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Moduli-Dependence of String Loop Corrections to Gauge Coupling Constants

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ABSTRACT

We consider one-loop corrections Δ_a to inverse gauge couplings g_a^{-2} in supersymmetric vacua of the heterotic string. The form of these corrections plays an important role in scenarios for dynamical supersymmetry breaking in string theory. Specifically, we calculate the exact functional dependence of $\Delta_a(U)$ on any untwisted modulus field U of an orbifold vacuum; it has the universal form $\Delta_a(U,\bar{U}) = A_a \cdot \log(|\eta(U)|^4 \cdot \mathrm{Im}\, U) + \mathrm{const.}$, where A_a are easily computable rational constants. The dependence is nontrivial ($A_a \neq 0$) only if some sectors of the orbifold Hilbert space have precisely $N = 2$ spacetime supersymmetry. The expression for Δ_a has an expected invariance under modular transformations of U, since these are symmetries of the orbifold vacuum state. However, Δ_a is not the real part of a holomorphic function, in seeming contradiction with the existence of a supersymmetric effective Lagrangian. The apparent paradox is an infrared problem, and can occur not just in string theory but in renormalizable supersymmetric field theories as well. We show how the paradox is resolved in the field theory case and argue that the same resolution applies also to the string theory case.

1. Introduction

Heterotic string theory[1] is currently the best candidate for a fundamental theory of all particle interactions. The first step in deducing phenomenology from string theory is to derive an effective four-dimensional quantum field theory for particles that are light compared to the string scale. This theory describes particle interactions at energies just below the string scale, but once it has been obtained from string theory, ordinary field-theoretical techniques can be used to deduce an effective theory valid at much lower (*e.g.* electroweak) energies. At present, there is a huge number of candidate vacua of string theory, each leading to a somewhat different effective field theory; this sad state of affairs necessitates a general treatment of such theories.

Consider an effective (Euclidean) Lagrangian for a general local field theory in four dimensions. Bosonic terms with at most two space-time derivatives can be summarized in the following formula:

$$\mathcal{L}_{\rm eff}^{\rm bose} = \frac{R}{2\kappa^2} + \left(\frac{1}{4g^2(\phi)}\right)_{ab} F^a_{\mu\nu}F^{b\mu\nu} + \frac{i\Theta_{ab}(\phi)}{32\pi^2} F^a_{\mu\nu}\tilde{F}^{b\mu\nu} + \tfrac{1}{2}G_{ij}(\phi)\, D^\mu\phi^i D_\mu\phi^j + V(\phi). \tag{1.1}$$

Here V is the scalar potential, $G_{ij}(\phi)$ is the metric on a Riemannian manifold spanned by the scalar fields ϕ^i, and the matrices $g^{-2}_{ab}(\phi)$ and $\Theta_{ab}(\phi)$ are generalized scalar-field-dependent inverse gauge couplings and gauge vacuum angles, respectively. If the effective four-dimensional field theory is $N=1$ supersymmetric, then all fermionic terms in its Lagrangian are completely determined by the bosonic terms (1.1), and the bosonic terms themselves have to obey certain constraints. In particular, the manifold spanned by the scalar fields must be Kähler, with an appropriate metric and complex coordinates Φ^i and $\overline{\Phi^\imath}$,⋆ and the

⋆ In this article we call scalar fields chiral or anti-chiral according to the space-time chirality of their fermionic superpartners; in our notations we distinguish them by using Φ^i for the former and $\overline{\Phi^\imath}$ for the latter. Lower-case ϕ^i are used to denote any scalar field in the effective four-dimensional theory — chiral, anti-chiral or mixed.

complex functions

$$f_{ab}(\phi) \stackrel{\text{def}}{=} \left(\frac{1}{g^2(\phi)}\right)_{ab} - \frac{i\Theta_{ab}(\phi)}{8\pi^2} \tag{1.2}$$

must be holomorphic functions of the coordinates Φ.[2] In a generic $d=4, N=1$ supergravity there is only one other restriction on the matrix-valued function $f_{ab}(\Phi)$ — it has to be gauge covariant. The latter requirement implies that if we limit our attention to the dependence of f_{ab} on scalar fields that are neutral with respect to the gauge symmetry, we can write $f_{ab} = \delta_{ab} \cdot f_a$, with equal f_a for all gauge bosons a that belong to the same simple gauge group.

In this article we focus on $N=1$ and $N=2$ supersymmetric string vacua and investigate the dependence of f_a on the moduli fields — massless gauge-neutral scalar fields whose effective potential is classically and perturbatively flat. (They are called moduli fields because of their relation to the continuous parameters, or moduli, of a family of classical string vacua.) At the tree level of the string theory f_a depend only on the four-dimensional dilaton/axion field S, through the universal formula[3−6]

$$f_{ab}^{\text{tree}}(\Phi) = k_a \delta_{ab} \cdot S, \tag{1.3}$$

where k_a is the level of the appropriate Kac-Moody algebra. However, radiative corrections to eq. (1.3) — the result of integrating out massive, charged string states — do depend on the moduli scalars. Moduli-dependence of the one-loop corrections $f_a^{\text{1-loop}}$ was first investigated in ref. [7,8]; however, that study employed a Peccei-Quinn symmetry for the moduli, which is spoiled by world-sheet instantons. In this article we determine the exact functional dependence of $f_a^{\text{1-loop}}$ for a large class of orbifold compactifications of the heterotic string.

There are two basic reasons for studying the moduli-dependence of the one-loop corrections to f_a. First, asymptotically-free gauge couplings provide a mechanism for an effective quantum field theory that is weakly interacting at the string scale to become strongly interacting at some hierarchically lower energy scale, typically of the order $M_{\rm string} \cdot \exp\left(\frac{-8\pi^2 C}{g^2(M_{\rm string})}\right)$, where C is an $O(1)$ constant. A moduli-dependent one-loop correction to the gauge coupling constant has an $O(1)$ effect on all physical quantities associated with this scale. In particular, if an effective potential is generated non-perturbatively, it is automatically moduli-dependent and thus lifts the degeneracy of string vacua corresponding to different vacuum expectation values (VEVs) of the moduli fields.

The other reason for studying field dependence of the $f_{ab}(\Phi)$ is that non-zero derivatives $\partial f_{ab}/\partial\Phi^i$ lead to various non-renormalizable interactions involving gauge bosons and their superpartners. Of particular importance are the non-derivative interactions involving gauginos but no other fermions[9]

$$\begin{aligned}\mathcal{L}_\lambda \;=\; G^{i\bar{\jmath}}(\Phi,\overline{\Phi})\, &\left[\tfrac{1}{8}\sum_a \frac{\partial f_{ab}}{\partial\Phi^i}\cdot\lambda^a\lambda^b \;+\; e^{K/2}\left(\frac{\partial W}{\partial\Phi^i}+W\frac{\partial K}{\partial\Phi^i}\right)\right]\\ \times\, &\left[\tfrac{1}{8}\sum_a \frac{\partial f^*_{ab}}{\partial\overline{\Phi}^{\jmath}}\cdot\bar{\lambda}^a\bar{\lambda}^b \;+\; e^{K/2}\left(\frac{\partial W^*}{\partial\overline{\Phi}^{\jmath}}+W^*\frac{\partial K}{\partial\overline{\Phi}^{\jmath}}\right)\right]\end{aligned} \tag{1.4}$$

(here $K(\Phi,\overline{\Phi})$ is the Kähler potential and $W(\Phi)$ is the superpotential of the scalar fields). When the chiral symmetry of the gauginos is spontaneously broken, the interactions (1.4) result in an effective potential for the gaugino condensates $\left\langle\lambda^a\lambda^b\right\rangle$; combined with the non-perturbative effects that cause formation of the condensates in the first place, this effective potential may lead to spontaneous breakdown of supersymmetry.[4,10,11] However, in order to verify that this mechanism indeed results in supersymmetry breaking rather than in runaway VEVs of moduli scalars, one must know how the moduli enter the non-perturbative effective potential (comprising (1.4) as well as other terms[12]); obviously, knowledge of the moduli-dependence of f_a is indispensable in such an analysis.

