Lecture 1 — chiral anomaly and its perturbative computation in QED

Noether theorem: continuous symmetries

=> conserved (Noether) current

$$\partial_\mu j^\mu = 0$$

=> conserved charge $Q = \int d^3x\, j^0$, $\partial_t Q = 0$

Anomaly: $\partial_\mu j^\mu = \underbrace{\hbar}_{=1} \mathcal{A}$ ← anomaly

2 possibilities:

1) j^μ is current of global (flavour) symmetry

=> $\mathcal{A}$ induces "new" physical processes (e.g. $\pi \to \gamma\gamma$)

2) j^μ is current of local (gauge) symmetry

=> Ward identity broken (anomalous)

=> quantum theory inconsistent

=> $\boxed{\text{gauge theories have to be anomaly free}}$

Example : QED

$$\mathcal{L}_{QED} = \bar{\Psi}\,(i\not{D} - m)\,\Psi - \frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu}$$

where $\not{D}\Psi = \gamma^\mu D_\mu \Psi = \gamma^\mu(\partial_\mu - ieA_\mu)\Psi$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} = 2\begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\Rightarrow \quad \mathcal{L}_{QED} = \bar{\Psi}\,(i\not{\partial} - m)\,\Psi - e\,j^\mu A_\mu - \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu}$$

with $\quad j^\mu = \bar{\Psi}\gamma^\mu\Psi$

eq. of motion:
- $i(\not{D} - m)\,\Psi = 0 \qquad$ (Dirac eq.)
- $\partial_\mu F^{\mu\nu} = -e\,j^\nu$

$$\Rightarrow \quad \partial_\nu\,\partial_\mu F^{\mu\nu} = -e\,\partial_\nu j^\nu = 0 \quad \checkmark$$

Symmetries of $\mathcal{L}_{QED}$:

- global $U(1)_V$ symmetry

$$\left.\begin{array}{l} \psi \to \psi' = e^{i\alpha}\psi \,, \quad \alpha \in \mathbb{R} \\ A_\mu \to A'_\mu = A_\mu \end{array}\right\} \Rightarrow \mathcal{L}_{QED}(\psi, A) = \mathcal{L}_{QED}(\psi', A')$$

Noether current $j^\mu = \bar{\psi}\gamma^\mu\psi$ conserved $\quad \partial_\mu j^\mu = 0$

- local $U(1)$ gauge symmetry

$$\psi \to \psi' = e^{i\alpha(x)}\psi \,, \quad x^\mu: \text{ space-time coordinates}$$

$$A_\mu \to A'_\mu = A_\mu + e^{-1}\partial_\mu \alpha(x)$$

- global axial $U(1)_A$ symmetry for $m = 0$

$$\psi \to \psi' = e^{i\gamma^5\alpha}\psi \,, \qquad \gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$A_\mu \to A'_\mu = A_\mu \qquad (\gamma^5)^2 = \mathbb{1}\,, \quad \{\gamma^5, \gamma^\mu\} = 0$$

Noether current: $j^\mu_{5/A} = \bar{\psi}\gamma^\mu\gamma^5\psi$

$$\partial_\mu j^\mu_5 = 2im\bar{\psi}\gamma^5\psi \overset{m=0}{=} 0 \quad \text{classically}$$

j^μ_5 is anomalous: $\partial_\mu j^\mu_5 = \mathcal{A}$
(for $m=0$)

$$\boxed{\mathcal{A} = -\frac{e^2}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}}$$

$\mathcal{A}$ is called Abelian, global, $U(1)_A$ anomaly

perturbative computation of A

$q = k_1 + k_2$

$$T^{\mu\nu\lambda} := -\int \frac{d^4p}{(2\pi)^4}\, \mathrm{tr}\left(\frac{i}{\not p}\, \gamma^\lambda \gamma^5 \frac{i}{\not p - \not q}\, \gamma^\nu \frac{i}{\not p - \not k_1}\, \gamma^\mu \right) + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}$$

$\sim \int d^4p\, p^{-3} \Rightarrow$ linear div. integral

compute $\partial_\lambda j_5^\lambda$ via $q_\lambda T^{\mu\nu\lambda}$

use: $\not q \gamma^5 = (-\not p + \not q + \not p)\gamma^5 = \gamma^5(\not p - \not q) + \not p \gamma^5$

$\{\gamma^5, \gamma^\mu\} = 0$

$$\Rightarrow q_\lambda T^{\mu\nu\lambda} = i \int \frac{d^4p}{(2\pi)^4}\, \mathrm{tr}\left(\frac{1}{\not p} \left(\gamma^5 (\not p - \not q) + \not p \gamma^5\right) \frac{1}{\not p - \not q}\, \gamma^\nu \frac{1}{\not p - \not k_1}\, \gamma^\mu \right) +$$

$$+\, \mathrm{tr}\left(\frac{1}{\not p} \left(\gamma^5 (\not p - \not q) + \not p \gamma^5\right) \frac{1}{\not p - \not q}\, \gamma^\mu \frac{1}{\not p - \not k_2}\, \gamma^\nu \right)$$

$$= \Delta_{14}^{\mu\nu} + \Delta_{32}^{\mu\nu}$$

$$\Delta_{14}^{\mu\nu} = i \int \frac{d^4p}{(2\pi)^4} \left(\mathrm{tr}\left(\frac{1}{\not p}\, \gamma^5 \gamma^\nu \frac{1}{\not p - \not k_1}\, \gamma^\mu \right) + \mathrm{tr}\left(\gamma^5 \frac{1}{\not p - \not q}\, \gamma^\mu \frac{1}{\not p - \not k_2}\, \gamma^\nu \right) \right)$$

$$= i \int \frac{d^4p}{(2\pi)^4} \left(\mathrm{tr}\left(\frac{1}{\not p}\, \gamma^5 \gamma^\nu \frac{1}{\not p - \not k_1}\, \gamma^\mu \right) - \mathrm{tr}\left(\frac{1}{\not p - \not k_2}\, \gamma^5 \gamma^\nu \frac{1}{\not p - \not q}\, \gamma^\mu \right) \right)$$

$$\Delta_{32}^{\mu\nu} = i \int \frac{d^4p}{(2\pi)^4}\, \mathrm{tr}\left(\frac{1}{\not p}\, \gamma^5 \gamma^\mu \frac{1}{\not p - \not k_2}\, \gamma^\nu \right) - \mathrm{tr}\left(\frac{1}{\not p - \not k_1}\, \gamma^5 \gamma^\mu \frac{1}{\not p - \not q}\, \gamma^\nu \right)$$

$$\Delta_{14}(k_1 = k_2 = 0) = \Delta_{32}(k_1 = k_2 = 0) = 0$$

In 1st term of $\Delta^{\mu\nu}_{14}$ shift: $p \to p - k_2$, $d^4p \Rightarrow d^4p$

$\Rightarrow \Delta^{\mu\nu}_{14} = 0$

In 1st term of $\Delta^{\mu\nu}_{32}$ shift $p \to p - k_1$, $d^4p = d^4p$

$\Rightarrow \Delta_{32} = 0$

but: too naive as integrals are linearly divergent

$\Rightarrow$ need to regulate before shifting

warm up: 1d integral

$$\Delta(k) := \int_{-\infty}^{+\infty} dp \left(f(p+k) - f(p) \right)$$

$$\overset{\text{Taylor}}{=} \int_{-\infty}^{+\infty} dp \left(k f' + \frac{k^2}{2} f'' + \dots \right)$$

$$= k \left(f(\infty) - f(-\infty) \right) + \frac{k^2}{2} \left(f'(\infty) - f'(-\infty) \right) + \dots$$

$$= \begin{cases} 0 & \text{if } \Delta(k) \text{ convergent (or log-div) as } f(\pm\infty) = f'(\pm\infty) = \dots = 0 \\ \text{const} & \text{if } \Delta(k) \text{ linear div as } f(\pm\infty) \neq 0 \text{ possible} \end{cases}$$

4d: $$\Delta(k) := \int \frac{d^4p}{(2\pi)^4} \left(f(p+k) - f(p) \right)$$

$$= \int \frac{d^4p}{(2\pi)^4} \left(k^\mu \frac{\partial}{\partial p^\mu} f + \frac{1}{2} k^\mu k^\nu \underbrace{\partial_{p\mu} \partial_{p\nu} f}_{\sim p^{-5} \text{ for } f \sim p^{-3}} + \dots \right)$$

$\Rightarrow$ no surface term

$$\overset{\text{Gauss}}{=} \lim_{|p| \to \infty} k^\mu \int_{S^3(|p|)} \frac{|p|^3}{(2\pi)^4} \underbrace{\frac{dp_\mu}{|p|}}_{= d\hat{p}_\mu} f$$

$\Delta^{\mu\nu}$ are precisely of this form

$$\Delta^{\mu\nu}_{14} = i\int\frac{d^4p}{(2\pi)^4}\, f^{\mu\nu}(p-k_2, k_1) - f^{\mu\nu}(p, k_1)$$

for $f^{\mu\nu}(p,k_1) = -\operatorname{tr}\frac{1}{\not p}\gamma^5\gamma^\nu\frac{1}{\not p-\not k_1}\gamma^\mu$

$$= -\frac{1}{p^2(p-k_1)^2}\, p_\sigma (p-k_1)_\lambda \underbrace{\operatorname{tr}\gamma^\sigma\gamma^5\gamma^\nu\gamma^\lambda\gamma^\mu}_{-4i\epsilon^{\sigma\nu\lambda\mu}}$$

o due to ϵ-Tensor

$$= 4i\,\epsilon^{\sigma\nu\lambda\mu}\frac{p_\sigma(p-k_1)_\lambda}{p^2(p-k_1)^2} = 4i\,\epsilon^{\sigma\nu\lambda\mu}\frac{p_\sigma k_{1\lambda}}{p^2(p-k_1)^2}$$

analogously

$$\Delta^{\mu\nu}_{32} = 4i\int\frac{d^4p}{(2\pi)^4}\left(f^{\nu\mu}(p-k_1, k_2) - f^{\nu\mu}(p, k_2)\right)$$

$$\Delta^{\mu\nu}_{14} = \lim_{|p|\to\infty} -4k_2^{\,g}\; i\int_{S^3}\frac{dp_g}{(2\pi)^4}\,\cancel{p^2}\,\epsilon^{\sigma\nu\lambda\mu}\frac{p_\sigma k_{1\lambda}}{\cancel{p^2}(p-k_1)^2}$$

$$= -\frac{i}{4\pi^4}\epsilon^{\sigma\nu\lambda\mu}k_{1\lambda}k_2^{\,g}\underbrace{\lim_{|p|\to\infty}\int_{S^3}\frac{dp_g\, p_\sigma}{(p-k_1)^2}}$$

$$= \lim_{|p|\to\infty}\frac{\frac{1}{4}p^2\eta_{g\sigma}\cdot 2\pi^2}{(p-k_1)^2} = \frac{\pi^2}{2}\eta_{g\sigma}$$

$$\rightarrow \Delta^{\mu\nu}_{14} = -i\epsilon^{\sigma\nu\lambda\mu}\frac{1}{8\pi^2}k_{1\lambda}k_{2\sigma}$$

analog

$$\Delta^{\mu\nu}_{32} = -\frac{i}{8\pi^2}\epsilon^{\sigma\mu\lambda\nu}k_{2\lambda}k_{1\sigma}$$

$$\Delta^{\mu\nu}_{14} + \Delta^{\mu\nu}_{32} = +\frac{i}{4\pi^2}\epsilon^{\mu\nu\sigma\lambda}(k_{1\lambda}k_{2\sigma})$$

I posit. space the yields

$$\mathcal{A} \sim \epsilon_{\mu\nu\rho\sigma}\, \partial^\mu A^\nu \partial^\rho A^\sigma \sim \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$$

Remarks:

- $\mathcal{A}$ can also be calculated using
  - dim. reg. [PS]
  - Pauli-Villars [B]
- higher loop corrections do not contribute (Adler-Bardeen theorem)
  $\Rightarrow$ $\mathcal{A}$ is one-loop exact!
- can add local counter-term to effective quantum action Γ
  $\Rightarrow$ coefficient of $\mathcal{A}$ is ambiguous

However there is no local counter term which can keep j^μ_5 and j^μ anomaly-free. [W]

Lecture 2: Non-Abelian Anomalies

Consider gauge theories with non-Abelian gauge group G (e.g. $G = SU(n)$)

Distinguish 2 cases:

1.) vector-like gauge theories

$$\mathcal{L} = \bar{\Psi}^i (\not{D} - m)_{ij} \Psi^j - \frac{1}{4c(\underset{\sim}{r})} \mathrm{tr}_{\underset{\sim}{r}} (F_{\mu\nu} F^{\mu\nu}), \quad i = 1, \dots, \dim(\underset{\sim}{r})$$

$\underset{\sim}{r}$: rep of G, $\dim(\underset{\sim}{r}) = r$ (e.g. $\underset{\sim}{n}$ of $SU(n)$)

$$= \bar{\Psi}^i (\not{\partial} - m)_{ij} \Psi^j - g\, \mathrm{tr}_{\underset{\sim}{r}} (j^\mu A_\mu) - \frac{1}{4} \frac{1}{c(\underset{\sim}{r})} \mathrm{tr}_{\underset{\sim}{r}} (F_{\mu\nu} F^{\mu\nu})$$

g: gauge coupling constant

where

$$\gamma^\mu D_{\mu ij} \Psi^j := \gamma^\mu (\partial_\mu \Psi^i - g A_\mu{}^i{}_j \Psi^j)$$

$$A_\mu{}^i{}_j = A_\mu^a \, t_{\underset{\sim}{r}}^a{}^i{}_j, \quad a = 1, \dots, \dim(\mathrm{adj}(G)) \quad \text{(e.g. } SU(n)\text{: } \dim(\mathrm{adj}\, SU(n)) = n^2 - 1)$$

$\uparrow$ generators of G $\quad [t_{\underset{\sim}{r}}^a, t_{\underset{\sim}{r}}^b] = i f^{[abc]} t_{\underset{\sim}{r}}^c$

e.g. $U(n)$: $t_{\underset{\sim}{r}}^a = t_{\underset{\sim}{r}}^{a\dagger}$

$SU(n)$: $\mathrm{tr}\, t_{\underset{\sim}{r}}^a = 0$

normalization: $\mathrm{tr}(t_{\underset{\sim}{r}}^a t_{\underset{\sim}{r}}^b) \equiv \mathrm{tr}_{\underset{\sim}{r}}(t^a t^b) = c(\underset{\sim}{r}) \delta^{ab}$

$c(\underset{\sim}{r})$: index of rep. (e.g. $c(\underset{\sim}{n}) = \frac{1}{2}$)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\,[A_\mu, A_\nu] = F_{\mu\nu}^a t^a$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$$

$$\mathrm{tr}\, j^\mu A_\mu = \sum_{a=1}^{\dim(\mathrm{adj}(G))} j^{\mu a} A_\mu^a \quad , \quad j^{\mu a} = \bar\psi^i \gamma^\mu t^a_{ij} \psi^j \quad (*)$$

$\mathcal{L}$ is gauge inv. under $SU(n)$ gauge trafo.

$$\psi^i \to \psi'^i = U^i{}_j(x)\, \psi^j \quad , \quad UU^\dagger = \mathbb{1}$$

$$A_\mu \to A_\mu' = U A_\mu U^\dagger - \frac{i}{g} (\partial_\mu U) U^\dagger$$

$$\Rightarrow \quad F_{\mu\nu} \to F_{\mu\nu}' = U F_{\mu\nu} U^\dagger$$

inf. version:

$$U^i{}_j = e^{i\alpha^a(x) t^{a\,i}{}_j} = \delta^i_j + i\alpha^a(x) t^{a\,i}{}_j + O(\alpha^2) \quad , \quad \alpha^a \text{ real}$$

$$\delta\psi^i = \psi'^i - \psi^i \approx i\alpha^a(x) t^{a\,i}{}_j \psi^j$$

$$\delta A_\mu^a = \frac{1}{g}\partial_\mu \alpha^a - \alpha^b A_\mu^c f^{bca}$$

$j^{\mu a}$ given in $(*)$ is Noether current of global $SU(n)_V$.

$$D_\mu j^{\mu a} = 0$$

(follows from eq. of motion $D_\mu F^{\mu\nu a} = j^{\nu a}$)

Note: There exists an "improved" current $J^{\mu a} := j^{\mu a} - F^{\mu\nu c} A_\nu^b f^{abc}$

which obeys: $\partial_\mu J^{\mu a}$ (follows from $\partial_\mu F^{\mu\nu a} = J^{\nu a}$)

global symmetries of $\mathcal{L}$: (for $G = SU(n)$)

- $SU(n)_V$
- $U(1)_V$: $\psi^i \to \psi'^i = e^{i\alpha}\psi^i$, $\bar{\psi}^i \to \bar{\psi}'^i = e^{-i\alpha}\bar{\psi}^i$

  $A_\mu \to A_\mu' = A_\mu$

  (corresponds to t^a, together $SU(n)_V \times U(1)_V = U(n)_V$)

- $m = 0$: $U(n)_A$

  $$\psi^i \to \psi'^i = \left(e^{i\gamma^5 \beta^a t^a}\right)^i{}_j \psi^j \quad , \ \beta^a \text{ real, const}$$

  $$\bar{\psi}^i \to \bar{\psi}'^i = \bar{\psi}^j \left(e^{+i\gamma^5 \beta^a t^a}\right)$$

  $$A_\mu \to A_\mu' = \left(e^{+i\gamma^5 \beta^a t^a}\right) A_\mu \left(e^{-i\gamma^5 \beta^a t^a}\right)$$

Noether currents: $j_A^{\mu a} = \bar{\psi}^i \gamma^\mu \gamma^5 t^a{}_{ij} \psi^j$

$$D_\mu j_A^{\mu a} = 0 \qquad \text{(also covariantly conserved!)}$$

together $\mathcal{L}$ has global $U(n)_V \times U(n)_A$ symmetry aside: rewrite $\mathcal{L}$ in terms of chiral fermions

$$\psi_{D(irac)} = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \quad \bar{\psi} = \psi^+ \gamma^0 = (\bar{\psi}_R, \bar{\psi}_L)$$

$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$ in chiral rep.

