currents $J^\mu_{\rm em}, J^\mu_Z$ are invariant. However the charged currents change due to

$$\overline{u}^I_L \sigma^\mu d^I_L = V_{IJ} \overline{u}'^I_L \sigma^\mu d'^J_L \ , \qquad \overline{d}^I_L \sigma^\mu u^I_L = V^*_{IJ} \overline{d}'^I_L \sigma^\mu u'^J_L \ , \tag{22.10}$$

where

$$V_{IJ} := S^\dagger_{uIK} S_{dKJ} \tag{22.11}$$

is the unitary CKM (Cabbibo-Kobayashi-Maskawa) matrix. Since V is unitary it depends a priori on 9 real parameters, three rotation angles and six phases. However, $\mathcal{L}_m$ of (22.5) is even for diagonal mass matrices invariant under the six phase rotation

$$u^I_{L,R} \to e^{-i\alpha_I} u^I_{L,R} \ , \qquad d^I_{L,R} \to e^{-i\beta_I} d^I_{L,R} \ . \tag{22.12}$$

These rotation change V as can be seen from (22.10) except for the global rotation with $\alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3$. Thus the phase rotations (22.12) remove an additional five phases from V leaving three rotation angles and one phase.

The three angles are measured for example in semi-leptonic quark decays while the phase parametrizes CP-violation observed for example in $K^0 - \bar{K}^0$ mixing. For further details see [8]. Finally a similar phenomenon is taking place in the leptonic sector where the parameters are measured in neutrino oscillations. (See also [8].)

22.3 Measurements of the SM parameters

Let us first list the parameters of the SM.[10]

- Three gauge couplings: g_s, g, g' or equivalently $g_s, e, \sin\theta_w$,
- one Higgs VEV: $v = \sqrt{2\mu^2/\lambda}$,
- one parameter of the Higgs potential: λ,
- six quark masses, three charged lepton masses and three or six neutrino masses,
- four CKM parameters,
- four-ten mixing parameters in the neutrino sector.

These parameters are overdetermined by the experimental measurements and thus apart from determining the above parameters as precisely as possible one has a lot of consistency checks. The bottom line is that the SM works very well, some of the prediction are meet at the pro-mille level. For further details we again refer to [8]. The only so far unobserved part of the SM is the Higgs boson. Its mass is not predicted by the SM and it has not yet been directly observed. However electro-weak precisions measurements at LEP strongly constrains m_h as it contributes to the one-loop correction in the process $Z^0 Z^0 \to f\bar{f}$ [8].

[10] Additionally one sometimes adds three θ-angles, the cosmological constant and the Newton constant.

23 Lecture 24: Anomalies in QFT

Any classical symmetry implies via the Noether-theorem a conserved current $\partial_\mu j^\mu = 0$. If this symmetry is broken at the quantum level the symmetry is called <u>anomalous</u> and one has

$$\partial_\mu j^\mu = \hbar \mathcal{A} \ . \tag{23.1}$$

$\mathcal{A}$ is called the <u>anomaly</u>.

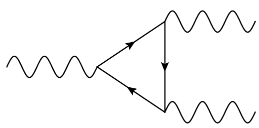
There are two possible anomalies:

1) An anomalous global symmetry, i.e. j^μ does not couple to a gauge field. This leads to new physical processes such as $\pi^0 \to \gamma\gamma$.

2) An anomalous (local) gauge symmetry, i.e. j^μ does couple to a gauge field. In this case the Ward-identity is broken, renormalizability is lost and the theory becomes quantum inconsistent.

As a consequence physical gauge theories have to be anomaly free.

The Feynman diagram which contributes to the anomaly is the triangle graph



(23.2)

One finds

$$D_\mu j^{a\mu} = -\tfrac{g^2}{16\pi^2}\, \epsilon^{\mu\nu\rho\sigma} F^b_{\mu\nu} F^c_{\rho\sigma} \mathcal{A}^{abc} \ , \tag{23.3}$$

with

$$\mathcal{A}^{abc}(\mathbf{r}) = \mathrm{Tr}_{\mathbf{r}} \left(t^a_{\mathbf{r}} \left\{ t^b_{\mathbf{r}}, t^c_{\mathbf{r}} \right\} \right) . \tag{23.4}$$

Thus the anomaly vanishes for all representations with $\mathcal{A}^{abc} = 0$. Furthermore, the <u>Adler-Bardeen-theorem</u> states that if $\mathcal{A}^{abc} = 0$ than $D_\mu j^{a\mu} = 0$ holds at all orders in perturbation theory.