Precisely because of its universality, formula (1.3) is too crude an approximation to use in a study of dynamical supersymmetry breakdown triggered by gaugino condensation, especially when there is no supersymmetry breaking at the tree level of the effective field theory. In the latter case, the only known way to stabilize the dilaton/axion VEV at a reliable weak-coupling value $\langle \mathrm{Re}\, S\rangle \gg 1$ is to have two or more independent gaugino condensates occurring at roughly the same scale.[13,14] In such a scenario the stable value of $\langle S\rangle$ so obtained is extremely sensitive to the differences between the f_a's of the gauge groups involved in the gaugino condensation. Moduli-dependence of these differences (which arises at the same one-loop level as the differences themselves) thus leads to moduli-dependence of all quantum effects in the effective four-dimensional theory. The question of whether and how supersymmetry is broken in this scenario, once the dilaton VEV is fixed, may also depend on the functional form of $f_a^{\text{1-loop}}(\Phi)$.[12,14,15]

At the one-loop level, the renormalized gauge couplings of the effective field theory can be written as

$$\frac{16\pi^2}{g_a^2(\mu)} \;=\; k_a\cdot\frac{16\pi^2}{g^2_{\mathrm{GUT}}} \;+\; b_a\cdot\log\frac{M^2_{\mathrm{GUT}}}{\mu^2} \;+\; \Delta_a\,, \tag{1.5}$$

where $\mu \ll M_{\mathrm{GUT}} \simeq M_{\mathrm{string}}$ is the renormalization scale, $g^{-2}_{\mathrm{GUT}} = \mathrm{Re}\, S + O(1)$ (*cf.* eq. (1.3)), and b_a are related to the one loop β-functions via $\beta_a = b_a\cdot g_a^3/16\pi^2$. Finally, Δ_a are the specific one-loop threshold corrections for each g_a^{-2}, which we would also like to identify as $16\pi^2\cdot \mathrm{Re}\, f_a^{\text{1-loop}}$. Ref. [16] gives a general formula for Δ_a in terms of the spectrum of all massive states of the string theory.⋆ Orbifolds provide an example of string vacua for which the entire massive spectrum is known exactly; moreover, all the masses can be written as analytic functions of the moduli that preserve the orbifold nature of the vacuum (*i.e.*, the moduli arising from the untwisted sector of the orbifold). Therefore, for orbifolds we can

⋆ The β-functions for some $N=1$ supersymmetric string vacua were first calculated in ref. [17], and for $N=2$ vacua in ref. [18]; the latter reference also carried out some of the analysis used in appendix A of this article.

derive exact formulæ for $f_a^{\text{1-loop}}$ as explicit functions of the untwisted moduli, and this is exactly what we shall do in the next section of this article.

Specifically, we will show that the one-loop threshold corrections Δ_a depend non-trivially on the untwisted moduli of an $N=1$ supersymmetric orbifold if and only if the orbifold group contains a subgroup that by itself would produce an orbifold with exactly $N=2$ space-time supersymmetry. Moreover, the functional form of this dependence can also be obtained from studying such $N=2$ orbifolds, which are examples of six-dimensional, $N=1$ supersymmetric vacua that have been toroidally compactified to four dimensions. We then consider the entire class of such vacua and compute Δ_a as functions of the moduli of the torus. To relieve the tedium, some of the calculations are presented in appendices A and B.

The main result of section 2 is that for any untwisted modulus U upon which Δ_a do depend non-trivially, the functional form of this dependence is given by

$$\Delta_a(U,\overline{U}) \;=\; A_a \cdot \log\left(|\eta(U)|^4 \cdot \operatorname{Im} U\right) \;+\; U\text{-independent terms}, \qquad (1.6)$$

where A_a are rational constants, computable from the massless spectrum alone.[†] Formula (1.6) has the expected invariance under modular ($PSL(2,\mathbf{Z})$) transformations of the complex U field, which are symmetries of the string vacuum under consideration.[19] On the other hand, the effective $g_a^{-2}(U,\overline{U})$ obtained from (1.6) cannot be written as the real part of a holomorphic function $f_a(U)$.

At first glance, this lack of a holomorphic effective $f_a^{\text{1-loop}}(U)$ appears to indicate some bizarre stringy effect that breaks the space-time supersymmetry at

† Actually, in order to compute A_a for an $N=1$ supersymmetric orbifold we need to know the massless spectra of related $N=2$ orbifolds and not of the $N=1$ orbifold itself. See section 2.3 for details.

the one-loop level. However, we find that the same problem can occur in ordinary four-dimensional supersymmetric quantum field theories. To our knowledge this phenomenon has not been treated in the literature, and so we devote section 3 to a discussion of the *renormalized* $f_a^{\text{1-loop}}$ and their dependence on the scalar fields. Our main point is that in a gauge theory with massless charged fermions, the renormalized gauge coupling is divergent in the infrared limit while the renormalized Θ angle is simply not well-defined. On the other hand, the infrared contributions to non-renormalizable effective interactions between two gauge bosons and a neutral scalar field ϕ^i are finite (at least at the one-loop level); the coefficients of these interactions can be interpreted as 'effective derivatives' of the gauge couplings g_a^{-2} and Θ_a angles with respect to ϕ^i, which we collectively denote by $\{\partial f_a/\partial\phi^i\}$. However, there is no guarantee that these effective derivatives are integrable, that is, can be written as derivatives of some renormalized functions $f_a(\phi)$ with respect to ϕ^i. We find that the effective derivatives of the *real* parts of $f_a^{\text{1-loop}}$ are in fact integrable, and even satisfy the naive relations

$$\{\partial\,\mathrm{Re}\, f_a/\partial\phi^i\}(p^2;\langle\phi\rangle) \;=\; \frac{\partial g_a^{-2}(p^2;\langle\phi\rangle)}{\partial\,\langle\phi^i\rangle} \tag{1.7}$$

where $g_a(p^2)$ are the running gauge couplings, but the *imaginary* parts of $f_a^{\text{1-loop}}$ are not integrable. Supersymmetry, if present, requires that

$$\{\partial f_a^*/\partial\Phi^i\} \;=\; \{\partial f_a/\partial\overline{\Phi^\imath}\} \;=\; 0 \tag{1.8}$$

for any chiral scalar Φ^i or anti-chiral $\overline{\Phi^\imath}$; however, eqs. (1.8) and (1.7) can be satisfied without the renormalized $g_a^{-2}(\langle\phi\rangle)$ being the real parts of some holomorphic functions of VEVs of the chiral fields Φ.

Most of the content of section 3 is field theoretical. We discuss the general theory behind eqs. (1.7) and (1.8) and the possibility of non-holomorphic dependence of the effective gauge couplings on the chiral scalars. We also give a simple example of a renormalizable gauge theory in which the 'effective derivatives' $\{\partial\Theta_a/\partial\phi^i\}$ are not integrable and the dependence of g_2^{-2} on the chiral scalars is not holomorphic. However, in the last subsection of section 3 we go back to string theory and calculate $\{\partial f_a/\partial\phi^i\}^{\text{1-loop}}$ for supersymmetric orbifolds directly from the string S-matrix elements. The results of this calculation explicitly verify that eqs. (1.7) and (1.8) hold true in the orbifold case, and therefore that the non-integrability of $\{\partial\Theta_a/\partial\phi^i\}$ is the right explanation of the non-holomorphicity of eq. (1.6).

Finally, in section 4 we summarize our results and compare them to previous calculations of loop corrections to the gauge couplings in four-dimensional supersymmetric vacua of the heterotic string.

2. Threshold Corrections for Orbifolds and $N = 2$ String Vacua

2.1. Threshold Corrections for Orbifolds

After all these preliminaries we are now ready to calculate the one-string-loop threshold corrections Δ_a for the orbifold vacua. Our starting point is the general formula of ref. [16]: for any four-dimensional, tachyon-free vacua of the heterotic string,

$$\Delta_a \;=\; \int\limits_\Gamma \frac{d^2\tau}{\tau_2}\,\left(\mathcal{B}_a(\tau,\bar\tau)\;-\;b_a\right), \tag{2.1}$$

where

$$\mathcal{B}_a(\tau,\bar\tau) \;=\; |\eta(\tau)|^{-4}\cdot\sum_{\text{even s}}(-)^{s_1+s_2}\,\frac{dZ_\Psi(\mathrm{s},\bar\tau)}{2\pi i\,d\bar\tau}\cdot\mathrm{Tr}_{s_1}\left(Q_a^2\cdot(-)^{s_2F}q^{H-\frac{11}{12}}\bar q^{\bar H-\frac{3}{8}}\right)_{\text{int}} \tag{2.2}$$

and

$$b_a \equiv \lim_{\tau_2\to\infty} \mathcal{B}_a = -\tfrac{11}{6}\,\mathrm{tr}_V(Q_a^2) \;+\; \tfrac{1}{3}\,\mathrm{tr}_F(Q_a^2) \;+\; \tfrac{1}{12}\,\mathrm{tr}_S(Q_a^2) \qquad (2.3)$$

Here $\tau = \tau_1 + i\tau_2$ is the modulus of the world-sheet torus, and is integrated over the usual fundamental domain $\Gamma = \{\tau_2 > 0,\ |\tau_1| < \frac{1}{2},\ |\tau| > 1\}$; $q = e^{2\pi i\tau}$ and $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$; Q_a measures the charge under some generator in the a^{th} factor of the gauge group; indices s_1 and s_2 each take values 0 & 1 corresponding to the Neveu-Schwarz and Ramond boundary conditions on the world sheet and the restriction 'even $\mathbf{s}$' excludes the Ramond-Ramond case $(s_1, s_2) = (1,1)$. The derivation of eqs. (2.1) – (2.3) used the fact that every four-dimensional string vacuum is composed of two pieces: free world-sheet bosons X^μ and fermions Ψ^μ, which constitute the spacetime degrees of freedom ($\mu = 0,1,2,3$), and some internal superconformal field theory (SCFT) of central charge $(c,\bar{c}) = (22,9)$. The trace in eq. (2.2) is taken over this internal SCFT, whereas the factors $|\eta|^{-4}$ and Z_Ψ denote the light-cone gauge partition functions of X^μ and Ψ^μ respectively. Z_Ψ, which is equivalently the partition function of one *complex* free fermion, is given by

$$Z_\Psi(\mathbf{s},\bar\tau) \;=\; \frac{1}{\eta(\bar\tau)}\cdot \begin{cases} \vartheta_3(\bar\tau) & \text{for } \mathbf{s} = (0,0), \\ \vartheta_4(\bar\tau) & \text{for } \mathbf{s} = (0,1), \\ \vartheta_2(\bar\tau) & \text{for } \mathbf{s} = (1,0), \\ 0 & \text{for } \mathbf{s} = (1,1). \end{cases} \qquad (2.4)$$

Actually, formula (2.1) only has physical import for the difference $\frac{\Delta_{a_1}}{k_{a_1}} - \frac{\Delta_{a_2}}{k_{a_1}}$ between threshold corrections for two different gauge group factors, because an uncalculated constant (denoted by Y in ref. [16]) appears in the relation between the bare string coupling constant and g_{GUT} as defined in eq. (1.5). This caveat will play a role below.