$\psi_{L,R} :=$ 2 component Weyl-spinors

chiral rep. $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \sigma^\mu = (\mathbb{1}, \vec{\sigma}), \quad \bar{\sigma}^\mu = (\mathbb{1}, -\vec{\sigma})$

Pauli

$$\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \quad \psi_{L,R} = \tfrac{1}{2}(\mathbb{1} \mp \gamma^5)\psi_D$$

$$\Rightarrow \bar{\psi}_D (i\gamma^\mu D_\mu - m)\psi_D = i\bar{\psi}_L \sigma^\mu D_\mu \psi_L + i\bar{\psi}_R \bar{\sigma}^\mu D_\mu \psi_R$$
$$- m(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L)$$

$$D_\mu \psi^i_{L,R} = \partial_\mu \psi^i_{L,R} - g A^a_\mu t^{a\,i}{}_j \psi^j$$

$m=0$: global $U(n)_L \times U(n)_R$

$$\delta\psi_L = i\alpha^a_L t^a \psi_L, \quad \delta\psi_R = i\alpha^a_R t^a \psi_R$$

$t^a_{L,R}$: generators of $U(n)_{L,R}$, $\quad \alpha = \tfrac{1}{2}(\alpha_L + \alpha_R)$
$\quad \beta = \tfrac{1}{2}(\alpha_L - \alpha_R)$

Noether currents: $\quad j^{\mu a}_L = \bar{\psi}_L \sigma^\mu t^a \psi_L = \tfrac{1}{2}(j^{\mu a} + j^{\mu a}_A)$

$$j^{\mu a}_R = \bar{\psi}_R \bar{\sigma}^\mu t^a \psi_R = \tfrac{1}{2}(j^{\mu a} - j^{\mu a}_A)$$

conservation: for $m=0$ $\quad D_\mu j^{\mu a}_L = 0 = D_\mu j^{\mu a}_R$

2) chiral gauge theories

$G = G_L \times G_R$ (e.g. $SU(n)_L \times SU(m)_R$, $n \neq m$)

$$\mathcal{L} = i\bar{\psi}_L \sigma^\mu D_\mu \psi_L + i\bar{\psi}_R \bar{\sigma}^\mu D_\mu \psi_R - \frac{1}{4C_L(x)} \mathrm{tr}\left[F_{\mu\nu} F^{\mu\nu}\right]_L - \frac{1}{4C_R(x)} \mathrm{tr}\left[F_{\mu\nu} F\right.$$

$$D_\mu \psi_L = \partial_\mu \psi_L - g_L A^a_{\mu L} t^a_L \psi_L \quad , \quad t^a_{L,R} = \text{generators of } G_{L,R}$$

$$D_\mu \psi_R = \partial_\mu \psi_R - g_R A^a_{\mu R} t^a_R \psi_R$$

$$\mathcal{L} = i\bar{\psi}_L \sigma^\mu \partial_\mu \psi_L + i\bar{\psi}_R \bar{\sigma}^\mu \partial_\mu \psi_R - g_L \mathrm{tr}\, j^\mu_L A_{\mu L} - g_R \mathrm{tr}\, j^\mu_R A_{\mu R} \dots$$

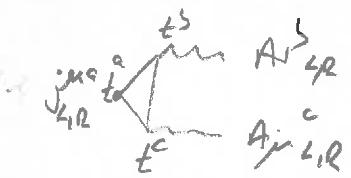
$\Rightarrow$ both currents $j^\mu_{L,R}$ couple to independent gauge fields $A_{\mu L,}$

vector-like theories: $\underset{\sim}{\bar{r}}(\psi_L) = \underset{\sim}{\bar{r}}(\bar{\psi}_R)$ or $\underset{\sim}{r}(\psi_L) = \underset{\sim}{r}(\psi_R)$

such that $\bar{\psi}_R \psi_L$ is inv.

perturbative calculation of non-Abelian anomaly

one-loop Feynman diagrams:



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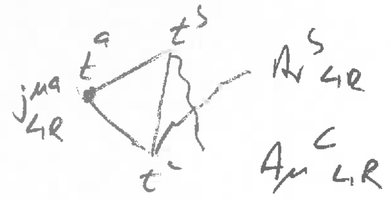

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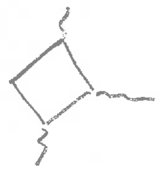


+ ...

$\sim O(A^3)$

- momentum integral identical
  t^a-factors new

Result: $$D_\mu j^{\mu a}_{L,R} = \mathcal{A}^a_{L,R}$$

$$\mathcal{A}^a_{L,R} = \mp \frac{g^2}{32\pi^2}\,\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}_r\left(t^a_{L,R}\,\partial_\mu\left(A_\nu\partial_\rho A_\sigma - \frac{i}{4}A_\nu[A_\rho,A_\sigma]\right)\right)_{L,R}$$

$$= \mp \frac{g^2}{32\pi^2}\,\epsilon^{\mu\nu\rho\sigma}\,D^{abc}_{L,R}\,\partial_\mu\left(A^b_\nu\partial_\rho A^c_\sigma - \frac{i}{2}A^b_\nu[A_\rho,A_\sigma]^c\right)_{L,R}$$

(where $D^{abc}_{L,R}(r) = \mathrm{tr}_r\left(t^a_{L,R}\{t^b,t^c\}_{L,R}\right)$)

uses $\epsilon^{\mu\nu\rho\sigma}[A_\nu,[A_\rho,A_\sigma]] = 0$ due to Jacobi id.

Remarks

- ψ_L, ψ_R transform differently under Lorentz trans.
  (in $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ rep.

but $\bar{\psi}_R$ transforms as ψ_L (in $(\frac{1}{2},0)$)!

convention in computation:
all fermions in same Lorentz rep!

- vector-like theories: $\tilde{r}(\psi_L) = \bar{r}(\bar{\psi}_R)$

$$\Rightarrow \quad A_L = -A_R \quad \Rightarrow \quad D_\mu j^{\mu a} = D_\mu \left(j_L^{\mu a} + j_R^{\mu a} \right) = 0 \,!$$

$\Rightarrow$ all vector-like theories have no gauge anomaly!
(of course there can be a global anomaly)

- In this case

$$D_\mu j_A^{\mu a} = D_\mu \left(j_L^{\mu a} - j_R^{\mu a} \right) \neq 0 \qquad (\text{if } D^{abc} \neq 0)$$

decompose $U(n)_A = SU(n)_A \times U(1)_A$

$$\underset{\underset{U(1)_A}{\uparrow}}{\partial_\mu j_A^\mu} = -\frac{g^2}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} \partial_\mu \operatorname{tr} \left(A_\nu \partial_\rho A_\sigma - \frac{i}{4} A_\nu [A_\rho, A_\sigma] \right)$$

L3 : chiral anomalies and the WZ consistency condition

recall chiral gauge theories $\quad G = G_L \times G_R$

$$\mathcal{L} = i\bar{\Psi}_L \sigma^\mu \partial_\mu \Psi_L + i \bar{\Psi}_R \bar{\sigma}^\mu \partial_\mu \Psi_R - g_L j_L^{\mu a} A_{\mu L}^a - g_R j_R^{\mu a'} A_{\mu R}^{a'}$$

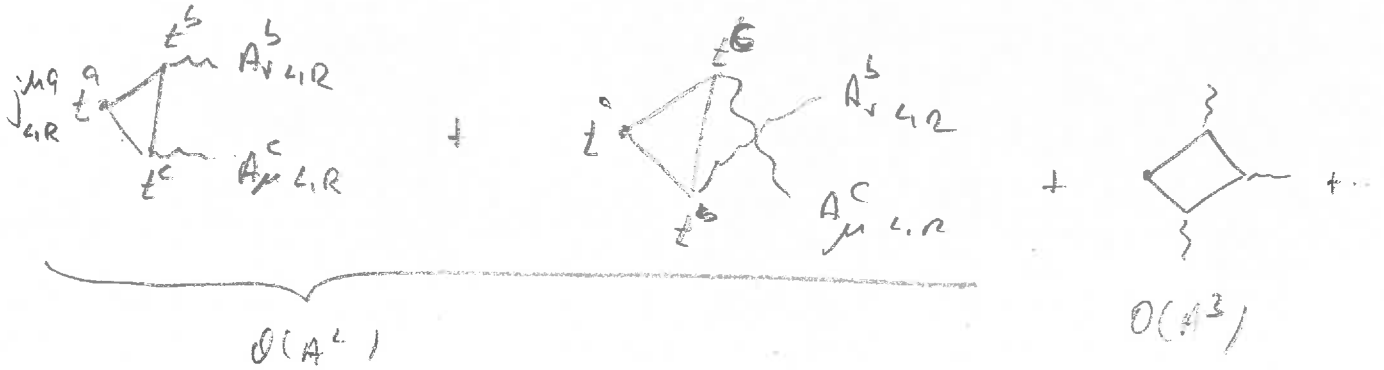
$$\underbrace{-\frac{1}{4c_L} \mathrm{tr}(F_{\mu\nu}F^{\mu\nu})_L - \frac{1}{4c_R} \mathrm{tr}(F_{\mu\nu}F^{\mu\nu})_R}_{\substack{\approx 0 \text{ if } A_{L,R} \text{ is treated} \\ \text{as a background gauge field } A_{L,R} \sim \text{const}}}$$

$a = 1, \dots, \dim(\mathrm{ad}(G_L))$
$a' = 1, \dots, \dim(\mathrm{ad}\, G_R))$

$$j_L^{\mu a} = \bar{\Psi}_L^i \sigma^\mu t_{L ij}^a \Psi_L^j \quad , \quad j_R^{\mu a'} = \bar{\Psi}_R^{i'} \bar{\sigma}^\mu t_{R i'j'}^a \Psi_R^{j'}$$

$$D_\mu j_{L,R}^{\mu a} = 0 + \mathcal{A}_{L,R}^a$$

anomaly computed from triangle diagram



momentum integral identical t^a factors <u>new</u>

Result

$$\mathcal{A}^a_{L,R} = \mp \frac{g^2_{L,R}}{32\pi^2}\, \epsilon^{\mu\nu\rho\sigma}\, \mathrm{tr}\left(t^a_{L,R}\, \partial_\mu \left(A_\nu \partial_\rho A_\sigma - \frac{i}{2} A_\nu [A_\rho, A_\sigma] \right)_{L,R} \right)$$

use $\left\{ \begin{array}{l} \epsilon^{\mu\nu\rho\sigma} \\ A_\nu = A^a_\nu t^a \end{array} \right. [A_\nu, [A_\rho, A_\sigma]] = 0$ due to Jacobi id.

$$\Rightarrow \mathcal{A}^a_{L,R} = \mp \frac{g^2_{L,R}}{32\pi^2}\, \epsilon^{\mu\nu\rho\sigma}\, D^{abc}_{L,R}\, \partial_\mu \left(A^b_\nu \partial_\rho A^c_\sigma - \frac{i}{2} A^b_\nu [A_\rho, A_\sigma]^c \right)_{L,R}$$

where $D^{abc}_{L,R}(\underset{\sim}{r}) = \mathrm{tr}_{\underset{\sim}{r}} \left(t^a_{L,R} \{ t^b_{L,R}, t^c_{L,R} \} \right)$

Remarks

- ψ_L, ψ_R transform differently under Lorentz trans. (in $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ resp.)

but $\bar{\psi}_R$ transforms as ψ_L (in $(\frac{1}{2},0)$)!

convention in computation: <u>all</u> fermi in same Lorentz rep!

vector-like theories: $\tau(\psi_L) = \bar{\tau}(\psi_R)$

$$\Rightarrow D_L^{abc} = D_R^{abc}$$

$$\Rightarrow A_L = -A_R \quad \Rightarrow \quad D_\mu j^{\mu a} = D_\mu (j_L^{\mu a} + j_R^{\mu a}) = 0 \,!$$

$\Rightarrow$ all vector-like theories have no gauge anomaly! (of course there can be a global anomaly)

In this case

$$D_\mu j_A^{\mu a} = D_\mu (j_L^{\mu a} - j_R^{\mu a}) \neq 0 \quad (\text{if } D^{abc} \neq 0)$$

decompose $U(n)_A = SU(n)_A \times U(1)_A$

$$\underset{\substack{\uparrow \\ U(1)_A}}{\partial_\mu j_A^\mu} = -\frac{g^2}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} \partial_\mu \operatorname{tr}\left(A_\nu \partial_\rho A_\sigma - \frac{i}{3} A_\nu [A_\rho, A_\sigma]\right)$$

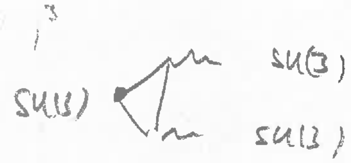
- $\mathcal{A}$ has two factors: D^{abc} and its A_μ^a dep.
  - D^{abc} is <u>non-universal</u> and depends on matter content
  - A-dep is <u>universal</u> and determined by WZ-consistency condition

example: compute D^{abc} for Standard Model (SM)

fermion rep of SM:

	$SU(3)$ ×	$SU(2)$ ×	$U(1)_Y$	$U(1)_{em}$
$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\underset{\sim}{3}$	$\underset{\sim}{2}$	$\frac{1}{6}$	$\begin{pmatrix} 2/3 \\ -1/3 \end{pmatrix}$
$\bar{u}_R$	$\underset{\sim}{\bar{3}}$	1	$-\frac{2}{3}$	$-\frac{2}{3}$
$\bar{d}_R$	$\underset{\sim}{\bar{3}}$	1	$+\frac{1}{3}$	$+\frac{1}{3}$
$L_L = \begin{pmatrix} \nu_e \\ e_L \end{pmatrix}$	$\underset{\sim}{1}$	$\underset{\sim}{2}$	$-\frac{1}{2}$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$
$\bar{e}_R$	$\underset{\sim}{1}$	$\underset{\sim}{1}$	$+1$	$+1$

SU(3) - anomaly



quarks on is vectorlike rep of SU(3)

$\Rightarrow D_L = D_R \Rightarrow A = 0$

$$D^{abc} = \mathrm{tr}\,(\overset{\text{Pauli}}{\sigma^a}\underbrace{\{\sigma^b,\sigma^c\}}_{\delta^{bc}\mathbb{1}}) = \delta^{bc}\,\mathrm{tr}\,\sigma^a = 0$$

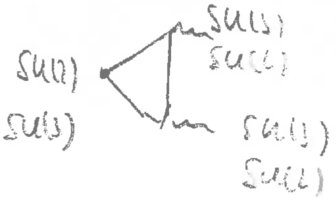
(reflects that $\underset{\sim}{2}$ of SU(2) is real rep)

$$D = \sum_q q_Y^3$$

$$= 2\,q_Y^3(L_L) + q_Y^3(\bar e_R) + 3\cdot 2\,q_Y^3(Q_L) + 3\,q_Y^3(\bar u_R) + 3\,q_Y^3(\bar d_R)$$