Let us compute $\mathcal{A}^{abc}$ for the complex conjugate representation $\bar{\mathbf{r}}$

$$\begin{aligned}
\mathcal{A}(\bar{\mathbf{r}}) &= \mathrm{Tr}_{\bar{\mathbf{r}}} \left(t^a_{\bar{\mathbf{r}}} \left\{ t^b_{\bar{\mathbf{r}}}, t^c_{\bar{\mathbf{r}}} \right\} \right) = -\,\mathrm{Tr}_{\mathbf{r}} \left(t^{a*}_{\mathbf{r}} \left\{ t^{b*}_{\mathbf{r}}, t^{c*}_{\mathbf{r}} \right\} \right) \\
&= -\left(t^{a*}_{\mathbf{r}\,ij} t^{b*}_{\mathbf{r}\,jk} t^{c*}_{\mathbf{r}\,ki} + t^{a*}_{\mathbf{r}\,ij} t^{c*}_{\mathbf{r}\,jk} t^{b*}_{\mathbf{r}\,ki} \right) \\
&= -\left(t^{a\dagger}_{\mathbf{r}\,ji} t^{b\dagger}_{\mathbf{r}\,kj} t^{c\dagger}_{\mathbf{r}\,ik} + t^{a\dagger}_{\mathbf{r}\,ji} t^{c\dagger}_{\mathbf{r}\,kj} t^{b\dagger}_{\mathbf{r}\,ik} \right) \\
&= -\left(t^{a}_{\mathbf{r}\,ji} t^{c}_{\mathbf{r}\,ik} t^{b}_{\mathbf{r}\,kj} + t^{a}_{\mathbf{r}\,ji} t^{c}_{\mathbf{r}\,kj} t^{b}_{\mathbf{r}\,ik} \right) \\
&= -\,\mathrm{Tr}_{\mathbf{r}} \left(t^a_{\mathbf{r}} \left\{ t^b_{\mathbf{r}}, t^c_{\mathbf{r}} \right\} \right) = -\mathcal{A}(r)
\end{aligned}$$

Thus vector-like theories, i.e. theories with fermions in the $\mathbf{r} \oplus \bar{\mathbf{r}}$ representation, are automatically anomaly free. Similarly, fermions in real representations do not lead to an anomaly.[11]

[11] Note that in $\mathcal{A}$ one needs to compare fermions in the same Lorentz representation (i.e. all left-handed or all right-handed). This can be done by noting that $\bar{\psi}_R$ transforms as ψ_L under the Lorentz group.

Let us now check that the SM is anomaly free. For the $SU(3)$ only the quarks contribute in the loop. u_L, d_L transform in the $\mathbf{3}$ while $\bar{u}_R, \bar{d}_R$ transform in the $\bar{\mathbf{3}}$. Thus the $SU(3)$ part of the SM is vector-like and $\mathcal{A}$ vanishes. For the $SU(2)$ one has

$$\mathcal{A}^{abc} = \mathrm{Tr}\left(\sigma^a \underbrace{\left\{\sigma^b, \sigma^c\right\}}_{2\delta^{bc}\cdot \mathbb{1}}\right) = 2\delta^{bc}\underbrace{\mathrm{Tr}\,\sigma^a}_{=0} = 0\ . \tag{23.5}$$

The reason behind this is that the fundamental $\mathbf{2}$ of $SU(2)$ is a real representation. For the $U(1)_Y$ one has

$$\mathcal{A} \sim \sum_f y_f^3 \tag{23.6}$$

where the sum runs over all fermions f charged under $U(1)_Y$ and Y_f is their hypercharge. For each family of the SM one has

$$\sum_f y_f^3 = 2\cdot\left(-\tfrac{1}{2}\right)^3 + 1^3 + 6\cdot\left(\tfrac{1}{6}\right)^3 + 3\cdot\left(-\tfrac{2}{3}\right) + 3\cdot\left(\tfrac{1}{3}\right) = 0\ . \tag{23.7}$$

(This is called the cubic anomaly.)

In addition to the above three anomalies one also has <u>mixed anomalies</u> which generically occur for gauge groups which contain various factors. The $SU(2)-SU(3)-SU(3)$ anomaly vanishes due to $\mathcal{A} \sim \mathrm{tr}(\sigma^a) = 0$ while the $SU(2)-SU(2)-SU(3)$ anomaly vanishes due to $\mathcal{A} \sim \mathrm{tr}(t^{\hat{a}}) = 0$. For the $U(1)_Y - SU(3) - SU(3)$ anomaly one has

$$\mathcal{A} \sim \sum_{f=\text{quarks}} y_f = 2y(Q_L) + y(\bar{u}_R) + y(\bar{d}_R) = \tfrac{2}{6} - \tfrac{2}{3} + \tfrac{1}{3} = 0\ . \tag{23.8}$$

For the $U(1)_Y - SU(2) - SU(2)$ anomaly one has

$$\mathcal{A} \sim \sum_{f_l} y_{f_L} = y(E_L) + 3y(Q_L) = -\tfrac{1}{2} + \tfrac{3}{6} = 0\ . \tag{23.9}$$