The trace over the internal sector in eq. (2.2) is model-dependent and in general cannot be simplified further. However, great simplification is possible for orbifold vacua, which can be described by starting from a ten-dimensional vacuum of the heterotic string in which six out of ten space-time dimensions form a flat torus $\mathbf{T}^6$, and then dividing the world-sheet conformal theory describing such a vacuum by a discrete symmetry group $\mathbf{G}$. In order to preserve an $N=1$ supersymmetry in space-time, $\mathbf{G}$ should be a subgroup of $SU(3)$. Also, $\mathbf{G}$ should be an isometry of the $\mathbf{T}^6$; given the action of $\mathbf{G}$, this is a constraint on the shape of the torus, *i.e.*, on the constant background metric and antisymmetric tensor fields on the $\mathbf{T}^6$; the parameters describing the shape of the $\mathbf{T}^6$ that are not fixed by this constraint constitute the untwisted moduli of the orbifold. The trace in eq. (2.2) decomposes into sectors with boundary conditions (g,h) along the two cycles of the world-sheet torus, according to

$$\mathrm{Tr}_{s_1}\left(Q_a^2\cdot(-)^{s_2F}q^{H-\frac{11}{12}}\bar{q}^{\bar{H}-\frac{3}{8}}\right)_{\mathrm{int}}=\frac{1}{|\mathbf{G}|}\sum_{\substack{g,h\in\mathbf{G}\\ gh=hg}}\mathrm{Tr}_{(g,s_1)}\left(Q_a^2\cdot h\cdot(-)^{s_2F}q^{H-\frac{11}{12}}\bar{q}^{\bar{H}-\frac{3}{8}}\right). \tag{2.5}$$

Each (g,h) sector preserves a certain amount of four-dimensional supersymmetry, either $N=4$, $N=2$ or $N=1$. The only $N=4$ supersymmetric sector is the completely untwisted sector ($g=h=1$), which also represents compactification on the torus $\mathbf{T}^6$. This sector gives a vanishing contribution to both the beta functions[20] and the Δ_a.[16] (The latter result holds because the spin-structure-dependent part of the trace in eq. (2.5) for $g=h=1$ is just the partition function $Z_\Psi^3(\mathbf{s},\bar\tau)$ of six real (three complex) untwisted world-sheet fermions; hence the contribution to $\mathcal{B}_a$ is proportional to

$$\sum_{\text{even }\mathbf{s}}(-)^{s_1+s_2}\,Z_\Psi^3(\mathbf{s},\bar\tau)\cdot\frac{d}{d\bar\tau}Z_\Psi(\mathbf{s},\bar\tau) \;=\; \tfrac{1}{4}\frac{d}{d\bar\tau}\sum_{\text{even }\mathbf{s}}(-)^{s_1+s_2}\,Z_\Psi^4(\mathbf{s},\bar\tau) \;=\; 0\,, \tag{2.6}$$

where the last equation uses the identity $\vartheta_3^4-\vartheta_4^4-\vartheta_2^4=0$ which is also responsible for the vanishing of the partition function.)

Since g and h commute, we can choose complex coordinates on the torus $\mathbf{T}^6$ that diagonalize the action of g and h. We refer to these three complex directions as 'complex planes'. Each (g,h) sector is also characterized by the number of complex planes that are *not* rotated by the action of g and h on $\mathbf{T}^6$: All 3 planes are fixed for the $N=4$ sector, 1 is fixed in the $N=2$ sectors, and none is fixed in the $N=1$ sectors. (No $g \in SU(3)$ can fix exactly 2 of the 3 planes.) As argued in ref. [16], the sectors that rotate all planes are not sensitive to the geometry of the torus. Indeed, the charges Q_a of string states do not depend on the moduli,⋆ while the values of H and $\bar{H}$ depend on the untwisted moduli only for states with non-trivial six-momenta and/or winding numbers. In a sector where g rotates all three planes, the (twisted) states have neither six-momenta nor winding numbers. In a sector where g rotates only two planes, states do have momenta and winding numbers associated with the third plane, but if that third plane is rotated by h, then all states with non-trivial momenta and/or winding numbers are projected out of the trace. Only the sectors in which a plane is fixed by g and h simultaneously are sensitive to untwisted moduli. Thus while both b_a and Δ_a receive contributions from $N=1$ and $N=2$ sectors, only the $N=2$ sectors provide for moduli-dependence of the threshold corrections. In particular, for orbifolds with no $N=2$ sectors, such as those with $\mathbf{G}=\mathbf{Z_3}$ or $\mathbf{Z_7}$, Δ_a are completely independent of the untwisted moduli.

Let us focus on an $N=2$ supersymmetric (g,h) sector, or more precisely an orbit of such sectors under the action of the modular group $PSL(2,\mathbf{Z})$ on τ. All (g,h) in such an orbit act on the three complex planes in the following way:

$$g = \mathrm{diag}(\alpha_g, \alpha_g^*, 1), \qquad h = \mathrm{diag}(\alpha_h, \alpha_h^*, 1), \tag{2.7}$$

where the phases α_g, α_h are not simultaneously equal to 1. (As an example

⋆ This is true not only for the untwisted moduli of an orbifold, but for all moduli of any heterotic string vacuum with exactly $N=1$ spacetime supersymmetry.[21]

consider the $\mathbf{Z_4}$ orbifold, where $\theta = (i, i, -1)$ generates the $\mathbf{Z_4}$. It contains an N=2 orbit consisting of $(g, h) = (1, \theta^2), (\theta^2, 1), (\theta^2, \theta^2)$.) We now require for simplicity that $\mathbf{T}^6 = \mathbf{T}^4 \oplus \mathbf{T}^2$, where $\mathbf{T}^2$ refers to the third complex plane in eq. (2.7); also, if any translations (shifts) of $\mathbf{T}^6$ accompany g and h, they must not affect $\mathbf{T}^2$.† Otherwise we allow for arbitrary action of $\mathbf{G}$, on the $E_8 \times E_8$ or Spin(32)/Z_2 current algebra and even on the torus $\mathbf{T}^4$. Under these conditions the contribution of the $N = 2$ supersymmetric (g, h) orbit is equivalent to that of a toroidal compactification (on $\mathbf{T}^2$) of a vacuum with $N = 1$ supersymmetry in six dimensions. Therefore we now consider one-loop gauge-coupling corrections for such vacua.

2.2. Threshold Corrections for Toroidal Compactifications of Six-dimensional Theories

In this subsection we study one-string-loop contributions to g_a^{-2} for all four-dimensional $N = 2$ supersymmetric vacua that are toroidal compactifications of six-dimensional $N = 1$ supersymmetric vacua. The previous discussion motivates our interest in these vacua; however, we stress that the results in this subsection apply to arbitrary six-dimensional vacua, not just orbifolds. In ref. [18] it was shown that the β-function for such $N = 2$ supersymmetric theories can be extracted from an index calculation. However, the same analysis can also be used to determine the threshold effects. For completeness and consistency with our notation we repeat the necessary steps in appendix A, where we find that $\mathcal{B}_a$, defined in eq. (2.2), is given by

$$\mathcal{B}_a(\tau, \bar{\tau}) \;=\; \hat{Z}_{\text{torus}}(\tau, \bar{\tau}) \cdot \mathcal{C}_a(\tau)\,. \tag{2.8}$$

† It is possible to relax these conditions, but parts of the analysis and the end result then become significantly more complicated.

Here the factor

$$\hat{Z}_{\rm torus}(\tau,\bar{\tau}) \;\equiv \sum_{(p_L,p_R)\in\Gamma_{2,2}} q^{p_L^2/2}\bar{q}^{p_R^2/2} \tag{2.9}$$

is contributed by the zero modes of the two toroidal dimensions X^I, while the factor $\mathcal{C}_a(\tau)$, which accounts for all other string degrees of freedom, is a holomorphic function of τ. $\Gamma_{2,2}$ is an even self-dual (2,2)-dimensional Lorentzian lattice to be described below.