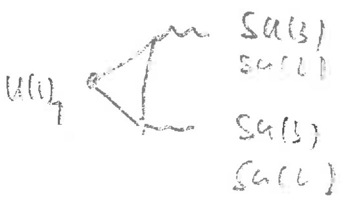
$$= 2\left(-\tfrac{1}{2}\right)^3 + 1^3 + 6\left(\tfrac{1}{6}\right)^3 + 3\left(-\tfrac{2}{3}\right)^3 + 3\left(\tfrac{1}{3}\right)^3$$

$$= -\frac{2}{8} + 1 + \frac{1}{36} - \frac{8}{9} + \frac{1}{9} = \frac{1}{72}\left(-18 + 72 + 2 - 64 + 8\right) = 0$$



$$D \sim \underbrace{\mathrm{tr}\,\sigma^a}_{=0}\,\mathrm{tr}(\lambda^b\lambda^c) = 0$$

$$\sim \underbrace{\mathrm{tr}\,\lambda^a}_{=0}\,\mathrm{tr}(\sigma^b\sigma^c) = 0$$



$$D^{133} \sim \sum_q q_Y(Q_L, u_L, \ldots)\,\mathrm{tr}\,\lambda^a\lambda^b = (\mathrm{tr}\,\lambda^a\lambda^b)\big(2\,\underset{1/6}{q_Y(Q_L)} + \underset{-2/3}{q(\bar u_R)} + \underset{1/3}{q(\bar d_R)}\big)$$

$$= \mathrm{tr}(\lambda^a\lambda^b)\left(\tfrac{1}{3} - \tfrac{2}{3} + \tfrac{1}{3}\right) = 0$$

$$D^{122} \sim \mathrm{tr}\,\sigma^a\sigma^b\big[q_Y(L_L) + 3\,q_Y(Q_L)\big] = \mathrm{tr}\,\sigma^a\sigma^b\left(-\tfrac{1}{2} + 3\,\tfrac{1}{6}\right) = 0$$

WZ consistency condition

- treat A_μ^a as background gauge field and denote effective quantum action $\Gamma[A]$

$$\Gamma[A] = S_{cl} + \hbar \ldots$$

- write gauge inv in terms of a functional operator $T^a_{(x)}$

$$\delta A_\mu^a = \frac{1}{g}\partial_\mu \alpha^a(y) - \alpha^b A_\mu^c f^{bca} \equiv D_\mu \alpha^a$$

$$= i\int d^4x\, \alpha^b(x)\, T^b(x)\, A_\mu^a(y)$$

for $iT^b(x) = -\frac{1}{g}\partial_\nu \frac{\delta}{\delta A_\nu^b(x)} - f^{bcd} A_\nu^c(x) \frac{\delta}{\delta A_\nu^d(x)} \equiv -D_\mu \frac{\delta}{\delta A_\mu^b}$

where $\frac{\delta}{\delta A_\nu^b(x)} A_\mu^a(y) = \delta_\mu^\nu \delta_b^a\, \delta(x-y)$ (functional derivative)

check: $i\int d^4x\, \alpha^b(x) T^b(x) A_\mu^a(y) = \int d^4x\, \alpha^b(x) \Big(-\frac{1}{g}\partial_\nu \delta_\mu^\nu \delta_b^a\, \delta(x-y)$

$$- f^{bcd} A_\nu^c(x)\, \delta_\mu^\nu \delta_d^a\, \delta(x-y)\Big)$$

$$= \frac{1}{g}\partial_\mu \alpha^a(y) - \alpha^b f^{bcd} A_\nu^c \delta_\mu^\nu \delta_d^a$$

$$= \frac{1}{g}\partial_\mu \alpha^a - \alpha^b A_\mu^c f^{bca} \quad \checkmark$$

also check [B] (uses Jacobi id) $\left[T^a(x), T^b(y)\right] = -i f^{abc}\, T^c(x)\, \delta(x-y)$ (*)

Can define anomaly as variat. of Γ

$$\delta_\alpha \Gamma[A] = i\int d^4x\, \alpha^a(x)\, T^a(x)\, \Gamma[A] = -\int d^4x\, \alpha^a(x)\, \mathcal{A}^a(x)$$

consistent with definition of current

$$j^{\mu a}(x) := \frac{\delta \Gamma[A]}{\delta A_\mu^a(x)} \qquad \left(\text{classically } \mathcal{L} \sim j^{\mu a} A_\mu^a\right)$$

$$\Rightarrow \delta_\alpha \Gamma = -\int d^4x\, \alpha^a D_\mu \frac{\delta \Gamma}{\delta A_\mu^b} = -\int d^4x\, \alpha^a D_\mu j^{\mu a}$$

$$\Rightarrow \begin{cases} D_\mu j^{\mu a} = \mathcal{A}^a \\ i T^a_{(x)} \Gamma[A] = -\mathcal{A}^a_{(x)} \quad (**) \end{cases}$$

$$(**) \Rightarrow \left(T^a(x) T^b(y) - T^b(x) T^a(y)\right)\Gamma[A] = iT^a(x)\, \mathcal{A}^b(y) - iT^b(x)\, \mathcal{A}^a(y)$$

$$\overset{(*)}{=} -i f^{abc}\, T^c(x)\, \delta(x-y)\, \Gamma[A]$$

$$\Rightarrow \boxed{T^a(x)\, \mathcal{A}^b(y) - T^b(x)\, \mathcal{A}^a(y) = -i f^{abc}\, \delta(x-y)\, \mathcal{A}^c(x)}$$

Wess-Zumino consistency condition

L4 Solution of WZ condition

1^{st} step: rewrite gauge symmetry as BRST-symmetry (Becchi, Rouet, Stora, Tyutin)

4.1
$$\alpha^a(x) = \theta\, \omega^a(x)$$

$\uparrow$ gauge parameter

θ: anticommuting Grassmann variable, $\theta^2 = 0$

$\omega^a(x)$: " Grassmann valued ghost field

$$\omega^a(x)\,\omega^b(x) = -\omega^b(x)\,\omega^a(x)\ , \quad \theta^2 = 0$$

variation $\delta = \theta\, s$

s: BRST operator

recall gauge transf.

$$\delta A^a_\mu = \frac{1}{g}\partial_\mu \alpha^a(x) - \alpha^b A^c_\mu f^{bca}$$

$$\Rightarrow \theta\, s A^a_\mu = \frac{\theta}{g}\partial_\mu \omega^a(x) - \theta\, \omega^b(x) A^c_\mu f^{bca}$$

$$\Rightarrow s A^a_\mu = \frac{1}{g}\partial_\mu \omega^a(x) - \omega^b(x) A^c_\mu f^{bca} \equiv D_\mu \omega^a$$

$$s^2 = 0 \quad \text{for} \quad s\,\omega^a(x) = -\frac{1}{2} f^{abc}\,\omega^b(x)\,\omega^c(x)$$

define $s(AB) = (sA)B \pm A\, sB$ — $+$: A commuting var., $-$: A anticomm. var.

exercise: check $s^2\omega^a = s^2 A^a_\mu = 0$

(use Jacobi id for f^{abc})

Define $\mathcal{A}[\omega, A] := \int d^4x \; \omega^a(x) \, \mathcal{A}^a(A, x)$
↑ anomaly

compute

$$s\mathcal{A} = \int d^4x \left[(s\omega^a)\, \mathcal{A}^a - \omega^b \, s\mathcal{A}^b \right]$$

$$s\mathcal{A}^b(x) = \int d^4y \; \frac{\delta \mathcal{A}^b(x)}{\delta A^d_\mu(y)} \, s A^d_\mu(y) = \int d^4y \; \frac{\delta \mathcal{A}^b}{\delta A^d_\mu} \, D_\mu \omega^d(y)$$

$$\Rightarrow s\mathcal{A}[\omega, A] = \int d^4x\, d^4y \; \omega^b(x)\, \omega^c(y) \Big(-\tfrac{1}{2} f^{abc} \, \delta(x-y)\, \mathcal{A}^a(x)$$
$$- \underbrace{\left(-\frac{1}{g} \frac{\partial}{\partial y^\mu} \delta^d_c - f^{ced} A^e_\mu \right) \frac{\delta}{\delta A^d_\mu} \mathcal{A}^b(x)}_{= i T^c_{(y)} \mathcal{A}^b(x)}$$

$$= \int d^4x\, d^4y \; \omega^b(x)\, \omega^c(y) \left(-\tfrac{1}{2} f^{abc} \, \delta(x-y)\, \mathcal{A}^a - i T^c_{(y)} \mathcal{A}^b(x) \right)$$

$\omega^b \omega^c = -\omega^c \omega^b$

$$\stackrel{\downarrow}{=} -\frac{i}{2} \int d^4x\, d^4y \; \omega^b(x)\, \omega^c(y) \underbrace{\Big(T^c_{(y)} \mathcal{A}^b(x) - T^b_{(y)} \mathcal{A}^c(x) + i f^{cba} \mathcal{A}^a \, \delta(x-y) \Big)}_{= 0 \text{ due to WZ}}$$

$\Rightarrow$ In BRST formalism the WZ consistency cond. reads

$$\boxed{s\,\mathcal{A}[\omega, A] = 0}$$

Remarks

- $\mathcal{A}$ is BRST closed functional of ghost # = 1 (= linear in ω)
- trivial solution :

$$\mathcal{A}_{loc} = +s\, G[A]$$

local functional with ghost # = 0

corresponds to adding local counterterm to $\Gamma[A]$

$$s\,\Gamma[A] = -\mathcal{A}[\omega, A]$$

$$s(\Gamma - G) = -\mathcal{A} - \mathcal{A}_{loc}$$

$\Rightarrow$ $\mathcal{A}$ is <u>not</u> unique, local counterterm adds "trivial" piece $\hat{=}$ choice of regulator

$\Rightarrow$ If $\mathcal{A} = \mathcal{A}_{loc} = sF \Rightarrow$ anomaly can be removed!

$\Rightarrow$ nontrivial ("real") anomaly satisfies

$$s\mathcal{A} = 0 \quad \text{and} \quad \mathcal{A} \neq sG$$

$\Rightarrow$ $\mathcal{A}$ is in cohomology class of s at ghost # = 1 $\quad H^1_s$

use diff forms to construct solution

p-form: $J = \frac{1}{p!} J_{\underbrace{\mu_1 \ldots \mu_p}}(x) \, dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_p}$

wedge prod.: $dx^\mu \wedge dx^\nu = - dx^\nu \wedge dx^\mu$

1-form gauge field: $A = A_\mu dx^\mu$

Lie algebra valued 1-form: $A = i A_\mu^a t^a dx^\mu$

ghost 0-form: $\omega = i \omega^a t^a$

exterior derivation: $d = dx^\mu \partial_\mu \quad , \quad d^2 = 0$

2-form field strength: $F = dA - A^2$

BRST transformations: $sA = -d\omega + [A, \omega]$

$s\omega = \omega^2$

convention: $ds + ds = 0$

$dx^\mu \omega + \omega dx^\mu = 0$

Abelian anomaly: $\mathcal{A} \sim \quad F \wedge F = \frac{1}{4} F_{\mu\nu} F_{\rho\sigma} \underbrace{dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma}_{\epsilon^{\mu\nu\rho\sigma} d^4x}$

$= \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \, d^4x$

construct solution via auxiliary manifold $M_4 \times D$
2dim. disk
(θ, ρ as coordinates)

$$A = A_\mu dx^\mu + A_\rho d\rho + A_\theta d\theta$$

$$d = dx^\mu \partial_\mu + d\rho\, \partial_\rho + d\theta\, \partial_\theta$$

$$F = dA - A^2$$

check:

i) $dF = d(dA - A^2) = -(dA)A + A\,dA = -FA - A^3 + AF + A^3$

$= [A, F]$

ii) $d(\mathrm{tr}\, F^3) =: d(\mathrm{tr}\, F\wedge F\wedge F) = \mathrm{tr}\left(dF\wedge F^2 + F\wedge dF\wedge F + F^2 dF\right)$

$= 3\,\mathrm{tr}((dF)F^2) \overset{i)}{=} 3\,\mathrm{tr}([A,F]F^2) = 3\,\mathrm{tr}(AF^3 - FAF^2)$

$= 3\,\mathrm{tr}(AF^3 - AF^3) = 0 \quad \to \quad \boxed{d\,\mathrm{tr}\, F^3 = 0}$

iii) $\boxed{s(\mathrm{tr}\, F^3) = 0}$

ii) & iii) imply the <u>descent equation</u> [Stora - Zumino]

ii) $d\,\mathrm{tr}F^3 = 0 \Rightarrow$ $\underline{\mathrm{tr}\, F^3 = dQ_5^1}$ locally or on simply connected!
Chern - Simons Term

$0 = s\,\mathrm{tr}F^3 = s\,dQ_5^1 = -d\,s\,Q_5 \Rightarrow$ $\underline{sQ_5 = dQ_4^1}$ should be locally

$0 = s^2 Q_5 = s\,dQ_4^1 = -d\,s\,Q_4^1 \Rightarrow$ $\underline{sQ_4^1 = dQ_3^2}$ locally

<u>ex</u>: check $\quad Q_5 = \mathrm{tr}\left(AF^2 - \frac{1}{2}A^3F + \frac{1}{10}A^5\right)$

$$Q_4^1 = \mathrm{tr}\left(\omega\, d\left(A\,dA + \frac{1}{2}A^3\right)\right)$$

last descent equation gives anomaly

$$\mathcal{A}[\omega, A] \sim \int_{M_4} Q_4^1$$

need to show that $Q_4^1 \neq s\, Q_4^0$

assume $Q_4^1 = s\, Q_4^0 \Rightarrow s Q_5 = d Q_4^1 = d\, s Q_4^0 = s\, d Q_4^0$

$\Rightarrow s(Q_5 + d Q_4^0) = 0$

$\Rightarrow Q_5 + d Q_4^0$ gauge inv. but no odd gauge inv. form can exist!

$\Rightarrow Q_5 = - d Q_4^0 \Rightarrow d Q_5 = 0$

but $d Q_5 = \operatorname{tr} F^3 \Rightarrow$ contradiction

L5: Anomalies in the Path Integral Formalism

[Fujikawa]

fermionic PI

$$e^{i\Gamma[A]} := \int [D\psi][D\bar\psi]\, e^{iS(\psi,\bar\psi,A)} \qquad (\text{Euclidean } M_4:\ i \to -)$$

PI measure: integration over all field configurations

$$S = \int d^4x\, \bar\psi\, i\not{D}\psi = \int d^4x \left(\bar\psi\, i\not\partial\psi - g\, j^\mu A_\mu\right)$$

Dirac spinors (anticommuting)

S gauge inv. $\Rightarrow$ anomaly can only be in PI measure

expand ψ: $\quad \psi(x) = \sum_m \theta_m\, \phi_m(x)$

θ_m: const Grassmann variable; ϕ_m: eigenfct of $\not{D}$

$$i\not{D}\,\phi_m = \lambda_m\,\phi_m \qquad (\lambda_m \text{ real in Euclidean } M_4)$$

$$\int d^4x\, \phi_m^\dagger\,\phi_{m'} = \delta_{mm'} \qquad (\text{orthogonality})$$

$$\sum_m \phi_m(x)\,\phi_m^\dagger(y) = \delta(x-y) \qquad (\text{completeness})$$

$$\Rightarrow\ e^{i\Gamma[A]} = c\cdot \det\left(i\not{D}(A)\right) = c\prod_m \lambda_m$$

determine transformation law of measure,

$$e^{i\Gamma[A]} = \int [D\psi][D\bar\psi]\, e^{i\int \bar\psi i\not{D}\psi} = c \cdot \det(i\not{D})$$

$$= \int [D\psi'][D\bar\psi']\, e^{i\int \bar\psi' i\not{D}\psi'} = \int [D\psi][D\bar\psi] J^2 e^{i\int \bar\psi u^+ i\not{D} u\psi}$$

$\psi' = u\psi$

$[D\psi'] = J[D\psi]$

$u \neq u(x)$

$$\overset{\downarrow}{=} J\bar J \det(u^+ i\not{D} u) = J\bar J (\det u^+)\det u \det(i\not{D})$$

$$\Rightarrow J = (\det u)^{-1}$$

vector-like gauge transf. $u = e^{i\alpha^a(x) t^a}$, $uu^+ = 1$

$$\Rightarrow \det u^+ \det u = 1 \Rightarrow J\bar J = 1$$

chiral gauge transf: $u = e^{i\gamma_5 \alpha^a(x) t^a} = u^+$

$\rightarrow$ measure not inv. !

use $\det M = e^{\mathrm{tr}\ln M}$

$$\Rightarrow \ln\det u = \mathrm{tr}\ln u$$

inf: $u = 1 + i\gamma_5 \alpha^a t^a + O(\alpha^2)$, $\ln(1+x) \approx x$

$$\Rightarrow \ln\det u = +i\int d^4x\, \alpha(x) \sum_n \phi_n^+ \gamma_5 t^a \phi_n + \dots$$

$$\sim \text{“}\mathrm{tr}\,\gamma_5\text{”}, \quad \mathrm{tr}\,\gamma_5 = 0$$

but also infinite sum over eigenstates !