Finally one can also compute the gravitational anomaly with

$$\mathcal{A} \sim \sum_{\mathrm{all} f} y_f = 2y(E_L) + y(\bar{e}_R) + 6y(Q_L) + 3y(\bar{u}_R) + 3y(\bar{d}_R) = -\tfrac{2}{2} + 1 + \tfrac{6}{6} - \tfrac{6}{3} + \tfrac{3}{3} = 0\ . \tag{23.10}$$

Thus we showed that the SM is anomaly free. One can ask if the absence of all anomalies determines the hyper charges of the SM particles. Indeed we have four equations (23.7)–(23.10) for the five unknowns $y(E_L), y(\bar{e}_R), y(Q_L), y(\bar{u}_R), y(\bar{d}_R)$. Fixing $y(\bar{e}_R) = 1$ (corresponding to $Q_{\mathrm{em}}(e) = -1$) one obtains a unique solution, up to the ambiguity $y(\bar{u}_R) \leftrightarrow y(\bar{d}_R)$. Furthermore only a completely neutral chiral fermion such as ν_R can be added without upsetting the anomaly freedom.

24 Lecture 25: Theories beyond the Standard Model

The SM is experimentally very well confirmed. However, theoretically one expects it to be an effective theory of a more fundamental theory. This is partly due to the following (unanswered) questions.

- Why is $G = SU(3) \times SU(2) \times U(1)_Y$?
- What determines the spontaneous symmetry breaking $G \to SU(3) \times U(1)_{\text{em}}$ and sets the scale of the breaking?
- What determines the particle spectrum?
- What determines the parameters of the SM?
- What is the Dark Matter component?
- How does one couple the SM to (quantum) gravity?

Theories beyond the Standard Model (BSM) attempt to generalize or extend the SM. There are basically two possibilities.

1. Change $\mathcal{L}$ so that some of the above questions are answered. Examples are:
   - Supersymmetric theories,
   - Grand Unified Theories (GUTs),
   - Technicolour theories.
2. Change the formalism of the QFT. Here the example is string theory.

24.1 Supersymmetric theories

In supersymmetric theories the Poincare space-time symmetry is enlarged by a fermionic symmetry generator Q with (anti-) commutation relations

$$\{Q, \bar{Q}\} \sim \sigma^{\mu} P_{\mu} \ , \qquad [Q, P_{\mu}] = 0 \ , \qquad [Q, J^{\mu\nu}] \sim \sigma^{\mu\nu} Q \ , \tag{24.1}$$

where P_{μ} is the momentum operator generating space-time translations and $J^{\mu\nu}$ is the Lorentz-generator. The representation of this superalgebra are super multiplets which combine bosonic and fermionic fields. For example the chiral multiplet (ϕ, ψ) contains a complex scalar ϕ and a Weyl fermion ψ. Under a supersymmetry transformation they transform into each other $Q\phi \sim \psi, Q\psi \sim \partial\phi$. The vector multiplet (A_{μ}, λ) contains a gauge boson A_{μ} and a gauge fermion λ.

With these multiplets a supersymmetric Standard Model can be constructed with the following properties:

- it contains new scalar and fermionic particles,
- it predicts a 'light' Higgs with $m_h \leq 200\text{GeV}$,

- it contains a weakly interacting massive particle (WIMP) as a candidate for Dark Matter,
- it fits the electro-weak precision data,
- it solves the naturalness problem.

24.2 Grand Unified Theories (GUTs)

In this class of theories the gauge symmetry is enlarged in that the SM gauge group G_{SM} is embedded in a larger gauge group G_{GUT} which is spontaneously broken at M_{GUT} by a Higgs mechanism $G_{GUT} \to G_{SM}$. Examples are $G_{GUT} = SU(5), SO(10)$. In the first case one family of the SM precisely fits into a $\mathbf{5} \oplus \bar{\mathbf{10}}$ representation of $SU(5)$. In the second case one family including a right-handed neutrino sit in the **16** spinor representation of $SO(10)$. These theories predict proton decay and the unification of gauge couplings

$$g_{5,10} = g_s = g = \sqrt{5/3}g' \tag{24.2}$$

at M_{GUT}. Using (10.9) this prediction can be compared to the measured gauge couplings of the SM, for example at m_z. For the SM this prediction fails while in the supersymmetric SM it works perfectly.

24.3 String Theory

The basic idea of string theory is to replace a classical point-like particle by an extended object: a string. String theory can then be viewed as the quantum theory of extended objects. Upon quantization one finds a finite number of massless modes with spins $0, \frac{1}{2}, 1, \frac{3}{2}, 2$ and a infinite number of massive modes with masses $M \sim nM_s, n \in \mathbb{N}$. M_s is the characteristic scale of string theory related to the tension of the string. The massless spin-2 excitation of the string can be identified with the graviton of General Relativity if $M_s \sim M_{Pl} = \sqrt{\hbar c/G_N} \sim 10^{19}$GeV. Furthermore, due to the extended nature of the string the theory becomes UV finite and thus is a candidate for a quantum gravity.

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