Now consider the behavior of $\mathcal{B}_a(\tau,\bar{\tau})$ under modular transformations of the world sheet. In ref. [16] $\tau_2\cdot\mathcal{B}_a(\tau,\bar{\tau})$ was derived from a regulated two-point correlation function on the world sheet. The correlation function of the Kac-Moody currents involved — $\langle J_a(\zeta)\cdot J_a(0)\rangle$ — is modular invariant, but the regulator that removes the double pole of this correlation function at $z\to 0$ is not. Consequently, $\tau_2\cdot\mathcal{B}_a$ is not quite modular invariant; *modular anomalies* of this kind have been discussed extensively in refs. [22], [23] and [18], though perhaps with a slightly different emphasis. On the other hand, the regulator term is proportional to k_a and is otherwise independent of the choice of a gauge group factor a; therefore, the differences $\frac{\tau_2}{k_{a_1}}\mathcal{B}_{a_1}-\frac{\tau_2}{k_{a_2}}\mathcal{B}_{a_2}$ are proportional to unregulated and hence modular-invariant correlation functions $\left\langle k_{a_1}^{-1}J_{a_1}(\zeta)\,J_{a_1}(0)-k_{a_2}^{-1}J_{a_2}(\zeta)\,J_{a_2}(0)\right\rangle$ and are modular invariant themselves.

Finally, consider the modular properties of $\mathcal{C}_a(\tau)$. Since $\tau_2\cdot\hat{Z}_{\rm torus}(\tau,\bar{\tau})$ is manifestly modular invariant, it follows from eq. (2.8) that the differences $k_{a_1}^{-1}\mathcal{C}_{a_1}(\tau)-k_{a_2}^{-1}\mathcal{C}_{a_2}(\tau)$ are modular invariant too. On the other hand, these differences are holomorphic functions of τ and are no more singular than q^0 as $q\to 0$ (*i.e.*, $\tau\to i\infty$) — there is no q^{-1} term because the $SL(2,\mathbf{C})$-invariant vacuum has $Q_a=0$. Under these circumstances, the theory of modular forms requires these functions to be constants,[24] which by eq. (2.3) must be equal to $\frac{b_{a_1}}{k_{a_1}}-\frac{b_{a_2}}{k_{a_1}}$ (note that $\hat{Z}_{\rm torus}(\tau=i\infty)=1$). As we already mentioned, only the differences $\frac{\Delta_{a_1}}{k_{a_1}} - \frac{\Delta_{a_2}}{k_{a_2}}$ are computed by eq. (2.1) in any case, so for the case of a six-dimensional supersymmetric string vacuum compactified on $\mathbf{T}^2$ that equation reduces to

$$\Delta_a \;=\; b_a \cdot \int_\Gamma \frac{d^2\tau}{\tau_2} \,\left(\hat{Z}_{\rm torus}(\tau,\bar\tau) \,-\, 1\right). \tag{2.10}$$

An immediate corollary of formula (2.10) is that Δ_a does not depend on any of the moduli of the $(c,\bar{c}) = (20,6)$ SCFT, but only on the moduli of $\mathbf{T}^2$. This is consistent with the $N = 2$ supersymmetry in four space-time dimensions, which allows gauge couplings to depend on the scalars that belong to vector multiplets but not on the scalars belonging to hypermultiplets.[25] Moduli of the $\bar{c} = 6$ SCFT give rise to massless scalar supermultiplets in six spacetime dimensions, which under further compactification on $\mathbf{T}^2$ yield massless hypermultiplets of the $N = 2, D = 4$ supersymmetry. On the other hand, the moduli of $\mathbf{T}^2$ belong to vector multiplets of the $N = 2$ supersymmetry; their vector partners are generated by the two world-sheet currents ∂X^I.

The dependence of $\hat{Z}_{\rm torus}$ and hence of Δ_a on the geometry of the two-torus is implicit in the definition of $\Gamma_{2,2}$. Given a constant background metric G_{IJ} (with inverse G^{IJ}) and a constant antisymmetric tensor B_{IJ} on $\mathbf{T}^2$, the lattice vectors (p_L, p_R) are given by[26]

$$p^I_{L,R} \;=\; \pm n^I + \tfrac{1}{2} G^{IJ} m_J - G^{IJ} B_{JK} n^K, \qquad m_I, n^I \in \mathbf{Z}, \tag{2.11}$$

and $p^2_{L,R} \equiv p^I_{L,R} G_{IJ} p^J_{L,R}$. (We have set $\alpha' = \frac{1}{2}$.) Following ref. [19], we group the four real degrees of freedom in $G_{IJ} = G_{JI}$ and $B_{IJ} = b\epsilon_{IJ}$ into two complex fields T and U,

$$T = T_1 + iT_2 = 2\left(b + i\sqrt{\det G}\right), \qquad U = U_1 + iU_2 = \left(G_{12} + i\sqrt{\det G}\right)/G_{11}\,. \tag{2.12}$$

Note that T is a (1,1) form for the orbifold $\mathbf{T}^6/\mathbf{G}$, while U is a (1,2) form. In terms of T and U,

$$\hat{Z}_{\rm torus}(\tau,T,U) = \sum_{m_{1,2},n^{1,2}\in\mathbf{Z}} e^{2\pi i\tau(m_1n^1+m_2n^2)} \times \\ \times \exp\left(-\frac{\pi\tau_2}{T_2U_2}\cdot\left|TUn^2+Tn^1-Um_1+m_2\right|^2\right). \tag{2.13}$$

At this point we can write an explicit formula for the one-loop threshold corrections Δ_a by simply substituting eq. (2.13) into eq. (2.10) and calculating the integral. The integral is performed in appendix B; the result is

$$\Delta_a(T,\overline{T},U,\overline{U}) = -b_a\cdot\log\left[\frac{8\pi e^{1-\gamma_E}}{3\sqrt{3}}\cdot T_2\left|\eta(T)\right|^4\cdot U_2\left|\eta(U)\right|^4\right] \tag{2.14}$$

(γ_E is the Euler-Mascheroni constant). Note that besides its obvious symmetry with respect to an exchange of the two complex moduli of the torus, eq. (2.14) is also invariant under (separate) $PSL(2,\mathbf{Z})$ modular transformations of T and of U.⋆ The latter invariance manifests itself via the modular properties of the η function and makes eq. (2.14) consistent with the identification of the moduli space of the toroidal compactification as[19]

$$\frac{(SU(1,1)/U(1))}{PSL(2,\mathbf{Z})}\times\frac{(SU(1,1)/U(1))}{PSL(2,\mathbf{Z})}. \tag{2.15}$$

Another noteworthy feature of the formula (2.14) for toroidal compactifications of supersymmetric six-dimensional string vacua is that it gives the same ratio $\Delta_a(T,U)/b_a$ for all gauge couplings a in the four-dimensional theory. Therefore, we can completely absorb the specific threshold corrections Δ_a

⋆ Both symmetries are shared by the partition function (2.13); in fact, $\tau_2\hat{Z}_{\rm torus}(\tau,T,U)$ is invariant under separate $PSL(2,\mathbf{Z})$ modular transformations of T, of U and of τ and under any permutations of τ, T and U.[19]

in the formula (1.5) into a redefinition of $M_{\rm GUT}$: If we replace the definition[16] $M^2_{\rm GUT}\equiv e^{1-\gamma_E}/6\sqrt{3}\pi\alpha'$ with a new definition

$$\tilde{M}_{\rm GUT} \;\equiv\; \frac{1}{4\pi\sqrt{\alpha'}}\cdot\frac{1}{\sqrt{T_2}\,|\eta(T)|^2}\cdot\frac{1}{\sqrt{U_2}\,|\eta(U)|^2}\,, \tag{2.16}$$

then formula (1.5) becomes

$$\frac{16\pi^2}{g_a^2(\mu)} \;=\; k_a\cdot\frac{16\pi^2}{g^2_{\rm GUT}} \;+\; b_a\cdot\log\frac{\tilde{M}^2_{\rm GUT}}{\mu^2}\,, \tag{2.17}$$

without any additional threshold corrections. From the Grand Unification point of view, this would be the most convenient definition of the GUT scale, even though the mass (2.16) itself has no physical meaning — there are no massive particles whose mass is equal or even proportional to $\tilde{M}_{\rm GUT}$. In this article, however, we are interested in the moduli-dependence of the gauge couplings and their differences at some fixed mass scale $\mu\ll M_{\rm GUT}$; for this purpose it is more convenient to use a moduli-independent definition of $M_{\rm GUT}$ and have explicit moduli-dependent threshold corrections such as (2.14).