$\Rightarrow$ regulate the sum

$$\sum_n \phi_n^+ \gamma^5 t^a \phi_n = \lim_{M\to\infty} \sum_n \phi_n^+ \gamma^5 t^a \phi_n \, e^{\lambda_n^2/M^2}$$

sign of exp:
free Dirac: $\lambda^2 \sim (k^0)^2 - \vec{k}^2$
$\Rightarrow \lambda^2 < 0$ for Euclid space-time

$$= \lim_{M\to\infty} \sum_n \phi_n^+ \gamma^5 t^a e^{\frac{(i\not{D})^2}{M^2}} \phi_n$$

$$(i\not{D})^2 = i^2 \gamma^\mu (\partial_\mu - ieA_\mu) \gamma^\nu (\partial_\nu - ieA_\nu)$$

$$= -D^\mu D_\mu + \frac{e}{2} \Sigma^{\mu\nu} F_{\mu\nu}, \qquad \Sigma^{\mu\nu} := \frac{i}{2} [\gamma^\mu, \gamma^\nu]$$

to get non-zero result need 4 γ-Matrices

$$\Rightarrow \sum_n \phi_n^+ \gamma^5 t^a \phi_n = \lim_{M\to\infty} \sum_n \phi_n^+(x) \gamma^5 t^a \frac{1}{2!} \left(\frac{e}{2M^2} \Sigma^{\mu\nu} F_{\mu\nu}\right)^2 e^{-\frac{D^2}{M^2}} \phi_n(x)$$

insert complete set of state

$$\sum_n \phi_n^+ t^a \gamma^5 \phi_n = \lim_{M\to\infty} \sum_{n,m} \phi_n^+(x) t^a \gamma^5 \frac{1}{2!} \left(\frac{e}{2M^2} \Sigma^{\mu\nu} F_{\mu\nu}\right)^2 \underbrace{\int d^4y \, \phi_m(y) \phi_m^+(y)}_{\delta_{nm}} e^{-\frac{D^2}{M^2}} \phi_n$$

$$= \lim_{M\to\infty} \int d^4y \left(\sum_n \phi_n^+(x) t^a \gamma^5 \frac{1}{2!} \left(\frac{e}{2M^2} \Sigma^{\mu\nu} F_{\mu\nu}\right)^2 \phi_n(y) \right) \sum_m \phi_m^+(y) e^{-\frac{D^2}{M^2}} \phi_m(x)$$

completes

$$= \lim_{M\to\infty} \mathrm{tr}\Big(\gamma^5 t^a \frac{1}{2!}\Big(\frac{e}{2M^2}\Sigma^{\mu\nu}F_{\mu\nu}\Big)^2\Big) \int d^4y\, \delta(x-y) \sum_m \phi_m^\dagger(y)\, e^{-\frac{D^2}{M^2}} \phi_m(x)$$

in γ-space

$$\sim \lim_{x\to y} \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)}\, e^{\frac{k^2}{M^2}}$$

$$= i \int \frac{d^4k_E}{(2\pi)^4}\, e^{-\frac{k_E^2}{M^2}}$$

$$= i\,\frac{M^4}{16\pi^2}$$

Gauss

$$\sim \frac{e^2}{M^2}\, \underbrace{\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)}_{\sim\, \epsilon^{\mu\nu\rho\sigma}}\, \mathrm{tr}(t^a F_{\mu\nu} F_{\rho\sigma})$$

$\Rightarrow$

$$\sum_n \phi_n^\dagger \gamma^5 t^a \phi_n \sim e^2 \epsilon^{\mu\nu\rho\sigma}\, \mathrm{tr}(t^a F_{\mu\nu} F_{\rho\sigma}) \sim \mathcal{A}^a(x)$$

$\Rightarrow$ under chiral translation we have

$$\Gamma[A] \to \Gamma[A] - i \int d^4x\, \alpha^a(x)\, \mathcal{A}^a(x) \quad \checkmark$$

Recall:
for $m=0$ Dirac spinor split $\psi_0 = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$

$$\psi_{L,R} = \tfrac{1}{2}(1 \pm \gamma_5)\,\psi_D \quad , \quad \gamma_5\,\psi_{L,R} = \pm\,\psi_{L,R}$$

eigenvalues of $i\not{D}$: $\quad i\not{D}\,\phi_m = \lambda_m\,\phi_m$

$$\{\gamma^5, \gamma^\mu\} = 0 \quad \Rightarrow \quad i\gamma^5\gamma^\mu D_\mu\,\phi_m = -i\gamma^\mu D_\mu\,\gamma^5\phi_m = \lambda_m\,\gamma^5\phi_m$$

$\Rightarrow$ $\gamma^5\phi_m$ is eigenfct of $i\not{D}$ with eigenvalue $-\lambda_m$

$\Rightarrow$ 1) eigenvalues of $i\not{D}$ come in pairs $\pm\lambda_m$

2) $\sum_n \phi_n^+ \gamma^5 \dot{t}\phi_n = \sum_n (\phi_{nL}^+ \dot{t}\phi_{nL}) - (\phi_{nR}^+ \dot{t}\phi_{nR})$

$= 0$ as λ_n^2 is the same

$\Rightarrow$ only zero modes of $i\not{D}$ can contribute

$i\not{D}\,\phi_m = 0$ $\qquad$ they do not have to be paired!

$$\int d^4x\, \mathcal{A}(x) \approx \int d^4x\, \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = \int d^4x\, \partial_\mu \underset{\substack{\uparrow \\ \text{Chern-Simons Term}}}{Q^\mu} \approx n_+ - n_-$$

where $n_\pm$ = # of zero modes of $i\not{D}$ with ± 1 eigenvalue of γ_5

= # of zero modes of $\phi_{n\,L/R}$

Remarks:

- Result is a special case of Atiyah-Singer index theorem
- under variation of the gauge field integral cannot change smoothly but only by integers
  $\Rightarrow$ integral only depends on topology of gauge field configuration

L6 Anomalies and Index Theorems

Def: analytic index of operator $i\not{D}$

$$\mathrm{index}(i\not{D}) := \dim \ker(i\not{D}) - \dim \mathrm{coker}(i\not{D})$$

$$= \dim \ker(i\not{D}) - \dim \ker((i\not{D})^+)$$

(in general $i\not{D}$ = any Fredholm op. when $\ker(i\not{D})$ and $\mathrm{coker}(i\not{D})$ are finite dimensional)

Recall $\psi_D(x) = \sum_m \theta_m \phi_m(x)$

Grassmann, eigenfct of $i\not{D}$

$$i\not{D}\phi_m = \lambda_m \phi_m(x)$$

$$\int d^4x\, \phi_m^+ \phi_n = \delta_{mn} \quad , \quad \sum_m \phi_m(x)\phi_m^+(y) = \delta(x-y)$$

$$\ker(i\not{D}) = \{\phi_m \mid i\not{D}\phi_m = 0\}$$

$\rightarrow \dim \ker(i\not{D}) = n = \#$ of zero modes of $i\not{D}$

define $(i\not{D})^+$:

$$\int d^4x\, \phi_m^+ i\not{D}\phi_n = \int d^4x\, [(i\not{D})^+\phi_m^+]\, \phi_n$$

Dirac op: $(i\not{D})^+ = i\not{D} \Rightarrow \boxed{\mathrm{index}(i\not{D}) = 0}$

This changes if one considers Dirac op. acting on Weyl fermions.

Recall $\phi_{n L,R} = P_{L,R}\,\phi_n$, $P_{L,R} := \frac{1}{2}(1 \pm \gamma_5)$

$$\gamma_5 \phi_{n L,R} = \pm 1\, \phi_{n L,R}$$

$$i\not{D}\,\gamma_5 \phi_n = -\lambda_n\, \gamma_5 \phi_n$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}$$

and $\bar\Psi_D (i\not{D} - m)\Psi_D \overset{!}{=} (\bar\Psi_R, \bar\Psi_L) \begin{pmatrix} 0 & \overbrace{i\sigma^\mu \partial_\mu}^{\equiv i\not{D}_R} \\ \underbrace{i\bar\sigma^\mu \partial_\mu}_{\equiv i\not{D}_L} & 0 \end{pmatrix} \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix}$

$$= \bar\Psi_L\, i\not{D}_L \Psi_L + \bar\Psi_R\, i\not{D}_R \Psi_R - m(\bar\Psi_L \Psi_R + \bar\Psi_R \Psi_L)$$

$\Rightarrow$ $\bar\Psi_L \Psi_R$ and $\bar\Psi_R \Psi_L$ is Lorentz inv.

$\Rightarrow$ $i\not{D}_{L,R}\Psi_{L,R}$ transforms as $\Psi_{R,L}$

define $i\not{D}_{L,R} = i\not{D}\, P_{L,R}$, $S_{L,R} = \{\phi_{n L,R} = P_{L,R}\,\phi_n\}$
(Weyl op.) (space of L,R chiral fcts)

$\Rightarrow$ $i\not{D}_L : S_L \to S_R$

$i\not{D}_R : S_R \to S_L$

can also be seen from

$$\gamma^\mu P_{L,R} = \frac{1}{2}\gamma^\mu(1 \pm \gamma_5) = \frac{1}{2}(1 \mp \gamma_5)\gamma^\mu = P_{R,L}\,\gamma^\mu$$

adjoint op: $(i\not{D}_{L,R})^{\dagger} = i\not{D}_{R,L}$

$$\Rightarrow \boxed{\text{index}\,(i\not{D}_{L,R}) = n_{L,R} - n_{R,L}}$$

$n_{L,R}$ = # of zero modes of $i\not{D}_{L,R}$

<u>A-S index theorem</u> relates analytic index to topological index expressed in terms of characteristic classes.

$$\boxed{\text{index}\,(i\not{D}_L) = \int_{M_4} ch(F)}$$

chern character $ch(F) = \mathrm{tr}\, e^{\frac{i}{2\pi}F}$,

$$= \underbrace{r}_{\text{dim of group}} + \frac{i}{2\pi}\,\mathrm{tr}F + \frac{1}{2!}\left(\frac{i}{2\pi}\right)^2 \mathrm{tr}F^2 + \dots$$

$F = dA + A^2$

$$\Rightarrow \text{index}\,(i\not{D}_L) = \frac{1}{2}\left(\frac{i}{2\pi}\right)^2 \int_{M_4} \mathrm{tr}F^2$$

<u>next</u>: show that this is a topological quantity

we already showed

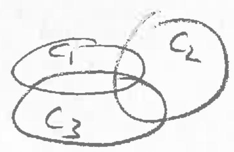
- $d\,\mathrm{tr} F^2 = 0$
- $\mathrm{tr} F^2 = dQ_3$, $Q_3 = \mathrm{tr}(AF - \frac{1}{3}A^3)$

now show: $\int \mathrm{tr} F^2$ invariant under deformations of A
i.e. topologically invariant

transformation laws

$$A \to A' = U^+ A U + U^+ dU \; , \quad F \to F' = U^+ F U$$

M_4 topologically non-trivial $\to$ cover M_4 by finite # of coordinate patches C_i



require $A_{(i=1,2,3,\dots)}$ to be related to $A_{(j)}$ on overlaps by a gauge transf.

$$A_{(i)} = U_{ij}^+ \,(A_{(j)} + d)\, U_{ij} \; , \quad F_{(i)} = U_{ij}^+ F_{(j)} U_{ij} \text{ on } U_i \cap U_j$$

↑ transition fct., encodes top. info. of gauge bdl.

<u>Def</u>: <u>characteristic class</u> P: is a local form on (compact) M_4 constructed from curvatures such that
$\int_M P$ is only sensitive to non-trivial transition fct.

Define $A_t = A_0 + t(A_1 - A_0)$, $F_t = dA_t + A_t^2$, $t \in [0$

where $A_{0,1}$ have same transition fct.

Then $\frac{\partial F}{\partial t} = d(A_1 - A_0) + (A_1 - A_0)A_0 + A_0(A_1 - A_0) + 2t(A_1 - A_0)^2$

$= d(A_1 - A_0) + (A_1 - A_0)A_t + A_t(A_1 - A_0)$

$= D_t(A_1 - A_0)$

$\frac{\partial}{\partial t} \operatorname{tr} F_t^2 = 2 \operatorname{tr}\left(\frac{\partial F_t}{\partial t} F_t\right) = 2 \operatorname{tr}\left(D_t(A_1 - A_0) F_t\right) = 2 D_t \operatorname{tr}\left((A_1 - A_0) F_t\right)$

$= 2 d \operatorname{tr}\left((A_1 - A_0) F_t\right)$

since $D_t F_t = dF_t - F_t A_t + A_t F_t$

$= dA_t A_t - A_t dA_t - dA_t A_t + A_t dA_t = 0$

Since A_1 and A_0 have same transition fcts we have

$A_{1(i)} - A_{0(i)} = u_{ij}^{+} (A_{1(j)} - A_{0(j)}) u_{ij}$

i.e. inhomogeneous term drops out!

$\Rightarrow \operatorname{tr}\left((A_1 - A_0) F_t\right)$ is globally defined

$\Rightarrow R := 2 \int_0^1 dt \operatorname{tr}(A_1 - A_0) F_t$ is globally defined

$\Rightarrow \int_{M_4} dR = 2 \int_{M_4} \int_0^1 dt \, d \operatorname{tr}(A_1 - A_0) F_t = \int_{M_4} \int_0^1 dt \, \partial_t \operatorname{tr} F_t^2$

$= \int_M \operatorname{tr} F_t \Big|_0^1 = \int_M \left(\operatorname{tr} F_1^2 - \operatorname{tr} F_0^2\right) = 0$ as R is globally defined

$\Rightarrow$ tr F^2 is only sensitive to non-trivial topology

Example $\int \mathrm{tr} F^2 =$ instanton winding # in instanton background

Relation with anomalies.

Last time: $$e^{i\Gamma[A]} = \int [D\psi][D\bar\psi]\, e^{i\int \bar\psi i \not{D} \psi} = c \det(i\not{D})$$

$$\delta\Gamma[A] = -i \int d^4x\, \alpha^a(x)\, \mathcal{A}^a(x) \quad , \quad \mathcal{A}^a \sim g^2 \epsilon^{\mu\nu\rho\sigma} \mathrm{tr}[t^a F_{\mu\nu} F_{\rho\sigma}$$

global $\overset{U(N)}{\text{anomaly}}$: $\alpha^a \neq \alpha^a(x)$

$U(N) \sim SU(N) \times U(1)$

$U(1)$ singlet anomaly : $\underset{\substack{\uparrow \\ \text{current of } U(1)}}{\partial_\mu j_5^\mu} \sim \alpha\, \epsilon^{\mu\nu\rho\sigma}\, \mathrm{tr} F_{\mu\nu} F_{\rho\sigma} \sim$

$\sim$ Chern character

next $\overset{\text{local}}{\vee}$ chiral anomaly

$$e^{i\Gamma[A_L]} = \int [D\bar\psi_L][D\psi_L]\, e^{i\int \bar\psi_L i\not{D}_L(A_L)\psi_L} \overset{?}{=} c \det(i\not{D}_L(A$$

ill defined

problem: no well-defined eigenvalue problem

as $i\not{D}_L$ maps $S_L \to S_R$ and $i\not{D}_L \phi_{nL} = \lambda_n \phi_{nL}$ is meaningless mathematical resolution: family index theorem

physical resolution [Alvarez-Gaumé, Ginsparg]:

Define $i\hat{\not{D}} = i\not{D}_L(A_L) + i\not{\partial}_R$

which acts on ψ_L and ψ_R → well defined eigenvalue problem

⇒ can redo Fujikawa and recover chiral anomaly [Alvarez-Gaumé, Ginsparg, Bertlmann]

"intuitive" relation with index theorem:

define 1-parameter family of gauge transf. $U = U(\theta)$

$$A^\theta = U^+(A + d)U(\theta)$$

extend to 2-parameter family

$$A^{t,\theta} = t A^\theta, \quad t \in [0,1]$$

(t, θ) coordinates of additional disc.