2.3. Back to $N=1$ Orbifolds.

Having completed our excursion into $N=2$ string vacua, we now return to the main subject of this article, the study of the $N=1$ orbifolds. We have already mentioned that for orbifold groups $\mathbf{G}$ such as $\mathbf{Z_3}$ or $\mathbf{Z_7}$ that contain no non-trivial twists with unit eigenvalues, the threshold corrections Δ_a do not depend on the untwisted moduli of the orbifold $\mathbf{T}^6/\mathbf{G}$. Now consider an orbifold group such as $\mathbf{Z_4}$ in which some twists $g\in\mathbf{G}$ ($g\neq 1$) have unit eigenvalues, but all such twists leave unrotated the same complex plane in six dimensions. Then all twists with unit eigenvalues form a subgroup $\mathbf{G}'$ of $\mathbf{G}$ — the little group of the unrotated plane — and the complete set of the $N\geq 2$ supersymmetric sectors of the orbifold $\mathbf{T}^6/\mathbf{G}$ form an $N=2$ orbifold $\mathbf{T}^6/\mathbf{G}'$. Combining the results of subsections 2.1 and 2.2, we immediately find that in this case

$$\begin{aligned}\Delta_a &= \frac{|\mathbf{G}'|}{|\mathbf{G}|}\cdot\Delta'_a \ + \ c_a \\ &= -\frac{b'_a\,|\mathbf{G}'|}{|\mathbf{G}|}\cdot\left[\log\left(|\eta(T)|^4\ \mathrm{Im}\,T\right) \ + \ \log\left(|\eta(U)|^4\ \mathrm{Im}\,U\right)\right] \ + \ c_a\,,\end{aligned} \tag{2.18}$$

where the $b'_a\cdot g_a^3/16\pi^2$ are the β-functions of the $N=2$ supersymmetric theory corresponding to the $\mathbf{T}^6/\mathbf{G}'$ orbifold, T and U are the moduli of the two-torus $\mathbf{T}^2$ fixed by $\mathbf{G}'$, and the moduli-independent term c_a comprises the contributions of the $N=1$ sectors as well as the constant part of eq. (2.14). Note that U is not always a modulus of the $N=1$ orbifold $\mathbf{T}^6/\mathbf{G}$ — the requirement that the six-torus $\mathbf{T}^6$ should be symmetric with respect to the full orbifold group $\mathbf{G}$ (and not just $\mathbf{G}'\subset\mathbf{G}$) may fix the shape of $\mathbf{T}^2$ and thus the value of U. On the other hand, if the orbifold $\mathbf{T}^6/\mathbf{G}$ has untwisted moduli other than T and U, their VEVs do not affect the one-loop threshold corrections to the gauge couplings.

Formula (2.18) looks almost identical to the $N=2$ formula (2.14); however, we would like to highlight the following difference: The coefficients $b'_a\,|\mathbf{G}'|\,/\,|\mathbf{G}|$ in eq. (2.18) are related to the β-functions of the $N=2$ theory $\mathbf{T}^6/\mathbf{G}'$ but generally have very little to do with the β-functions of the $N=1$ theory $\mathbf{T}^6/\mathbf{G}$ itself. To be precise, if one writes the β-functions of an $N=1$ orbifold as sums over all the $(g,h)\in\mathbf{G}\times\mathbf{G}$ sectors, then the contributions of the $N=2$ sectors to b_a amount to exactly $b'_a\,|\mathbf{G}'|\,/\,|\mathbf{G}|$; however, while the $N=1$ sectors do not contribute to the moduli-dependence of the threshold corrections, they do contribute to the β-functions. Hence $b_a\neq b'_a\,|\mathbf{G}'|\,/\,|\mathbf{G}|$ and there is no reason for the b_a and b'_a to be proportional to each other.⋆ Therefore, for an $N=1$ orbifold the ratios Δ_a/b_a

⋆ And in fact for all specific $N=1$ orbifold models we have considered b_a are not proportional to b'_a. For example, for the symmetric $\mathbf{Z_4}$ orbifold whose gauge group is $E_8\times E_6\times SU(2)\times U(1)$ one has $(b_8,b_6,b_2,b_1)=(-90,78,54,342)$ while $(b'_8,b'_6,b'_2,b'_1)=(-60,84,84,252)$ (here the $U(1)$ charges are normalized according to $k=3$).

generally differ for different gauge couplings and no redefinition of $M_{\rm GUT}$ would reduce eq. (1.5) to the form (2.17).

In general, different $g \in \mathbf{G}$ with a unit eigenvalue may leave unrotated different complex planes of the six-torus; for example, the $\mathbf{Z_2} \times \mathbf{Z_2}$ orbifold has three $N = 2$ twists, $(+1, -1, -1)$, $(-1, +1, -1)$ and $(-1, -1, +1)$, each leaving a different complex plane invariant. The complete set of $N = 2$ twists form a union $\bigcup_i \mathbf{G}^i$ of the little groups of all the unrotated planes, and the subgroups $\mathbf{G}^i \subset \mathbf{G}$ are disjoint — $\mathbf{G}^i \cap \mathbf{G}^j = 1$ for $i \neq j$ — because no non-trivial twist $g \in \mathbf{G} \subset SU(3)$ can fix two planes at once. Therefore, the complete set of $N = 2$ supersymmetric sectors (g, h) of the orbifold $\mathbf{T}^6/\mathbf{G}$ is a disjoint union of sets of all twisted sectors of $N = 2$ orbifolds $\mathbf{T}^6/\mathbf{G}^i$, which leads us to the formula

$$\Delta_a \;=\; -\sum_i \frac{b_a^i\,|\mathbf{G}^i|}{|\mathbf{G}|}\cdot \left[\log\left(|\eta(T_i)|^4\ \mathrm{Im}\, T_i\right) \,+\, \log\left(|\eta(U_i)|^4\ \mathrm{Im}\, U_i\right)\right] \,+\, c_a \quad (2.19)$$

as a generalization of eq. (2.18). Here $b_a^i \cdot g_a^3/16\pi^2$ are the β-functions of the $N = 2$ orbifold $\mathbf{T}^6/\mathbf{G}^i$, and T_i and U_i are the moduli of the two-torus fixed by $\mathbf{G}^i$. As in the previous case of a single little group $\mathbf{G}'$, U^i may or may not be moduli of the $N = 1$ orbifold $\mathbf{T}^6/\mathbf{G}$. However, for an abelian point group $\mathbf{G}$, all T_i that appear in eq. (2.19) always survive as untwisted moduli of $\mathbf{T}^6/\mathbf{G}$; specifically, they are among the diagonal untwisted (1,1) moduli (in the basis that diagonalizes all $g \in \mathbf{G}$). On the other hand, some of the diagonal untwisted (1,1) moduli may fail to appear in eq. (2.19) if no non-trivial $g \in \mathbf{G}$ fixes the appropriate complex plane. The untwisted moduli of abelian orbifolds that do appear in eq. (2.19) are summarized in table 1. Note that each of those moduli spans a separate $\left(SU(1,1)/U(1)\right)/PSL(2,\mathbf{Z})$ component[19] of the orbifold's untwisted moduli space, which explains why eq. (2.19) is invariant with respect to separate $PSL(2,\mathbf{Z})$ modular transformations of every modulus that appears in it. The gauge groups and constants b_a^i appearing in eq. (2.19) will of course depend on how the twists act on the $E_8 \times E_8$ (or $SO(32)$) current algebra.

For non-abelian $N = 1$ supersymmetric orbifolds the relation between the untwisted moduli and the parameters T_i and U_i that appear in eq. (2.19) can get rather complicated. On the one hand, there are fewer independent untwisted moduli than in the abelian case; for example, $\mathbf{T}^6/\Delta(3\cdot 3^2)$ has only one untwisted modulus T — the breathing mode (the group $\Delta(3\cdot 3^2)$ is the semidirect product of $\mathbf{Z_3}$ and $\mathbf{Z_3^2}$). On the other hand, complex planes that are fixed by non-commuting elements of the orbifold group need not be orthogonal to each other. Hence there can be more than three terms in eq. (2.19), and the parameters T_i and U_i that appear in those terms can be different linear combinations of the untwisted moduli. Moreover, usually for at least some of the little groups $\mathbf{G}^i$ the six-torus $\mathbf{T}^6$ cannot be decomposed into a direct sum $\mathbf{T}^4 \oplus \mathbf{T}^2$ with the $\mathbf{T}^2$ component lying in the unrotated plane. In this case formula (2.14) does not apply to the $N = 2$ orbifold $\mathbf{T}^6/\mathbf{G}^i$ and eq. (2.19) loses its validity altogether. It should not be too hard to derive a more general formula that applies to orbifolds of non-decomposable six-tori, but we do not wish to do so here.

We conclude our analysis of the one-loop threshold corrections Δ_a in supersymmetric orbifolds with a comment that the right hand side of eq. (2.19) is not the real part of any holomorphic function $f_a^{\text{1-loop}}(T_i, U_i)$ of the untwisted moduli fields. This non-holomorphic behavior is completely unexpected from the point of view based on tree-level supergravity theories in four dimensions.⋆ However, beyond the tree level supersymmetry does not require holomorphic $f_a(\Phi)$ in gauge

⋆ Strictly speaking, functions $f_{ab}(\Phi)$ are completely holomorphic only in $N = 1$ theories. In $N = 2$ supergravity theories such as toroidal compactifications of supersymmetric six-dimensional theories mixing of the graviphoton with other vector fields adds non-holomorphic terms to f_{ab}. However, for an unbroken gauge group a, the dependence of f_a on complex fields neutral with respect to a — in particular, on the moduli — is holomorphic.

TABLE 1. Abelian $N = 1$ orbifolds and their untwisted moduli.