Define Dirac op $i\not{D}_6$ on $M_4 \times \text{disc}$ and show

$$\boxed{\mathrm{ind}(i\not{D}_6) \sim \int_{M_4 \times \text{disc}} \mathrm{tr}\, F^3(x,t,\theta)}$$

we already showed that this quantity yields the anomaly via descent eq. in L4

7 Gravitational Anomalies

Coupling QFT to gravity: $\eta^{\mu\nu} \to g^{\mu\nu}(x)$

e.g. kinetic term for scalar field $\phi(x)$

$$\eta^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi \to g^{\mu\nu}(x) \, \partial_\mu \phi \, \partial_\nu \phi \quad , \quad \mu, \nu = 0, \dots 3$$

Lorentz transformations $\to$ general coordinate transformations

$$\eta_{\mu\nu} \to \eta'_{\mu\nu} = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma \eta_{\rho\sigma} = \eta_{\mu\nu} \quad , \quad \Lambda: \text{const. matrix}$$

$$g_{\mu\nu} \to g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x) \quad \text{for } x^\mu = x^\mu(x')$$

coupling fermions requires vierbein-formalism

$$g_{\mu\nu}(x) = e_\mu{}^a(x) \, e_\nu{}^b(x) \, \eta_{ab} \quad , \quad a, b = 0, \dots, 3$$

with transformations

$$e_\mu{}^a \to e_\mu{}^{a\prime}(x') = \frac{\partial x^\rho}{\partial x'^\mu} e_\rho{}^a \qquad \text{general coordinate transformation}$$

$$e_\mu{}^a \to e_\mu{}^{a\prime}(x') = e_\mu{}^b \Lambda_b{}^a(x) \qquad \underline{\text{local}} \text{ Lorentz transf. (additional redundancy of formalism, leaves } g_{\mu\nu} \text{ invariant)}$$

reason for introducing $e_\mu{}^a$:

fermions transform in spinor rep. of Lorentz group $SO(1,3)$ which do not exist for general coordinate transf.

kinetic term for Dirac fermion

$$\bar{\Psi}\, i\gamma^\mu \partial_\mu \Psi \to \bar{\Psi}\, i\gamma^a e_a^{\ \mu}(x)\, \partial_\mu \Psi \quad , \quad \{\gamma^a, \gamma^b\} = 2\eta^{ab}$$

transformation of Ψ:

$$\delta\Psi = i\, \lambda_{[ab]}(x)\, \Sigma^{ab}\, \Psi \quad , \quad \Sigma^{ab} := \tfrac{i}{4}[\gamma^a, \gamma^b]$$

6 parameters of <u>local</u> Lorentz trasf.

generators of $SO(1,3)$ in Dirac rep

$\Rightarrow$ also need connection $\omega_{\mu ab}(x)$ in fermionic $\mathcal{L}$

$$\bar{\Psi}\, i\not{D}\, \Psi = \bar{\Psi}\, i\gamma^a e_a^{\ \mu} \left(\partial_\mu - i g A_\mu^a t^a + \tfrac{1}{4}\omega_{\mu[ab]}\Sigma^{ab}\right)\Psi$$

$$\left(e_a^{\ \mu} e_\mu^{\ b} = \delta_a^b \quad , \quad e_\mu^{\ a} e_a^{\ \nu} = \delta_\mu^\nu \right)$$

with transformations

$$\omega_{\mu ab} \to \omega'_{\mu ab} = \frac{\partial x^\rho}{\partial x'^\mu}\, \omega_{\rho ab} \qquad \text{general coord. trasf.}$$

$$\omega_{\mu\ b}^{\ a} \to \omega'^{\ a}_{\mu\ b} = \Lambda_b^{\ c}\left(\omega_{\mu\ c}^{\ d} + \partial_\mu \delta_c^d\right)\Lambda_d^{\ a} \qquad \text{local Lorentz trasf.}$$

ω_μ is related to Christoffel symbol by

$$\partial_\nu e_\mu^{\ a} + \omega_{\nu\ b}^{\ a}\, e_\mu^{\ b} - e_\rho^{\ a}\, \Gamma^\rho_{\nu\mu} = 0$$

conservation laws

$$\mathcal{L} = \sqrt{g}\left(\bar{\Psi}\, i \not{D} \Psi + \frac{1}{2} R(g)\right)$$

↑ curvature scalar

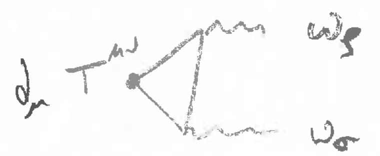
$$= \bar{\Psi}\, i \not{\partial} \Psi + g_{\mu\nu} T^{\mu\nu}(\Psi) + O(g^2)$$

↑ energy momentum tensor of Ψ

$$T^{\mu\nu} \approx i\bar{\Psi}\gamma^\mu D^\nu \Psi + i\Psi\gamma^\mu D^\nu \bar{\Psi} + \mu \leftrightarrow \nu$$

general coordinate invariance $\Rightarrow$ $D_\mu T^{\mu\nu} = 0$

Q: can there be a gravitational anomaly induced via



computation as for gauge anomaly with gauge group $G = SO(1,3)$ (or $SO(4)$ in Euclidean spacetime) (Lorentz group)

$\Rightarrow$ no anomaly as Weyl rep is a real rep. of $SO(1,3)$

real rep: $\bar{r} \simeq r$ $\sim$ $t^a_{\bar{r}} = S\, t^a_{r}\, S^{-1}$ $\forall a$

$$\delta\psi^i = i\alpha^a\, t^a_{r}{}^i{}_j\, \psi^j \qquad t^a = (t^a)^\dagger$$

$$\delta\bar{\Psi}^i = -i\alpha^a\, \bar{\Psi}^j \left(t^a_{r}{}_j{}^i\right)^\dagger \stackrel{\downarrow}{=} -i\alpha^a\, t^a_{r}{}^T{}^i{}_j\, \bar{\Psi}^j$$

$$\Rightarrow \quad t^a_{\bar{r}} = -\left(t^a_{r}\right)^T$$

anomaly coefficient

$$d^{abc}(\underset{\sim}{r}) = \mathrm{tr}_{\underset{\sim}{r}}\left(t^a_{\underset{\sim}{r}} \{ t^b_{\underset{\sim}{r}}, t^c_{\underset{\sim}{r}} \} \right)$$

$$d^{abc}(\bar{\underset{\sim}{r}}) = \mathrm{tr}\left((-t^a_{\underset{\sim}{r}})^T \{ (-t^b_{\underset{\sim}{r}})^T, (-t^c_{\underset{\sim}{r}})^T \} \right)$$

$$= -\mathrm{tr}\left(\left(\{ t^c_{\underset{\sim}{r}}, t^b_{\underset{\sim}{r}} \} t^a_{\underset{\sim}{r}} \right)^T \right)$$

$$= -\mathrm{tr}\left(\{ t^c_{\underset{\sim}{r}}, t^b_{\underset{\sim}{r}} \}, t^a_{\underset{\sim}{r}} \right) = - d^{abc}(\underset{\sim}{r})$$

if $\bar{\underset{\sim}{r}} \simeq \underset{\sim}{r} \Rightarrow d^{abc} = 0$

another way to see that Weyl rep is real:

$SO(4) \simeq SU(2) \times SU(2)$ $\qquad$ $\left(SO(1,3) \simeq SL(2) \times SL(2) \right)$

Weyl spin: $(\underset{\sim}{2}, 0)$ or $(0, \underset{\sim}{2})$

$\underset{\sim}{2}$ of $SU(2)$ is real! $\quad \underset{\sim}{2} \simeq \bar{\underset{\sim}{2}}$

For general $SO(d)$ or $SO(d-1,1)$

d odd: no Weyl rep $\rightarrow$ real rep $\rightarrow$ no anomaly

$d = 4k$; Weyl rep but real $\rightarrow$ no anomaly

$d = 2+4k$; ($k = 1,2,\ldots$) Weyl rep complex $\rightarrow$ anomaly in $d = 2, 6, 10, \ldots$ exists

Relation with index and descent eq.

In analogy with gauge theories define curvature two-form

Riemann Tensor: $R_{\mu\nu\rho}{}^{\sigma} = \partial_\mu \Gamma_{\nu\rho}^{\sigma} + \Gamma_{\mu\lambda}^{\sigma} \Gamma_{\nu\rho}^{\lambda} - (\nu \leftrightarrow \mu)$

curvature 2-form: $R_\rho{}^\sigma = R_{\mu\nu\rho}{}^\sigma dx^\mu \wedge dx^\nu = d\Gamma + \Gamma^2$

with spin-connection: $R_a{}^b = R_{\mu\nu a}{}^b dx^\mu dx^\nu$

$$= \tfrac{1}{2}\left(\partial_\mu \omega_{\nu a}{}^b + \omega_{\mu a}{}^c \omega_{\nu c}{}^b\right) dx^\mu \wedge dx^\nu$$

$$= d\omega + \omega^2$$

One has $\quad R_a{}^b = e_\mu^b \, e_a^\nu \, R_\nu{}^\mu$

AS-index theorem for curved manifolds M_{2n+2}

$$\text{index}(i \not{D}_2) = \int_{M_{2n+2}} I_{2n+2} \;, \quad I_{2n+2} = (-1)^{n+1} \left[\hat{A}(M)\, ch(F)\right]_{2n+2}$$

Chern character: $\quad ch(F) = \mathrm{tr}\, e^{\frac{i}{2\pi} F} = r + \frac{i}{2\pi} \mathrm{tr} F + \frac{1}{2}\left(\frac{i}{2\pi}\right)^2 \mathrm{tr} F^2 + \dots$

$\hat{A}$-roof-genus: $\quad \hat{A}(M) = \det\left(\dfrac{\frac{1}{2}\frac{1}{2\pi}R}{\sinh\left(\frac{1}{2}\frac{1}{2\pi}R\right)}\right)^{\frac{1}{2}} = \prod_{j=1}^{n} \dfrac{\frac{1}{2}X_j}{\sinh\left(\frac{X_j}{2}\right)}$

for $\quad \frac{1}{2\pi} R_{ab} = \begin{pmatrix} 0 & X_1 & & & \\ -X_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & X_n \\ & & & -X_n & 0 \end{pmatrix}$

I is characteristic class and satisfies descent eq.

$$d I_{2n+2} = 0 \Rightarrow I_{2n+2} = d Q_{2n+1}$$

$$s I_{2n+2} = 0 \Rightarrow 0 = s d Q_{2n+1} = -d s Q_{2n+1} \Rightarrow s Q_{2n+1} = d Q^{1}_{2n}$$

ghost #

$$\Rightarrow s d Q^1_{2n} = 0 \Rightarrow s Q^1_{2n} = d Q^1_{2n-1}$$

$\Rightarrow Q^1_{2n}$ satisfies WZ condition. and is candidate for anomaly

(s = BRST operator for $SO(1,5)$)

expand
$$\hat{A}(M) = 1 + \frac{1}{(4\pi)} \frac{1}{12} \operatorname{tr} R^2 + \frac{1}{(4\pi)^4} \left(\frac{1}{360} \operatorname{tr} R^4 + \frac{1}{288} (\operatorname{tr} R^2)^2 \right) + O(R^6)$$

$$\Rightarrow \hat{A}(M)\big|_{6\text{-form}} = 0 \Rightarrow \text{no gravitational anomaly!}$$

(only 4, 8, 12, ... – forms)

mixed anomalies

In four space-time dimensions mixed anomalies do exist

$$\Rightarrow \quad \partial_\mu j^\mu \sim a\, \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu a}{}^{b} R_{\rho\sigma b}{}^{a}$$

$$\sim a \operatorname{tr} R \wedge R$$

Remarks:

- j^μ can only be $U(1)$ current
- $a \sim \sum_f q_f \quad , \quad q_f = U(1)$ charge of fermion f
- $a = 0$ in Standard Model
- It can be obtained from I_6 by expanding

$$I_6 \sim \left(1 + \frac{1}{4\pi}\frac{1}{12} \operatorname{tr} R^2\right)\left(r + \frac{i}{2\pi} \operatorname{tr} F + \frac{1}{2}\left(\frac{i}{2\pi}\right)^2 \operatorname{tr} F^2 + \frac{1}{3}\left(\frac{i}{2\pi}\right)^3 \operatorname{tr} F^3 \ldots\right)$$

$$\supset \frac{1}{3}\frac{i}{(2\pi)} \operatorname{tr} F^3 + \frac{i}{8\pi^2}\frac{1}{12} \operatorname{tr} R^2 \operatorname{tr} F$$

$\uparrow$ gauge anomaly $\qquad\qquad$ $\uparrow$ mixed anomaly

L8. Supersymmetric derivation of index theorem

[Witten 82, Alvarez-Gaumé 8?, Alvarez-Gaumé, Witten 8?]

Recall Fujikawa method:

$$e^{i\Gamma[A]} := \int [D\psi][D\bar\psi]\, e^{-\overset{\text{Euclidean}}{S(\psi,\bar\psi,A)}} \underset{\uparrow}{=} c \cdot \det\left(i\not{D}(A)\right)$$

$$S = \int d^4x\, \bar\psi\, i\not{D}\psi$$

anomaly appears from transf. of measure

$$[D\psi][D\bar\psi] \to e^{i\int d^4x\, \alpha^a(x)\, \mathcal{A}^a(x)}\, [D\psi][D\bar\psi]$$

$$\mathcal{A}^a(x) \sim \sum_n \phi_n^+ \gamma_5 t^a \phi_n \qquad , \quad i\not{D}\phi_n = \lambda_n \phi_n$$

$$= \text{"tr}\, \gamma_5\text{"}$$

$$\overset{\text{regulat.}}{=} \lim_{M\to\infty} \sum_n \phi_n^+ \gamma^5 t^a e^{-\frac{(i\not{D})^2}{M^2}} \phi_n$$

$$\overset{M=\beta^{-1}}{=} \lim_{\beta\to 0} \sum_n \phi_n^+ \gamma^5 t^a e^{-\beta (i\not{D})^2} \phi_n$$

$$= \dots = \epsilon^{\mu\nu\rho\sigma}\, \mathrm{tr}\, t^a F_{\mu\nu} F_{\rho\sigma}$$

method cumbersome on curved manifolds

idea instead: find quantum mechanical system with $H = (i\not{D})^2$ and evaluate $\mathcal{A} \sim \mathrm{tr}\left(\gamma_5 e^{-\beta H}\right)$ as partition fct of qm system

Supersymmetric quantum mechanics

introduce N supercharges Q^I, $I = 1, \dots, N$

with superalgebra
$$\{Q^I, Q^{*J}\} = 2\delta^{IJ} H$$
$$\{Q^I, Q^J\} = 0 = \{Q^{*I}, Q^{*J}\}$$

Define fermion number operator $(-1)^F$

by $(-1)^F Q^I = -Q^I (-1)^F \Leftrightarrow \{(-1)^F, Q^I\} = 0$

Focus on $N=1$ and define $S = \frac{1}{\sqrt{2}}(Q + Q^*)$
$$S^2 = \frac{1}{2}(Q+Q^*)^2 = \frac{1}{2}(QQ^* + Q^*Q) = H \quad \Rightarrow \quad [H, S] = 0$$

Let $H|E\rangle = E|E\rangle$

$$H\, S|E\rangle = S^3|E\rangle = S\, H|E\rangle = E\, S|E\rangle$$

$\to$ $|E\rangle$ and $S|E\rangle$ have same energy but opposite fermion number

$S|b\rangle = \sqrt{E}|f\rangle$, $S|f\rangle = \sqrt{E}|b\rangle$

$\to$ all states with $E \neq 0$ appear in Fermi-Bose pairs!

$\to$ $|E\rangle$ and $S|E\rangle$ are $\hat{}$ rep. of superalgebra
(or $|b\rangle, |f\rangle$) 2 dim.

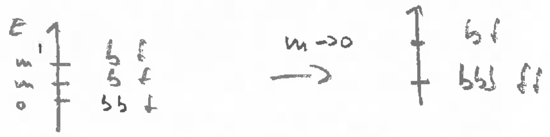
$\Rightarrow$ For $E = 0$ we have $H|0\rangle = 0$ and $S|0\rangle = 0$
(due to $H = S^2$)

$|0\rangle$ is 1-dim rep of superalgebra

$\Rightarrow$ For $E=0$ there is not necessarily a balance between fermions and bosons

$$\Rightarrow \quad \mathrm{Tr}\,(-1)^F = n_B^{E=0} - n_F^{E=0}$$

$n_{B/F}^{E=0}$ = number of bosonic / fermionic zero energy states



$\Rightarrow$ $\mathrm{Tr}(-1)^F$ unchanged as only b-f pairs can become add. zero modes

$\Rightarrow$ $\mathrm{Tr}\,(-1)^F$ is an index called <u>Witten index</u>

as it stands it is ill-defined and one needs to regulate the infinite sum

$$\mathrm{Tr}\,(-1)^F = \lim_{\beta\to 0} \underbrace{\mathrm{Tr}\left((-1)^F e^{-\beta H}\right)}_{\text{independent of } \beta \text{ as we already showed.}} \quad , \ \beta > 0, \text{ otherwise ab.}$$

Supersymmetric σ-model

4-dim σ-model $\mathcal{L} = \frac{1}{2} g_{ij}(\phi)\, \partial_\mu \phi^i_{(x)} \partial^\mu \phi^j_{(x)}$, ϕ^i: n scalar fields

1-dim. σ-model : $\mathcal{L} = \frac{1}{2} g_{ij}(\phi)\, \dot{\phi}^i_{(t)} \dot{\phi}^j_{(t)}$ $i = 1, \dots n$

$$\phi^i : \mathbb{R} \to M_n \leftarrow \text{target space}$$

g_{ij} : Riemannian metric on M

Supersymmetric σ-model : need to introduce fermion fields ψ^i

$$\mathcal{L} = \frac{1}{2} g_{ij}(\phi) \dot{\phi}^i \dot{\phi}^j + \frac{1}{2} g_{ij}\, \bar{\psi}^i\, i D_t \psi^j$$

$$D_t \psi^j = \frac{d\psi^j}{dt} + \Gamma^j_{ek} \dot{\phi}^e \psi^k$$

↑ Christoffel of g

For $\psi^a := e^a_i(\phi)\psi^i$, $\quad e^a_i e^b_j \eta_{ab} = g_{ij}$

one obtains

$$\mathcal{L} = \frac{1}{2} g_{ij} \dot{\phi}^i \dot{\phi}^j + \frac{1}{2} \delta_{ab} \psi^a (i D_t) \psi^b$$

$$D_t \psi^b = \frac{d\psi^b}{dt} + \omega_{e\ a}^{\ b} \dot{\phi}^e \psi^a$$

$$\omega_{e\ c}^{\ b} = -e^k_c \left(\partial_e e^b_k - \Gamma^m_{ek} e^b_m \right) \quad \text{spin connection}$$

supersymmetr. transf.

$$\delta\phi^i = \epsilon\psi^i \quad , \quad \delta\psi^i = i\epsilon\dot\phi^i - \epsilon\Gamma^i_{jk}\psi^j\psi^k$$

transforms $\mathcal{L}$ into total divergence

canonical quantization for: $\psi^a = e^a_i(\phi)\psi^i$

$$\{\psi^a, \psi^b\} = \delta^{ab} \quad , \quad [\phi^i_{(t)}, \pi^j_{(t')}] \sim \delta(t-t')\delta^{ij}$$

Clifford algebra on M $\Rightarrow$ $\psi^a \simeq \gamma^a$

$(-1)^F = \gamma_5$

compact H :

$$H = \dots = \tfrac{1}{2}(i\gamma^a D_a)^2 \;, \quad D_a = e_a{}^i(\partial_i + \tfrac{1}{2}\omega_{iab}\Sigma^{ab})$$

$$\Sigma^{ab} = \tfrac{1}{4}[\gamma^a, \gamma^b]$$

Dirac op. on M!

$\Rightarrow A \sim \lim_{\beta\to 0} \operatorname{tr} \gamma_5 e^{-\beta(iD)^2} = \lim_{\beta\to 0} \operatorname{tr}(-1)^F e^{-\beta H}$ = Witten index $= n_B^{E=0} - n_F^{E=0}$

compact classical eq. o. m. :

$$-g_{ij}\ddot\phi^j + \tfrac{1}{2}R_{jkil}\psi^j\psi^k\dot\phi^l = 0$$

$$D_t\psi^i = \dot\psi^i + \Gamma^i_{lk}\dot\phi^l\psi^k = 0$$

solved by
$$\left.\begin{aligned}\phi^i &= \phi^i_0 = \text{const}\\ \psi^i &= \psi^i_0 = \text{const.}\end{aligned}\right\} \Rightarrow \mathcal{L}(\phi_0, \psi^0) = 0$$

compute $\lim_{\beta\to 0} \operatorname{tr} (-1)^F e^{\beta H}$

Cecotti, Girardello : $\operatorname{tr} (-1)^F e^{\beta H} = \operatorname{Str} e^{\beta H} = \int_{\substack{\text{box} \\ \text{with} \\ \text{periodic b.c.}}} [\mathcal{D}\phi][\mathcal{D}\psi]\, e^{-S(\phi,\psi)}$ $\qquad S = \int_0^\beta \mathcal{L}(t)\,dt$

rescale time $t \to \beta t$ $\to$ $\mathcal{L} = \frac{1}{2\beta} g_{ij} \dot\phi^i \dot\phi^j + \frac{1}{2} g_{ij} \bar\psi^i (i\mathcal{D}_t) \psi^j$

$\Rightarrow$ for $\beta \to 0$ $\phi, \psi = \text{const}$ dominate PI !

expand $\mathcal{L}$ around x_0, ψ_0 using normal coordinates

$$\phi^i = \phi_0^i + \xi^i(\tau) \quad , \quad \psi^i = \psi_0^i + \eta^i(\tau)$$

such that $\quad g_{ij}(\phi) = \underbrace{g_{ij}(\phi_0)}_{\text{choose } = \delta_{ij}} + R_{ijkl}\,\xi^k \xi^l$

$\Updownarrow$ $\mathcal{L} = \frac{1}{2}\delta_{ij}\dot\xi^i\dot\xi^j + \frac{1}{2}\eta^a\dot\eta^a - \frac{1}{4} R_{ijab}\,\psi_0^a\psi_0^b\,\xi^i\dot\xi^j$

P.I. over ξ gives $\left(\det\left(-\delta_{ij}\frac{d^2}{dt^2} + R_{ijab}\psi_0^a\psi_0^b\frac{d}{dt}\right)\right)^{-\frac{1}{2}}$

" " η gives $\left(\det\left(-\delta_{ab}\frac{d}{dt}\right)\right)^{\frac{1}{2}}$

$$\Rightarrow \int [\mathcal{D}\phi][\mathcal{D}\psi]\, e^{-S} = \int d\phi_0\, d\psi_0 \cdot \frac{\det\left(-\delta_{ab}\frac{d}{dt}\right)^{1/2}}{\det\left(-\delta_{ij}\frac{d^2}{dt^2} + \dots\right)^{1/2}}$$

$$= \int d\phi_0\, d\psi_0 \left[\det\left(-\delta_{ij}\frac{d}{dt} + R_{ijab}\psi_0^a\psi_0^b\right)\right]^{-1/2}$$

$\underbrace{R_{ij}\,\text{as}\,\psi_0^q\,\psi_0^s}_{\equiv R_{ij}}$ is antisymmetric in $i \leftrightarrow j$

$$\to \quad R_{ij} = \begin{pmatrix} 0 & x_1 & & \\ -x_1 & 0 & & \\ & & \ddots & \\ & & & \boxed{\begin{matrix} 0 & x_{n/2} \\ -x_{n/2} & 0 \end{matrix}} \end{pmatrix}$$ ($n\times n$ matrix)

$$\to \left[\det\left(-\delta_{ij}\frac{d}{dt} + R_{ij}\right)\right]^{-\frac{1}{2}} = \prod_{\alpha=1}^{n/2} \left[\underbrace{\det\left(+\frac{d^2}{dt^2} + x_\alpha^2\right)}_{\text{partition fct. of harmonic osc.}}\right]^{-1/2}$$

Compute det. by using

$$\sqrt{\det(M)} = \int D\phi\, e^{-\frac{1}{2}\int \phi M \phi}$$

in Fourier space $\quad \phi = \sum_{k=-\infty}^{+\infty} \phi_k e^{-2\pi k t} \quad$ (P.B.C.)

$$\Rightarrow \det\left(\frac{d^2}{dt^2} + x_\alpha^2\right) = \prod_{k\neq 0}\left((2\pi k)^2 - x_\alpha^2\right) = \left(\prod_{k\geq 1} (2\pi k)^2\right)\left(\prod_{k\geq 1}\left(1+\frac{x_\alpha}{2\pi k}\right)^2\right),$$

$$= \left(\frac{\sinh \frac{1}{2}x_\alpha}{\frac{1}{2}x_\alpha}\right)^2$$

$$\Rightarrow \int (d\phi)(d\psi)\, e^{-S} = \int_M d\mathrm{Vol}(M)\, d\psi_0 \underbrace{\prod_{\alpha=1}^{n/2} \frac{\frac{1}{2}x_\alpha}{\sinh\frac{1}{2}x_\alpha}}_{\hat{A}(M)}$$

ψ_0 int: takes top form $\quad = \int_M d\mathrm{Vol}(M)\, \hat{A}(M)\Big|_{\text{top-form}}$ ∎

L9 Anomalies in higher-dimensional space-times

Recall gravitational anomaly only in space-time dimensions n where spinors are in complex rep of $SO(1, d-1)$

$\Rightarrow d = 2+4k = 2, 6, 10$
$= 2n$ for d even

$\mathcal{A} \sim Q^1_{2n}$ constructed via descent eq.

$$I_{2n+2} = d\,Q_{2n+1}$$
$$\delta Q_{2n+1} = d\,Q^1_{2n}$$

$$I_{2n+2} \sim \left(\hat{A}(M)\, ch(F)\right)\Big|_{2n+2}$$

$$ch(F) = \mathrm{tr}\, e^{\frac{i}{2\pi}F} = r + \frac{i}{2\pi}\mathrm{tr}F + \frac{1}{2}\left(\frac{i}{2\pi}\right)^2 \mathrm{tr}F^2 + \dots$$

$$\hat{A}(M) = \prod_{\alpha=1}^{n+1} \frac{\frac{1}{2}x_\alpha}{\sinh\left(\frac{x_\alpha}{2}\right)} = 1 + \frac{1}{4\pi}\,\frac{1}{12}\,\mathrm{tr}R^2 +$$

So far concentrated only on $s=\frac{1}{2}$ fields contributions to anomalies.

Other fields which can contribute: [Alvarez-Gaumé, Witten '83]

- $s=\frac{3}{2}$: gravitino (supersymmetric partner of graviton)

$$I_{2n+2}(s=\tfrac{3}{2}) = \left(\hat{A}(M)\left(\mathrm{tr}\, e^{\frac{i}{2\pi}R} - 1\right)\right)_{2n+2}$$

- antisymmetric tensor fields $C_{[\mu_1 \cdots \mu_p]}$, $\mu_i = 0,\dots,n-1$

($\hat{=}$ p-forms)

if field-strength obeys a self-duality condition

eg: $d=6$, $n=3$ $\quad C_{\mu\nu}$, field strength $\quad H_{\mu\nu\rho} = \partial_{[\mu} C_{\nu\rho]}$

(C_2) $\qquad (H_3 = dC_2)$

self-duality condition (anti)

$$\epsilon^{\mu_1\cdots\mu_6} H_{\mu_1\cdots\mu_3} = \pm H^{\mu_4\cdots\mu_6}$$

$$* H_3 = \pm H_3$$

$d=10$, $n=5$ $\quad C_4$, H_5, $\quad *H_5 = \pm H_5$

$I_{2n+2}(C) \sim L(M)$ (Hirzebruch Polynomial)

$$= 1 - \frac{1}{(4\pi)^2}\frac{1}{6}\,\mathrm{tr}\,R^2 + \frac{1}{(4\pi)^4}\left(-\frac{7}{180}\,\mathrm{tr}\,R^4 + \frac{1}{72}(\mathrm{tr}\,R^2)^2\right)$$

$$+ \cdots$$

(A-6, v) classical anomalies free theories in $d = 6, 10$

Define $$\det\left(1 - \frac{R}{2\pi}\right) = \sum_{k=0}^{\infty} \frac{P_k}{(2\pi)^{2k}}$$

$$P_0 = 1, \quad P_1 = \sum_\alpha x_\alpha^2, \quad P_2 = \sum_{\alpha<\beta} x_\alpha^2 x_\beta^2, \ldots$$

$d = 6$:

$$I_8(s=\tfrac{1}{2}) = \frac{1}{5760}\left(7P_1^2 - 4P_2\right)$$

$$I_8(s=\tfrac{3}{2}) = \frac{1}{5760}\left(275P_1^2 - 980P_2\right)$$

$$I_8(C) = \frac{1}{5760}\left(16P_1^2 - 112P_2\right)$$

$$I_{\text{total}} = n_{1/2}\, I_8(s=\tfrac{1}{2}) + n_{3/2}\, I_8(s=\tfrac{3}{2}) + n_c\, I(C)$$

there always is a solution (as 3 I's depending on 2 P's)

$$I_{\text{total}} = 0$$

simplest $\quad 21\, I(s=\tfrac{1}{2}) - I(s=\tfrac{3}{2}) + 8\, I(C) = 0$

$\Rightarrow$ field content of a specific $d=6$ super gravity $(2,0)$

$d = 10$:

$$I_{12}(s=\tfrac{1}{2}) = \frac{1}{967680}\left(-31P_1^3 + 44P_1P_2 - 16P_3\right)$$

$$I_{12}(s=\tfrac{3}{2}) = \frac{1}{967680}\left(225P_1^3 - 1620P_1P_2 + 7920P_3\right)$$

$$I_{12}(C) = \frac{1}{967680}\left(-256P_1^3 + 1664P_1P_2 - 7936P_3\right)$$

3 I's depending on 3 P's $\Rightarrow$ no solution expected

(as coeff have to be integer)

However: $-I(s=\frac{3}{2}) + I(s=\frac{1}{2}) + I(C) = 0$

$\hat{=}$ field content of Type IIA supergravity

- unique solution for chiral theory
- non-chiral theories (e.g. Type IIA) are anomaly free

$d > 10$: no anomaly free theories

way out: <u>Green - Schwarz mechanism</u> [G.S '84]

basic idea: <u>assign</u> non-trivial transformation to antisymmetric tensor and add appropriate local counterterm.

main example: two-form B_2, $B_2 = \frac{1}{2} B_{\mu\nu}\, dx^\mu dx^\nu$

standard field strength: $H_3 = dB_2$

modify H_3 as follows: $H_3 := dB_2 - Q_3(A,F) + Q_3(\omega,R)$

C-S term, gauge field, field strength, spin connect., curvature 2-form

$dQ_3(A,F) = \mathrm{tr}\, F\wedge F$, $dQ_3(\omega,R) = \mathrm{tr}\, R\wedge R$

gauge transf. of Q_3: $\delta Q_3 = dQ_2'$

H_3 is invariant for $\delta B_2 = Q_2'(A,F) - Q_2'(\omega,R)$

add to action local counter-term

$$\Delta\Gamma = \int B_2 \wedge X_{d-2}(F, R) \quad (\text{Green-Schwarz term})$$

with

$$\delta X_{d-2} = 0$$

$$X_{d-2} = dX_{d-3}$$

$$\delta X_{d-3} = d X'_{d-4}$$

$$\delta \Delta\Gamma = \int \delta B_2 \wedge X_{d-2} = \int \left(Q_2'(A,F) - Q_2'(\omega, R) \right) \wedge d X_{d-3}$$

$$= - \int d\left(Q_2'(A,F) - Q_2'(\omega,R) \right) \wedge X_{d-3}$$

$$= - \int \underbrace{\left(\delta Q_3(A) - \delta Q_3(\omega) \right)}_{= d Q_2'} \wedge X_{d-3}$$

can be obtained from $\Delta I_{d+2} = \left(\operatorname{tr} R^2 - \operatorname{tr} F^2 \right) \wedge X_{d-2}$

$$= d\left(Q_3(\omega) - Q_3(A) \right) \wedge X_{d-2}$$

$$= d\, Q_{d-1}$$

$$\delta Q_{d-1} = \left(\delta Q_3(\omega) - \delta Q_3(A) \right) \wedge X_{d-2} = \delta\left(Q_3(\omega) - Q_3(A) \right) \wedge d X_{d-3}$$

$$= d\left(\left(\delta Q_3(\omega) - \delta Q_3(A) \right) \wedge X_{d-3} \right) = d\, Q'_{d-2} = d\left(\delta \Delta\Gamma \right) \checkmark$$

$n=10$

$\Rightarrow$ If $I_{12} \sim \Delta I_{12}$ anomaly can be cancelled

$\Rightarrow$ I_{12} has to factorize !