Orbifold	Generators	# of untwisted moduli		moduli in (2.18)
		$h^{1,1}$	$h^{1,2}$	
$\mathbf{Z_3}$	$(e^{2\pi i/3}, e^{2\pi i/3}, e^{2\pi i/3})$	9	0	none
$\mathbf{Z_4}$	$(i, i, -1)$	5	1	T_3, U_3
$\mathbf{Z_6}$	$(e^{\pi i/3}, e^{2\pi i/3}, -1)$	3	1	T_2, T_3, U_3
$\mathbf{Z'_6}$	$(e^{\pi i/3}, e^{\pi i/3}, e^{4\pi i/3})$	5	0	T_3
$\mathbf{Z_7}$	$(e^{2\pi i/7}, e^{4\pi i/7}, e^{8\pi i/7})$	3	0	none
$\mathbf{Z_8}$	$(e^{\pi i/4}, e^{3\pi i/4}, -1)$	3	1	T_3, U_3
$\mathbf{Z'_8}$	$(e^{\pi i/4}, e^{5\pi i/4}, i)$	3	0	T_3
$\mathbf{Z_{12}}$	$(e^{\pi i/6}, e^{5\pi i/6}, -1)$	3	1	T_3, U_3
$\mathbf{Z'_{12}}$	$(e^{\pi i/6}, e^{7\pi i/6}, e^{2\pi i/3})$	3	0	T_3
$\mathbf{Z_2} \times \mathbf{Z_2}$	$(-1, -1, 1)$ $(1, -1, -1)$	3	3	$T_1, U_1, T_2, U_2, T_3, U_3$
$\mathbf{Z_4} \times \mathbf{Z_2}$	$(i, -i, 1)$ $(1, -1, -1)$	3	1	T_1, T_2, T_3, U_3
$\mathbf{Z_6} \times \mathbf{Z_2}$	$(e^{\pi i/3}, e^{-\pi i/3}, 1)$ $(1, -1, -1)$	3	1	T_1, T_2, T_3, U_3
$\mathbf{Z'_6} \times \mathbf{Z_2}$	$(e^{\pi i/3}, e^{\pi i/3}, e^{4\pi i/3})$ $(1, -1, -1)$	3	0	T_1, T_2, T_3
$\mathbf{Z_3} \times \mathbf{Z_3}$	$(e^{2\pi i/3}, e^{4\pi i/3}, 1)$ $(1, e^{2\pi i/3}, e^{4\pi i/3})$	3	0	T_1, T_2, T_3
$\mathbf{Z_6} \times \mathbf{Z_3}$	$(e^{\pi i/3}, e^{5\pi i/3}, 1)$ $(1, e^{2\pi i/3}, e^{4\pi i/3})$	3	0	T_1, T_2, T_3
$\mathbf{Z_4} \times \mathbf{Z_4}$	$(i, -i, 1)$ $(1, i, -i)$	3	0	T_1, T_2, T_3
$\mathbf{Z_6} \times \mathbf{Z_6}$	$(e^{\pi i/3}, e^{5\pi i/3}, 1)$ $(1, e^{\pi i/3}, e^{5\pi i/3})$	3	0	T_1, T_2, T_3

theories with massless charged fermions, and in the next section we shall provide both a field-theoretical explanation of this phenomenon and a string-theoretical proof that this explanation extends to the orbifold case.

3. Non-Holomorphic Field-Dependence of Effective Gauge Couplings

3.1. General Theory.

In classical $N = 1$ locally supersymmetric theories in four dimensions, the complex functions (1.2) must be holomorphic functions $f_{ab}(\Phi)$ of the chiral scalar fields Φ^i. Naively, this theorem should apply to renormalized quantum field theories as well, including effective field theories describing light particles in the superstring's spectrum; however, our result (2.19) is inconsistent with any holomorphic $f_a^{\text{1-loop}}(T^i, U^i)$. In this section we re-examine the assumptions of the theorem and explain why it fails for *quantum* gauge theories with massless charged fields. This failure has nothing to do with string theory; instead, it arises from infrared divergences that are purely field-theoretical in nature.

The argument for holomorphic $f_{ab}(\Phi)$ is usually made in terms of superfields. Supersymmetric gauge invariance requires the action for the gauge superfields to be a chiral superspace integral

$$\int d^4x\, d^2\theta\, \hat{E}(x,\theta) \cdot f_{ab}(\Phi)\, W^{a\alpha}(x,\theta)\, W^b_\alpha(x,\theta) \quad + \text{ h.c.,} \tag{3.1}$$

where W^a_α is the gauge covariant superfield that includes $F^a_{\mu\nu}$ and $\hat{E}$ is the superspace analog of the vierbein's determinant. Viewed as a (composite) superfield, $f_{ab}(\Phi)$ must be chiral and therefore must be a holomorphic function of the chiral superfields Φ^i. Note that from the superfield point of view eq. (1.2) is not a definition of f_{ab} but a result of expanding the action (3.1) in terms of ordinary gauge and scalar fields (and their superpartners) and then identifying the gauge couplings and Θ angles in the expanded action.

It is not however necessary to use superfields to prove that $f_{ab}(\Phi)$ should be holomorphic in the classical case, and we would like to briefly review an argument that uses only the ordinary fields.* For simplicity, we concentrate on the gauge-singlet f_a and their dependence on neutral scalars such as moduli; extending this argument to the general case is fairly straightforward. Consider a three-point Green's function involving a massless gauge boson A^a_μ, a gaugino λ^b and a neutral fermion ψ^i. Supersymmetry relates this Green's function to that involving two gauge bosons and Φ^i — the scalar superpartner of ψ^i; in terms of Weyl fermions this relation can be written as

$$\mathcal{A}(A^a_\mu, A^b_\nu, \Phi^i) \;=\; \tfrac{i}{2\sqrt{2}} p_{2\tau} (\sigma^\tau \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\tau)^\beta_{\ \gamma} \cdot \mathcal{A}(A^a_\mu, \lambda^{b\beta}, \psi^i_\gamma). \tag{3.2}$$

(We assume that gauginos are normalized to the same metric as gauge bosons and ψ^i to the same metric as Φ^i; such normalization is always possible in a four-dimensional supersymmetric theory, even off-shell.) The tree-level Green's functions can be read directly from the effective Lagrangian; in particular, given eq. (1.1) for the bosonic terms with up to two space-time derivatives, we have

$$\begin{aligned}\mathcal{A}^{\rm tree}(A^a_\mu, A^b_\nu, \phi^i) \;=\; \delta_{ab} \biggl(& (p_1^\nu p_2^\mu - g^{\mu\nu} p_1 \cdot p_2) \cdot \frac{\partial g_a^{-2}}{\partial \phi^i} \\ & - \tfrac{i}{8\pi^2} \epsilon^{\rho\mu\tau\nu} p_{1\rho} p_{2\tau} \cdot \frac{\partial \Theta_a}{\partial \phi^i} \;+\; O(p^4) \biggr) ,\end{aligned} \tag{3.3}$$

where $p_{1,2}$ are the momenta of the two gauge bosons and the $O(p^4)$ term is contributed by the higher-derivative terms not included in (1.1). Inserting the

$\star$ One reason for using ordinary fields rather than superfields is that our string calculations are all performed in terms of ordinary fields.

most general gauge invariant and Lorentz invariant form of the Green's function $\mathcal{A}(A^a_\mu, \lambda^{b\beta}, \psi^i_\gamma)$ into eq. (3.2) and comparing with (3.3) leads to

$$\frac{\partial f_a^*}{\partial \Phi^i} \equiv \frac{\partial g_a^{-2}}{\partial \Phi^i} + \frac{i}{8\pi^2} \frac{\partial \Theta_a}{\partial \Phi^i} = 0. \tag{3.4}$$

The last equation, or rather its complex conjugate, means that f_a is a holomorphic function of the complex scalar fields Φ^i.

In the absence of infrared divergences, an almost identical argument can be applied to quantum field theories, at least perturbatively. The role of the effective Lagrangian is now played by the generating function Γ — the sum of all 1PI Feynman diagrams (for effective theories we should also include the diagrams that are 1PI with respect to the light fields, but not with respect to the heavy fields that are integrated out). This generating function is not polynomial in fields and their space-time derivatives, but it can be expanded into a convergent power series. Let us collect the bosonic terms in that series that involve at most two space-time derivatives (and which are not related by gauge invariance to terms with more derivatives); this should give us an expression just like eq. (1.1), simply because the latter is the generic expression.† Hence, in the low momentum limit the Green's function involving two gauge bosons and one neutral scalar has to look just like eq. (3.3), with some effective $g_a^{\rm eff}(\phi)$ and $\Theta_a^{\rm eff}(\phi)$ replacing their tree-level counterparts. From this point on, we proceed exactly as in the classical case: formula (3.2) applies whenever there is unbroken supersymmetry, eq. (3.4) follows, and $f_a^{\rm eff}$ has to be a holomorphic function of the complex scalar fields Φ^i.

The loophole in this argument is that expanding 1PI Feynman diagrams into a power series in the particles' momenta — a procedure necessary for interpreting

† Actually, the truly generic formula allows for a scalar field dependent gravitational constant, *i.e.*, the first term in eq. (1.1) should really be $R/2\kappa^2(\phi)$. However, this generalization has no effect on the issue at hand (the arguments of ref. [27] do not apply here).

Γ as a local effective Lagrangian — yields a series whose radius of convergence is given by the mass of the lightest particle with non-derivative interactions. For quantum gauge theories with massless charged particles this radius is zero, so there is no *local* effective Lagrangian at all. In such theories the zero-momentum effective $1/g_a^2$ cannot be defined because of infrared divergences; the running effective couplings $1/g_a^2(p^2)$ — defined at some off-shell momentum $p^2 \neq 0$ — are commonly used instead. If we could similarly define running effective $\Theta_a(p^2)$ and relate the field dependence of the running $1/g_a^2(p^2;\phi)$ and $\Theta_a(p^2;\phi)$ to the off-shell two-vector-one-scalar Green's functions with a formula similar to eq. (3.3), then we would have holomorphic running $f_a(p^2;\Phi)$ just as we had holomorphic zero-momentum $f_a(\Phi)$ before. (Strictly speaking, we would need off-shell supersymmetry to maintain eq. (3.2) off shell; however, this is not a problem in four dimensions.) We will see, however, that defining a running field-dependent $\Theta_a(p^2;\phi)$ is often impossible and this is precisely what in a supersymmetric theory allows a running $1/g_a^2(p^2;\phi)$ not to be the real part of a holomorphic function of Φ's.