$\Rightarrow$ terms $\mathrm{tr}\, R^6$, $\mathrm{tr}\, F^6$ are disallowed

can be cancelled by choosing σ

cannot be cancelled but $\mathrm{tr}\, F^6$ or $\mathrm{tr}\, F^2 \mathrm{tr}\, F^4$, $(\mathrm{tr}\, F^2)^3$ could appear

<u>Solution</u> : $N=1$ supergravities with

$$G = SO(32) \text{ or } E_8 \times E_8$$

(precisely for these cases string theories exist)

$$\{ Q^I, \bar{Q}^J \} \sim \not{P}\, \delta^{IJ} \quad , \quad I = 1, \dots, N$$

type II : $N=2$ IIA $Q, \bar{Q}$ have opposite chiralities
IIB $Q, \bar{Q}$ " same "

type I : $N=1$

field content : $(g_{\mu\nu}, B_{\mu\nu}, \phi, \psi_\mu, \chi)$

(A_μ^a, λ^a)

L10 Anomalies and spontaneously broken global symmetries

Recall QCD-like non-Abelian gauge theories (Yang-Mills theories) with gauge group G (e.g. $G = SU(N)$)

$$\mathcal{L} = \bar{\Psi}^i (i\not{D} - m)_{ij} \psi^j - \frac{1}{4C(r)} \mathrm{tr}_r F_{\mu\nu} F^{\mu\nu}$$

$i, j = 1, \dots, \dim(r)$, r : rep of G

$$\not{D}\psi = \gamma^\mu \partial_\mu \psi_i - g \underbrace{A_\mu^a}_{\text{gluons in QCD}} t^a{}_{ij} \psi^j$$

g: gauge coupling constant

Theory can have global flavour symmetry G_f

$$\mathcal{L}_q = \bar{\Psi}^{iI} (i\not{D} - m)_{ij} \psi^{jI} \quad , \quad I = 1, \dots, n_f$$

e.g. $G_f = U(2)$ for $\bar{\Psi}^{iI} = \begin{pmatrix} u^i \\ d^i \end{pmatrix}$ ← up-quark ← down-quark

(if we ignore weak interaction)

for $m = 0$: $G_f = U(2)_L \times U(2)_R$ (chiral flavour sym.)

recall $\psi_{Dirac} = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$, here $\psi_L^{iI} = \begin{pmatrix} u_L^i \\ d_L^i \end{pmatrix}$, $\psi_R^{iI} = \begin{pmatrix} u_R \\ d_R \end{pmatrix}$

Strongly coupled gauge theories : $g = O(1)$

$\rightarrow$ no perturbation methods available

QCD: asymptotically free gauge theory

$\Rightarrow$ weakly coupled at high energies (UV)

strongly " at low " (IR)

reason: gauge coupling is energy dependent

$$\mu \frac{d}{d\mu} g(\mu) = \beta(g)$$

$$\beta(g) = -\frac{b_0 g^3}{(4\pi)^2} + O(g^5)$$

$$b_0 = \frac{11}{3} C_2(ad) - \frac{4}{3} n_f(\underset{\sim}{r})\, C(\underset{\sim}{r})$$

$$\mathrm{Tr}\left(t^a_{\underset{\sim}{r}}\, t^b_{\underset{\sim}{r}}\right) = C(\underset{\sim}{r})\, \delta^{ab}$$, e.g. $\underset{\sim}{n}$ of $SU(n)$

$C(\underset{\sim}{n}) = \frac{1}{2}$

$$\left(t^a_{\underset{\sim}{r}} t^a_{\underset{\sim}{r}}\right)_{ij} = C_2(\underset{\sim}{r})\, \delta_{ij}$$ e.g. $C_2(ad) = n$

$\rightarrow b_0(SU(n)) = \frac{11n}{3} - \frac{2}{3} n_f$

for $b_0 > 0$ } g increases in IR

$\beta < 0$ } $\rightarrow$ asymptotically free gauge theory leading order solution

$$\frac{1}{\alpha(\mu)} \equiv \frac{4\pi}{g^2(\mu)} = \frac{1}{4\pi} b_0 \ln\left(\frac{\mu^2}{\Lambda^2}\right)$$

cutoff

compare with measured value can estimate

scale Λ where $g = \mathcal{O}(1)$

$\Rightarrow \Lambda \sim 250$ MeV

compare quark / meson masses

u	2 MeV
d	5 "
s	100 "
c	1200 "
$\pi^\pm, \pi^0$	140 "
p, n	1000

$\Rightarrow m_\pi \ll m_{p,n}$

In the IR gauge theory becomes strong, and gauge neutral bound states (condensates) form:

Mesons $M^{IJ} = \langle \bar{\psi}^I_i \psi^J_i \rangle$ - (eg π)

Baryons $B^{I_1 \dots I_N} = \langle \epsilon^{i_1 \dots i_N} \psi^{I_1}_{i_1} \dots \psi^{I_N}_{i_N} \rangle$ (eg p, n)

$$B^{IJK} \overset{N=3}{=} \epsilon^{ijk} \psi^I_i \psi^J_j \psi^K_k$$

$\Rightarrow$ chiral flavour symmetry dynamically broken $U(N)_L \times U(N)_R \to U(N)_V$

Goldstone Theorem: Every spont. broken global symmetry has a massless scalar field (Goldstone boson)

Spont. symm. breaking: $\delta \mathcal{L} = 0$

but $\delta V|_{min} \neq 0$

Proof of Goldstone Theorem:

$$\delta V(\phi) = \frac{\partial V}{\partial \phi^i} \delta \phi^i = 0$$

mass matrix of ϕ^i: $m_{ij} = \frac{1}{2} \frac{\partial^2 V}{\partial \phi^i \partial \phi^j}\Big|_{min}$

$$0 = \left(\frac{\partial}{\partial \phi^j} \delta V\right)_{min} = \left(\frac{\partial^2 V}{\partial \phi^i \partial \phi^j} \delta \phi^i\right)_{min} + \left(\underbrace{\frac{\partial V}{\partial \phi^i}}_{=0} \frac{\partial}{\partial \phi^j} \delta \phi^i\right)_{min} = 2 m_{ij} \delta \phi^i\Big|_{min} \overset{def}{=} 2 m_{ij} \delta \phi^i$$

$\Rightarrow$ unbroken symmetry $\delta\phi^i|_{min} = 0 \Rightarrow m_i \neq 0$ allowed

$\delta\phi^i|_{min} \neq 0 \Rightarrow m_i = 0$ ☐

$\Rightarrow$ dynamically S.B. in QCD is spont.

$\Rightarrow$ G.T applies

$$U(2)_L \times U(2)_R \rightarrow U(2)_V$$

$\Rightarrow 4+4-4 = 4$ GB

explains three light pions $\pi^\pm, \pi^0$ but where is the fourth?

$\pi^\pm, \pi^0 \,\hat{=}\, \underset{\sim}{3}$ of $SU(2)_A$ $\Rightarrow$ no GB for $U(1)_A$

($U(1)_A$ problem)

rescue: anomaly

$$\partial_\mu j^\mu_A \sim \epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\,F_{\mu\nu}F_{\rho\sigma} = \epsilon^{\mu\nu\rho\sigma}\,\partial_\mu\,\omega^{CS}_{\nu\rho\sigma}$$

However $\int \epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\,F_{\mu\nu}F_{\rho\sigma} = 0$ for standard Yang-Mills gauge fields

Instanton solutions of Yang-Mills theories

solutions of (Euclidean) Yang-Mills eq.

which vanish as $A_i \underset{i=1,2,3}{\overset{r\to\infty}{\sim}} U^{-1}(\theta)\, \partial_i\, U(\theta) \sim \frac{1}{r}$

$F \sim 0$

$\theta \equiv$ coordinate of S^3 at $r=\infty$

$\Rightarrow \quad \int \epsilon^{\mu\nu\rho\sigma}\, \mathrm{tr}\, F_{\mu\nu} F_{\rho\sigma} \approx \underset{\underset{N_0}{\uparrow}}{n} =$ instanton winding #

$\rightarrow$ $U(1)_A$ broken!

Remarks:

- Renormalization to $U(3)_L \times U(3)_R \to U(3)_V$ for $\begin{pmatrix} u \\ d \\ s \end{pmatrix}$ possible other application $\pi^0 \to 2\gamma$

consider quarks to also carry electric charge

$$D_\mu \psi^i = \partial_\mu \psi^i - g_s A^a_\mu t^{a\,i}{}_j \psi^j - g_1 q_{em} A^0_\mu \psi^i$$

↑ photon

↓

breaks SU(2) flavor symmetry as $q_u \neq q_d$

but leaves $U(1)_V \times U(1)_A$

'naive' decay rate $\Gamma(\pi^0 \to 2\gamma) \sim 10^{13}\ s^{-1}$

observed $\Gamma \sim 10^{16}\ s^{-1}$

reason: anomaly

$$\partial_\mu j^\mu_A \sim \epsilon^{\mu\nu\rho\sigma} F^0_{\mu\nu} F^0_{\rho\sigma}$$

cures discrepancy! (and measures coefficient of anomaly)

Last time: $\mathcal{L}_\psi = \sum_{I=1}^{n_f} \bar{\psi}^{iI} (i\not{D})_{ij} \psi^{jI}$

with flavour symmetry $G_f = U(n_f)_L \times U(n_f)_R$

e.g. $\psi^I = \begin{pmatrix} u \\ d \end{pmatrix}$, $n_f = 2$

$\psi^I = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$, $n_f = 3$

Below dynamical scale Λ: bound states form

break $U(n_f)_L \times U(n_f)_R \to U(n_f)_V$

in general $G_f \to H_f$

$\Rightarrow$ $\underbrace{(\dim G_f - \dim H_f)}_{= n_g}$ Goldstone Bosons π^α, $\alpha = 1, \dots, n_g$

In addition: G_f anomalous generically

goal today: give $\mathcal{L}_{eff}(\pi^\alpha)$

let $[t^a, t^b] = i f^{[abc]} t^c$ be generators of H

and (X^α, t^a) be generators of G, X^α: broken generators

then $[t^a, X^\alpha] = i f^{a\alpha\beta} X^\beta$ (since $f^{a\alpha b} = 0$ if H is subgroup)

$[X^\alpha, X^\beta] = i f^{\alpha\beta a} t^a + i f^{\alpha\beta\gamma} X^\gamma$

finite group element $g = e^{i\alpha^a t^a} e^{i\xi^\alpha X^\alpha}$

$\delta\phi = i\alpha^a t^a \phi + i\xi^\alpha X^\alpha \phi$, $\delta\phi|_{min} = i\xi^\alpha X^\alpha \phi|_{min}$

GB can be parametrized by $U(x) = e^{i\pi^\alpha(x) X^\alpha}$

(e.g. $G = U(1)$: $U(x) = e^{i\pi}$ = phase of Mexican hat pot.)

$\mathcal{L}_{eff}(U) = -\frac{1}{16} F^2 \, \mathrm{Tr}\, \partial_\mu U \partial^\mu U^+$ + higher derivatives

$\uparrow$

π-decay constant

How do we take anomaly into account?

idea: ['t Hooft] make G_f local and anomaly free

by add fictitious gauge fields V_μ and fictitious fermions χ

in a weakly coupled theory

choose spectrum s.t. all anomalies cancel (e.g. leptons and weak int. in S.M.)

$\mathcal{L}_{eff}'$ below Λ: no quarks and gluons

ψ A

but χ, V_μ, π^α

$\Rightarrow$ $\mathcal{L}_{eff}$ cannot be invariant as χ produces anomaly which for $E \gg \Lambda$ was cancelled by ψ

$$\Rightarrow \quad T^{\alpha,a}\, \Gamma[\pi, V] = \mathcal{A}^a[V] \quad (*), \qquad \Gamma[\pi,V] = \int [D\chi]\, e^{i\mathcal{L}(\chi, V, \pi)}$$

generator of G_f in functional form; minus before; quantum eff. action

assume: $\mathcal{A}^a = 0$ (no anomaly in unbroken theory H)

we already showed that there is no local $\Gamma[0,V]$ which solves $(*)$

new ingredient: GBs π^α!

solution: [WZ '71]

$$\boxed{\Gamma_{WZ}[\xi, V] \propto \int_0^1 dt \int d^4y\, \pi^\alpha(y)\, \mathcal{A}^\alpha[V_{-t\pi}]}$$

Wess-Zumino term

where $V_{-t\pi\,\mu} = U(t\pi)\, V_\mu\, U^{-1}(t\pi) - i(\partial_\mu U)\, U^{-1}$

$$U(t\pi) \equiv e^{-it\pi^\alpha X^\alpha}$$

proof: [Weinberg, vol 2, p. 410]

properties of Γ_{WZ}:

- unique for $\Gamma[0,V] = 0$
- $U(1)$ symmetry: $V_{-ts\,\mu} = V_\mu$

$$\Gamma_{WZ} \sim \int \pi \; \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

transformation of π: $\pi \to \pi' = \pi + \text{const}$

$$\Rightarrow \delta\Gamma_{WZ} \sim \int \mathcal{A} \quad \checkmark$$

- For $V_\mu = 0 \Rightarrow V_{-ts\,\mu} = -i(\partial_\mu U)U^{-1}$

$$\Rightarrow F_{\mu\nu}(V_{-ts}) = 0$$

(check with forms: $F = dV - V^2$
for $V = idUU^{-1}$: $F = -i\,dUdU^{-1} - dUU^{-1}\underbrace{dUU^{-1}}_{-UdU^{-1}} = 0$)

$$\mathcal{A}^\alpha = \operatorname{tr} x^\alpha d\left(VdV + \tfrac{1}{2}V^3\right) \quad \text{in forms}$$

$$= \operatorname{tr} x^\alpha\left(dV dV + \tfrac{3}{2}(dV)V^2\right)$$

$$\underset{F = dV - V^2 = 0}{\sim} \operatorname{tr} x^\alpha V^4$$

$$\Rightarrow \Gamma_{WZ}[\pi, 0] \sim \mathrm{tr} \int d^4y\, \pi^\alpha X^\alpha \int_0^1 dt\, \epsilon^{\mu\nu\rho\sigma}\, V_{-t\pi\,\mu}\, V_{-t\pi\,\nu}\, V_{-t\pi\,\rho}\, V_{-t\pi\,\sigma}$$

$$V_{-t\pi\,\mu} = -i\, \partial_\mu U / U^{-1}$$

for small π : $U = 1 - it\,\pi^\alpha X^\alpha$

$$V_{-t\pi\,\mu} = -it\,\partial_\mu \pi^\alpha / X^\alpha$$

$$\Rightarrow \Gamma_{WZ}[\pi, 0] \sim \epsilon^{\mu\nu\rho\sigma}\, \mathrm{tr}(X^\alpha X^\beta X^\gamma X^\delta X^\epsilon)$$

$$\int d^4y \underbrace{\int_0^1 (-it)^4}_{\frac{1}{5}}\, \pi^\alpha \partial_\mu \pi^\beta \partial_\nu \pi^\gamma \partial_\rho \pi^\delta \partial_\sigma \pi^\epsilon$$

$$\sim \epsilon^{\mu\nu\rho\sigma}\, \mathrm{tr}(\quad) \int d^4y\, \pi^\alpha \partial_\mu \pi^\beta \partial_\nu \pi^\gamma \partial_\rho \pi^\delta \partial_\sigma \pi^\epsilon$$

- $\Gamma_{WZ}[\pi, 0] \overset{[\text{Witten}]}{\sim} \epsilon^{MNPQR} \int_{B_5} d^5y\, \mathrm{tr}(V_M V_N V_P V_Q V_R)$