Consider a field-dependent effective coupling such as $\Theta_a^{\text{eff}}(\phi)$. It is actually an infinite series of coupling constants $\Theta^{\text{eff}}_{a,i\ldots l}$ that appear as coefficients of the operators $\frac{i}{8\pi^2}F^a\tilde{F}^a\phi^i\cdots\phi^l$ in the effective Lagrangian of the theory. Each of the coefficients depends on the expectation values of the scalar fields, but *at zero momentum* the $\Theta^{\text{eff}}_{a,i\ldots l}(\langle\phi\rangle)$ are related to each other via

$$\Theta^{\text{eff}}_{a,i}(\langle\phi\rangle) \;=\; \frac{\partial}{\partial\langle\phi^i\rangle}\Theta^{\text{eff}}_{a}(\langle\phi\rangle)\,, \quad \Theta^{\text{eff}}_{a,ij}(\langle\phi\rangle) \;=\; \frac{1}{2}\frac{\partial}{\partial\langle\phi^i\rangle}\Theta^{\text{eff}}_{a,j}(\langle\phi\rangle)\,,\ \textit{etc.} \qquad (3.5)$$

which means that $\Theta_{a,i\ldots l}$ is simply the derivative $\frac{1}{n!}\partial^n\Theta_a/\partial\phi^i\cdots\partial\phi^l$. If a local effective Lagrangian does not exist, then the set of zero-momentum effective couplings $\Theta^{\text{eff}}_{a,i\ldots l}(\langle\phi\rangle)$ should be replaced by a set of running effective couplings $\{\Theta_{a,i\ldots l}\}(p^2;\langle\phi\rangle)$ (henceforth, curly brackets $\{\}$ will denote running couplings).

Because the classical analogues of these running couplings are derivatives, we often call them 'effective derivatives' or 'renormalized derivatives'

$$\tfrac{1}{n!}\{\partial^n\Theta_a/\partial\phi^i\cdots\partial\phi^l\}(p^2,\langle\phi\rangle)\ \equiv\ \{\Theta_{a,i\ldots l}\}(p^2;\langle\phi\rangle). \tag{3.6}$$

Our terminology and notations notwithstanding, running couplings (3.6) do not have to be derivatives of some running field-dependent $\{\Theta_a(\phi)\}(p^2)$ — if the running couplings do not obey eqs. (3.5), the field-dependent $\{\Theta_a(\phi)\}(p^2)$ cannot be consistently defined. Naturally, the same considerations apply to any other running field-dependent coupling such as $\{g_a^{-2}(\phi)\}(p^2)$, *etc.*

A priori, there is no reason why relations (3.5) should continue to hold at non-zero momenta (they hold at zero momenta because there is no difference between a zero-momentum external leg in a 1PI Feynman diagram and a vacuum insertion). What actually happens depends on the particular theory and the particular running field-dependent coupling under consideration. Specifically, we need answers to the following three questions:

- Are the running gauge couplings $\{g_a^{-2}\}(p^2;\langle\phi\rangle)$ and the running couplings $\{\partial g_a^{-2}/\partial\phi^i\}(p^2;\langle\phi\rangle)$ consistent with eqs. (1.7)?
- Are the running Θ angles $\{\Theta_a\}(p^2;\langle\phi\rangle)$ and the running axionic couplings $\{\Theta_{a,i}\}(p^2;\langle\phi\rangle)$ consistent with a similar relation? Actually, since the effective Θ angles cannot be obtained via Feynman diagrams (the effective $\{\Theta_a\}$ is the coefficient of the $\mathrm{tr}_a(F\tilde{F})$ operator in the effective Lagrangian, but that operator is a total space-time derivative), this question amounts to checking the integrability conditions

$$\frac{\partial}{\partial\langle\phi^j\rangle}\{\Theta_{a,i}\}(p^2;\langle\phi\rangle)\ \stackrel{?}{=}\ \frac{\partial}{\partial\langle\phi^i\rangle}\{\Theta_{a,j}\}(p^2;\langle\phi\rangle) \tag{3.7}$$

for the axionic couplings.

- What are the consequences of unbroken supersymmetry for the running $\{g_a^{-2}\}(p^2; \langle\phi\rangle)$ and related couplings and how do these consequences depend on the answers to the first two questions?

The last question, at least, can be answered generically. Eq. (3.2) is a direct consequence of unbroken supersymmetry and thus should hold at any order of perturbation theory or even non-perturbatively. The general form of the three-boson Green's function $\mathcal{A}(A_\mu^a, A_\nu^b, \phi^i)$ is constrained by the gauge invariance and Lorentz invariance to be just like eq. (3.3), except that the tree-level $\partial g_a^{-2}/\partial\phi^i$ and $\partial\Theta_a/\partial\phi^i$ are replaced by some momentum-dependent form factors. Let us identify those form factors as the running couplings $\{g_{a,i}^{-2}\}(p^2)$ and $\{\Theta_{a,i}\}(p^2)$; this amounts to a choice of the renormalization scheme. Eq. (3.2) further constrains the form of the bosonic Green's function $\mathcal{A}(A_\mu^a, A_\nu^b, \Phi^i)$; in terms of the form-factors $\{g_{a,i}^{-2}\}(p^2)$ and $\{\Theta_{a,i}\}(p^2)$ this constraint is

$$\{\partial g_a^{-2}/\partial\Phi^i\} \ + \ \tfrac{i}{8\pi^2}\{\partial\Theta_a/\partial\Phi^i\} \ = \ 0 \qquad (3.8)$$

(*cf.* eq. (3.4) for the classical case). In terms of $\{f_{a,i}\}$, eq. (3.8) and its complex conjugate become eqs. (1.8), as promised in the introduction.

Further consequences of unbroken supersymmetry for the running gauge couplings depend on the answers to our first two questions. Indeed, if eqs. (1.7) hold true (we believe this is generally the case), then eq. (3.8) and its complex conjugate give us the following formulæ for the running axionic couplings $\{\Theta_{a,i}\}$:

$$\{\partial\Theta_a/\partial\Phi^i\} \ = \ +8\pi^2 i\,\frac{\partial\{g_a^{-2}\}}{\partial\left\langle\Phi^i\right\rangle} \quad , \qquad \{\partial\Theta_a/\partial\overline{\Phi^\imath}\} \ = \ -8\pi^2 i\,\frac{\partial\{g_a^{-2}\}}{\partial\left\langle\overline{\Phi^\imath}\right\rangle} \ . \qquad (3.9)$$

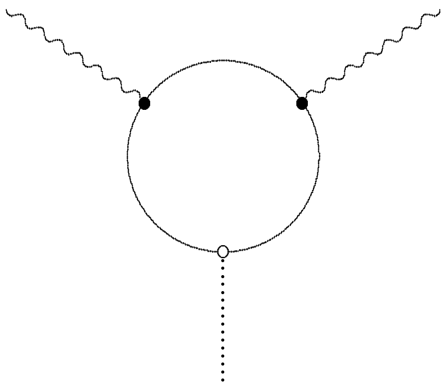
These formulæ are consistent with the integrability eqs. (3.7) if and only if

$$\frac{\partial}{\partial\left\langle\Phi^i\right\rangle}\,\frac{\partial}{\partial\left\langle\overline{\Phi^\jmath}\right\rangle}\,\{g_a^{-2}\}(p^2; \langle\Phi\rangle, \langle\overline{\Phi}\rangle) \ \equiv \ 0, \qquad (3.10)$$

that is, if and only if the dependence of the running $\{g_a^{-2}\}(p^2)$ on the scalar expectation values can be described by the real part of a holomorphic function of $\langle\Phi^i\rangle$. ($\frac{-1}{8\pi^2}\{\Theta_a\}$ will then be its imaginary part via eq. (3.9).) Thus we arrive at the following dichotomy for a supersymmetric theory: *either there is a running* $\{f_a(\Phi)\}(p^2)$ *that is a holomorphic function of the complex scalar fields and the running* $\{g_a^{-2}\}(p^2)$ *is its real part, or there is no well defined running* $\{\Theta_a(\phi)\}(p^2)$ *(i.e., eq. (3.7) does not hold) and the dependence of the running* $\{g_a^{-2}(\phi)\}(p^2)$ *on the scalar fields can be described by any real analytic function of* Φ^i *and* $\overline{\Phi^i}$. Clearly, it is the second alternative that is realized in the orbifold case. As we will see in the next subsection, the same phenomenon also occurs in renormalizable supersymmetric field theories.