$M = 0, \dots, 4$, $y^M = (y^\mu, t)$, $\pi^\alpha(y, t) \equiv t\,\pi^\alpha$

S_4 (= Euclid. M_4) $= \partial B_5$

properties:

- total div $\Rightarrow$ only depends on bound. i.e. space-time

$- \int_{B_5}$ not unique

S^4 B_5 but also S^4 B_5

$$\Rightarrow \quad \Gamma_{WZ} - \Gamma'_{WZ} = 2\pi \text{ integer}$$

so that $e^{-i\Gamma_{WZ}}$ is unique

$\Rightarrow$ coeff. of Γ_{WZ} has to be quantized!

last 2 lectures : asymptotically free gauge theory

$E \gg \Lambda$: description in terms of elementary fermions (quarks)

$E \lesssim \Lambda$: bound states

$$M^{IJ} \sim \langle \psi^I \psi^J \rangle$$

$$B^{IJK} \sim \langle \psi^I \psi^J \psi^K \rangle$$

masses $m \sim \Lambda$ general expectation

$m \ll \Lambda$ can occur : Goldstone bosons

today : can there be light composite fermions ?
(can quarks be composite ?)
light fermions arise when theory has global chiral symmetry as the mass term is forbidden

assume chiral symmetry unbroken $\Rightarrow$ no GB
but anomalous $D^{abc}_{(x)} = \mathrm{tr}_x \left(t^a \{ t^b, t^c \} \right) \neq 0$

as last time complete theory to fictitious gauge field V_μ
+ " fermion χ

s.t. $D^{abc}(\psi) + D^{abc}(\chi) = 0$

$A(\psi) + A(\chi) = 0$

$E < \Lambda$: $\mathcal{L}_{eff}(B, V, \chi)$

theory anomaly free $\Rightarrow$ $D(A) + D(\chi) = 0$

$\mathcal{A}(A) + \mathcal{A}(\chi) = 0$

more precisely

$$\boxed{\sum_{r_\psi} \mathcal{A}(r_\psi) = \sum_{r_B} \mathcal{A}(r_B)} \Leftrightarrow \sum_{\substack{\text{fundamental}\\ \text{fermions}}} \mathcal{A} = \sum_{\substack{\text{zero}\\ \text{mass}\\ \text{composite}\\ \text{fermions}}} \mathcal{A}$$

$E > \Lambda$ $\qquad$ $E < \Lambda$

't Hooft anomaly matching condition

Ex 1: $\psi_{L/R} = \begin{pmatrix} u_{L/R} \\ d_{L/R} \end{pmatrix}$ $\qquad$ $\mathcal{L} = \sum_{I=1}^{2} \bar{\psi}_L^I \, i\sigma^\mu D_\mu \psi_L^I + \bar{\psi}_R^I \, i\bar{\sigma}^\mu D_\mu \psi_R^I$

gauge: $G_f = SU(2)_L \times SU(2)_R \times U(1)_V$, $G = SU(3)$

$r(\psi_L)$: $(2,1)_{+1}$, $\qquad$ $r(\bar{\psi}_R) = (1,2)_{-1}$

charge of $U(1)_V$ $\qquad$ (convention: all fermions in same Lorentz rep.)

baryon: $\langle \psi^I \psi^J \psi^K \rangle$ $\quad$ $I = 1,2$

$\langle \underbrace{\psi_L^I \psi_R^J}_{\text{color triplet}} \psi_L^K \rangle \rightarrow (2,1)_3$

$\langle \underbrace{\bar{\psi}_L^I \bar{\psi}_R^J}_{\text{color singlet}} \bar{\psi}_R^K \rangle \rightarrow (1,2)_{-3}$

Anomalies	elem.		composite
$SU(2)_R$ – $SU(2)_{L,R}$ – $SU(2)_{L,R}$	$A \sim 0$	2 real $\sim \mathrm{tr}(\sigma^a\{\sigma^b,\sigma^c\}) = \mathrm{tr}\,\sigma^a\delta^{bc} = 0$	0
$SU(2)_{R,L}$ – $SU(2)_{L,R}$ – $SU(2)_{L,R}$	$A \sim 0$	$\mathrm{tr}\,\sigma^a = 0$	0
$U(1)_V$ – $U(1)_V$ – $U(1)_V$	$A \sim 0$	(vector $U(1)$)	0

$U(1)_V$ – $SU(2)_{L,R}$ – $SU(2)_{L,R}$:

$$\mathrm{tr}_r\,\sigma^a\sigma^b = \delta^{ab} C(r)$$

$$A \sim \mathrm{tr}(q\,\sigma^a\sigma^b) = \delta^{ab}\sum_r n_r\, q(r)\, C(r)$$

elem.:

$$= A(r_{q_L}) + A(r_{\bar q_L})$$
$$= 3\cdot 1\cdot \tfrac{1}{2} + 3\cdot(-1)\cdot 0 = \tfrac{3}{2}$$

colour, $q=1$, $C(2)=\tfrac{1}{2}$

composite:

$$= A(r_{b_L}) + A(r_{\bar b_L})$$
$$= 1\cdot 3\cdot\tfrac{1}{2} + 1(-1)\,0$$
$$= \tfrac{3}{2}\ ✓$$

remarks:

- 't Hooft anomaly matching is not very restrictive in terms of # of solutions (see [Weinberg, 22.5])
- good check together with other constraints e.g. Seiberg duality

<u>Ex2</u>: Seiberg duality (for N=1 supersymmetric theories)

electric theory: $G = SU(N_c)$

$G_f = SU(N_f)_L \times SU(N_f)_R \times U(1)$

N_f flavours of quarks ψ_L^I, ψ_R^I, $I = 1, \dots, N_f$

representation with respect to G_f

$r(\psi_L) = (\underset{\sim}{N_f}, 1)_{+1}$ $\quad r(\bar{\psi}_R) = (1, \underset{\sim}{\bar{N}_f})_{-1}$

dual magnetic theory: $G = SU(N_f - N_c)$

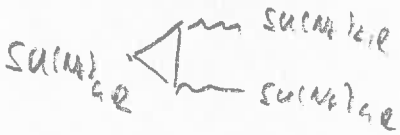
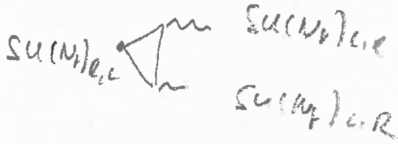
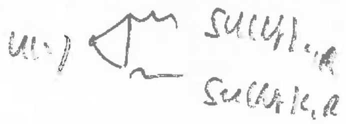
$G_f = SU(N_f)_L \times SU(N_f)_R \times U(1)$

N_f flavours of dual quarks $\tilde{\psi}_L^I, \tilde{\psi}_R^I$ + singlet m

$r(\tilde{\psi}_L) = (\bar{N}_f, 1)_q$, $r(\tilde{\psi}_R) = (1, N_f)_q$

$q = \frac{N_c}{N_f - N_c}$

$r(m) = (N_f, \bar{N}_f)_0$

anomali	eks.	mas.
	$A \sim N_c d^{abc}$	$[(N_f - N_c) + N_f]\, d^{abc}$
	$A \sim \mathrm{tr}\, t^a = 0$	
	$A = N_f \cdot N_c \left((+1)^3 + (-1)^3\right) = 0$	$= (N_f - N_c) N_f \left(q^3 + (-q)^3\right)$
	$A = \delta^{ab} \sum_x n_x\, q(x)\, C(x)$ $= \delta^{ab}\left(N_c \cdot 1 \cdot \frac{1}{2}\right)$	$\delta^{ab}\left(\underbrace{(N_f - N_c) \cdot q}_{= N_c} \frac{1}{2}\right)$

L13 Weyl / Conformal anomalies and the a-theorem

Weyl invariance: $g_{\mu\nu} \to g'_{\mu\nu} = e^{2\sigma(x)} g_{\mu\nu}$

$$\phi \to \phi' = e^{-\Delta\sigma(x)} \phi$$

$\Rightarrow$ local scale transformation

Δ_ϕ = conformal / Weyl weight of ϕ

example of inv. action

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{12} \phi^2 R \right) \quad \text{with } \Delta_\phi = 1$$

transf: $\sqrt{-g} \to \sqrt{-g}' = e^{4\sigma} \sqrt{-g}$

$$R(g) \to R(g') = e^{-2\sigma}\left(R(g) - 6\, \partial_\mu \sigma \partial^\mu \sigma - 6\, \Delta\sigma\right)$$

$$\phi \to \phi' = e^{-\sigma}$$

conservation law $\quad g^{\mu\nu} T_{\mu\nu} = T^\mu{}_\mu = 0$

T: energy momentum tensor

recall: general covariance: $\quad \partial^\mu T_{\mu\nu} = 0$

expressed as current conservation

$j^{a\mu} := \xi^a_\nu T^{\nu\mu}$, ξ^a_ν: Killing vectors. $D_\mu \xi^a_\nu \sim g_{\mu\nu}$

$$D_\mu j^{a\mu} = (D_\mu \xi^a_\nu) T^{\nu\mu} \sim g_{\mu\nu} T^{\nu\mu} = 0$$

conformal invariance

invariance of $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = 0$

$\Rightarrow$ Lorentz-transf. (6 generators $L_{\mu\nu}$)', translation (4 gen. P_μ)

+ $x^\mu \to x^{\mu'} = x^\mu + \xi^\mu$

s.t. $\delta\eta_{\mu\nu} = \partial_\nu \xi_\nu + \partial_\rho \xi_\mu = \Omega^2 \eta_{\mu\nu}$ (special conformal transf.)

$\uparrow$ conformal Killing vectors

+ $x^\mu \to x^{\mu'} = \lambda x$ dilatation

together $6+4+4+1 = 15$ generators

from the conformal group $SO(2,4)$ ($SO(2,d)$ in d-dim)

(recall Lorentz group: $SO(1,3)$ ($SO(1,d-1)$)

Theorem [Weiss '61]:

A general covariant and Weyl inv. theory is automatically conformal invariant in flat space

Example: Yang-Mills theory

$S = \int d^4x \sqrt{-g} \, \mathrm{tr} \, F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma}$, $\Delta(A_\mu) = 0$

Weyl inv. + conf. inv. in flat limit

quantum theory: conf. inv. generically broken

Yang Mills: $T^\mu{}_\mu = \beta(g)\, \mathrm{tr}\, F_{\mu\nu}F^{\mu\nu}$ conformal anomaly

β-function $\mu \frac{\partial}{\partial\mu} g = \beta(g)$

quantum conformal theories: $\beta = 0 \Rightarrow g \neq g(\mu)$

e.g. $N=4$ supersymmetric Yang-Mills theories

$\beta \neq 0 \to$ conformal anomaly

discuss anomalies in a Weyl inv. theory

coupled to external metric / gravitational field

$$e^{i\Gamma[g]} = \int [D\phi]\, e^{iS[\phi, g]}$$

$$\delta\Gamma[g] = -\int d^4x\, \sigma(x)\, \mathcal{A}(x, g)$$

WZ consistency condition

$$[\delta_{\sigma_1}, \delta_{\sigma_2}]\, \Gamma[g] = -\int (\sigma_1 \delta_{\sigma_2}\mathcal{A} - \sigma_2 \delta_{\sigma_1}\mathcal{A}) = 0$$

solution of wt consistency condition

$$\mathcal{A}(\sigma) = c \underbrace{C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}}_{\text{Weyl tensor}} - a \underbrace{E_4}_{\text{Euler density}}$$

$$C_{\mu\nu\rho\sigma} := R_{\mu\nu\rho\sigma} - \left(g_{\mu[\rho} R_{\sigma]\nu} - g_{\nu[\rho} R_{\sigma]\mu} \right) + \frac{1}{3} R \, g_{\mu[\rho} g_{\sigma]\nu}$$

inv. under Weyl transf.

$$E_4 := R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 = \partial_\mu j^\mu$$

$$\int_M E_4 \sqrt{g} \, d^4x = \chi(M) = \text{Euler } \#$$

	$\underline{a}$	$\underline{c}$	$(\times 90 (8\pi)^2)$
scalar	1	3	
Weyl fermion	$\frac{11}{2}$	9	
gauge field	62	36	

or $a = n_0 + \frac{11}{2} n_{\frac{1}{2}} + 62 n_1$

$c = 3 n_0 + 9 n_{1/2} + 36 n_1$

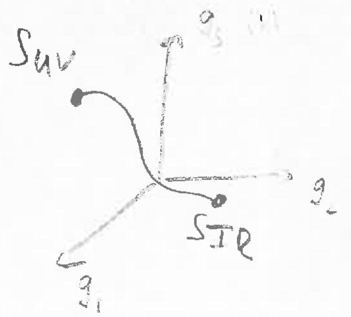
$n_{0,\frac{1}{2},1}$ = number of massless particles with spin $s = 0, \frac{1}{2}, 1$

a, c "measure" d.o.f. of a theory

a-theorem [Cardy '88, Komargodski, Schwimmer '11]
(generalizes $d=2$ c-theorem of [Zamolodchikov

The anomaly coefficient a decreases along renormalization group flows such that $a_{UV} > a_{IR}$

RG flow: $\mu \frac{d}{d\mu} g_i(\mu) = \beta_i(g_i)$ is a flow eq. in coupling constant space



intuition:
S_{IR} contains less d.o.f.
$\Rightarrow a_{IR} < a_{UV}$

(KS) proved this fact.

Remarks:

- $E \gg$ mass scales $\Rightarrow S_{UV}$ conformally invariant
  $E \ll$ mass scales $\Rightarrow S_{IR}$ " "

  ex: chiral QCD without quark masses.

  S_{UV}: SU(3) gauge theory with massless chiral quarks
  S_{IR}: theory of massless pions

- if $a_{UV} > a_{IR}$ flows are irreversible!
  $\Rightarrow$ "$M_{UV} > M_{IR}$"

distinguish two cases

- spont. breaking of conf. symm., i.e. $\langle O_\Delta \rangle = v^\Delta$
- explicit " " " " , i.e.

$$S_{CFT} \to S_{CFT} + m^{4-\Delta} \int O_\Delta$$

(this is the case we are really interested in)

for proof use conformal compensator / dilaton formalism

add a new field $\tau(x)$ to S in the following way:

$$g_{\mu\nu} \to \hat{g}_{\mu\nu} = e^{+2\tau(x)} g_{\mu\nu} \quad , \quad \phi \to \hat{\phi} = e^{-\Delta\tau} \phi$$

s.t. under Weyl transf.

$$\hat{g}_{\mu\nu} \to \hat{g}_{\mu\nu} \quad , \quad \hat{\phi} \to \hat{\phi}$$

$$\tau(x) \to \tau(x) - \sigma(x) \quad \Rightarrow \quad \tau \text{ is redundant variable}$$

and can (always) be removed by choosing σ appropriately

Note: τ transforms non-linearly, exactly as GB!

For

$\Rightarrow$ $S(\hat{g}, \hat{\phi})$ is Weyl invariant up to anomaly

$\Rightarrow$ add WZ-term with τ playing role of π-ions

$$S_{WZ}(g, \tau, a, c) = \int \sqrt{g} \Big(-a \big[\tau E_4 + 4 (R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R) \partial_\mu \tau \partial_\nu \tau - 4 (\partial \tau)^2 \Box \tau + 2 (\partial \tau)^4 \big] + c \, \tau \, C^2 \Big)$$

$$\Omega = e^{+\tau}$$

$$\delta S_{WZ} = \int d^4x \, \sigma \mathcal{A} \quad \Rightarrow \quad \delta \Gamma + \delta S_{WZ} = 0$$

Remarks:

- τ couples like π:

  for $\tau = \text{const}$: $S_{WZ} \sim \int \sqrt{g} (-a \tau E_4 + c \tau C^2)$

- for $g_{\mu\nu} = \eta_{\mu\nu}$ kinetic term for τ:

  $$S_{WZ} \sim 2\underline{\underline{a}} \int d^4x \, (\partial \tau)^4$$

- crucial subtlety: a, c changes along RG flow, as fields are integrated out.

  $\rightarrow$ anomaly matching

  $$\delta \Gamma^{UV} = \delta \Gamma^{IR} + \delta S_{WZ}^{IR} - \delta S_{WZ}^{UV}$$

N!

$\Rightarrow$ kinetic term of $\tilde{\tau}$ in IR

$$S \sim (a_{uv} - a_{IR}) \int d^4x \, (\partial \tilde{\tau})^4$$

unitary CFT demands

$$a_{uv} - a_{IR} > 0 \qquad \text{q.e.d.}$$