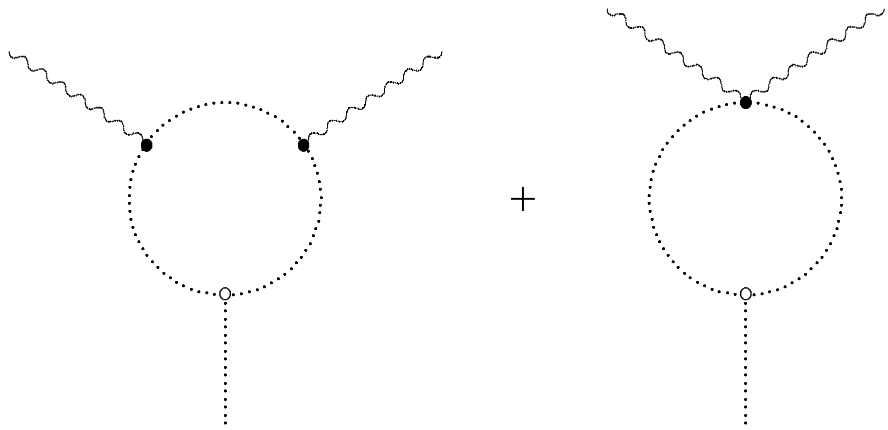
3.2. One Loop Results for Quantum Field Theories.

In this subsection we demonstrate that eqs. (3.7) often fail beyond the tree level while eqs. (1.7) continue to hold, at least to the one loop order. For simplicity, we only discuss renormalizable gauge theories and also disregard the possibility of partial gauge symmetry breakdown (*e.g.*, $SU(5)\mapsto SU(3)\times SU(2)\times U(1)$ at $M_{\rm GUT}$), although our results hold true in the general case as well. On the other hand, we allow the theories to be non-supersymmetric. Renormalizability requires $f_a^{\rm tree}=\rm const$ and also prohibits derivative couplings of scalars to each other (as in a σ-model) or to fermions. Thus there are only three kinds of one-loop Feynman diagrams contributing to the effective $\{\partial f_a/\partial\phi^i\}$:



(3.11)

where the internal propagators belong to charged fermions and the ∘ vertex is due to Yukawa couplings $\frac{1}{2}Y_{mni}\psi^m\psi^n\phi^i$ + c.c., and



(3.12)

where the internal propagators belong to charged scalars and the ∘ vertex is due to (superrenormalizable) three-scalar couplings $\frac{1}{6}T_{mni}\phi^m\phi^n\phi^i$ (here the external scalar field ϕ^i is neutral while the internal scalars ϕ^m and ϕ^n are charged). In terms of the field-dependent mass matrices $M_{mn}(\phi)$ for the charged fermions and $\mathcal{M}^2_{mn}(\phi)$ for the charged scalar fields, we have $Y_{mni} = \partial M_{mn}/\partial\phi^i$ and $T_{mni} = \partial\mathcal{M}^2_{mn}/\partial\phi^i$, so after computing the momentum integrals in diagrams (3.11) and (3.12), we arrive at the following results:

$$\begin{aligned}\{\partial g_a^{-2}/\partial\phi^i\}^{\text{1-loop}}(p^2;\langle\phi\rangle) \;&=\; \tfrac{-1}{24\pi^2}\,\mathrm{Tr}\left[Q_a^2\left(\frac{\partial M}{\partial\phi^i}\,M^\dagger \,+\, M\,\frac{\partial M^\dagger}{\partial\phi^i}\right)\frac{1}{MM^\dagger + O(p^2)}\right]\\ &\quad -\;\tfrac{1}{48\pi^2}\,\mathrm{Tr}\left[Q_a^2\,\frac{\partial\mathcal{M}^2}{\partial\phi^i}\,\frac{1}{\mathcal{M}^2 + O(p^2)}\right]\end{aligned} \tag{3.13}$$

and

$$\{\partial\Theta_a/\partial\phi^i\}^{\text{1-loop}}(p^2;\langle\phi\rangle) \;=\; \tfrac{-i}{2}\,\mathrm{Tr}\left[Q_a^2\left(\frac{\partial M}{\partial\phi^i}\,M^\dagger \,-\, M\,\frac{\partial M^\dagger}{\partial\phi^i}\right)\frac{1}{MM^\dagger + O(p^2)}\right]. \tag{3.14}$$

Here Q_a is a generator of the gauge group a, the traces are taken over Weyl fermions and complex scalars, and the precise form of the $O(p^2)$ expressions in the denominators is rather complicated but is irrelevant to our arguments. In case of a supersymmetric gauge theory, $\mathcal{M}^2 = MM^\dagger$ (gauginos can be excluded from our considerations here since they are massless and thus do not contribute to eqs. (3.13) and (3.14)) and the matrix elements of M are holomorphic functions of Φ^i, so formulæ (3.13) and (3.14) become

$$
\begin{aligned}
8\pi^2\{\partial g_a^{-2}/\partial\Phi^i\}^{\text{1-loop}} &= -i\{\partial\Theta_a/\partial\Phi^i\}^{\text{1-loop}} = \tfrac{-1}{2}\,\mathrm{Tr}\left[Q_a^2\,\frac{\partial M}{\partial\Phi^i}\,M^\dagger\,\frac{1}{MM^\dagger+O(p^2)}\right],\\
8\pi^2\{\partial g_a^{-2}/\partial\overline{\Phi^i}\}^{\text{1-loop}} &= +i\{\partial\Theta_a/\partial\overline{\Phi^i}\}^{\text{1-loop}} = \tfrac{-1}{2}\,\mathrm{Tr}\left[Q_a^2\,M\,\frac{\partial M^\dagger}{\partial\overline{\Phi^i}}\,\frac{1}{MM^\dagger+O(p^2)}\right],
\end{aligned}
\tag{3.15}
$$

in full agreement with eqs. (1.8). (In the supersymmetric case $\{f_{a,i}\}$ and $\{f_{a,\bar{\imath}}\}$ can also be computed using superfield Feynman rules,[28] and the result of this calculation is identical to eqs. (3.15).)

We now show that regardless of whether the theory in question is supersymmetric or not, the couplings (3.13) are integrable at any p^2 and furthermore are consistent with eq. (1.7). Indeed, to the one loop order the running effective gauge coupling is[29]

$$
\begin{aligned}
\{1/g_a^2\}^{\text{1-loop}}(p^2) \;=\; & \frac{1}{g^2_{a\,\text{bare}}} + \frac{11C_2}{48\pi^2}\log\frac{O(p^2)}{\Lambda^2}\\
& - \frac{1}{24\pi^2}\,\mathrm{Tr}\left(Q_a^2\,\log\frac{MM^\dagger+O(p^2)}{\Lambda^2}\right) - \frac{1}{48\pi^2}\,\mathrm{Tr}\left(Q_a^2\,\log\frac{\mathcal{M}^2+O(p^2)}{\Lambda^2}\right),
\end{aligned}
\tag{3.16}
$$

where Λ is the ultraviolet cutoff and C_2 is the second Casimir of the adjoint representation of the gauge group. Straightforward differentiation of this expression results in a formula for $\partial\{g_a^{-2}\}/\partial\langle\phi^i\rangle$ that looks identical to eq. (3.13), except for a possibly different specific form of the various $O(p^2)$ terms. However, the specific form of the $O(p^2)$ terms in eq. (3.16) depends on a particular choice of the renormalization scheme for the running gauge couplings. Similarly, in eq. (3.13) the exact definition of p^2, in terms of the momenta $p^2_{1,2,3}$ of the three particles involved, amounts to a choice of a renormalization scheme for the running $\{g^{-2}_{a,i}\}$ couplings. Therefore, given an appropriate choice of the renormalization schemes for the running couplings — both $\{g_a^{-2}\}(p^2)$ and $\{\partial g_a^{-2}/\partial\phi^i\}(p^2)$ — eq. (1.7) should hold exactly and for all renormalization scales p^2. In all other renormalization schemes eq. (1.7) holds whenever the particular values of the $O(p^2)$ terms are not important, that is, whenever the mass^2 of any charged particle is either much bigger than p^2 or much less than p^2. In particular, eq. (1.7) always holds in the infrared limit when $p^2 \ll$ mass^2 of the lightest massive charged fermion or scalar. (Note that in the infrared limit $\{g_a^{-2}\}(p^2 \to 0)$ diverge logarithmically, but their derivatives with respect to scalar fields ϕ^i remain finite. Correspondingly, $\{g^{-2}_{a,i}\}(p^2 = 0)$ do not suffer from infrared divergences.)

Now consider the running axionic couplings $\{\Theta_{a,i}\}(p^2; \langle\phi\rangle)$. (Like $\{g^{-2}_{a,i}\}(p^2)$, these couplings are infrared-convergent even at $p^2 = 0$.) At first glance eq. (3.14) looks just like the first line of eq. (3.13) except for the relative sign between the two terms in parentheses, but that minus sign is precisely what renders the axionic couplings non-integrable. Indeed, explicit differentiation of eq. (3.14) results in

$$\begin{aligned}\frac{\partial\{\Theta_{a,i}\}^{\text{1-loop}}(p^2;\langle\phi\rangle)}{\partial\langle\phi^j\rangle}\ &-\ (i\leftrightarrow j) \qquad (3.17)\\ &=\ i\,\mathrm{Tr}\left(\frac{O(p^2)}{M^\dagger M+O(p^2)}\cdot Q_a^2\,\frac{\partial M^\dagger}{\partial\phi^i}\,\frac{1}{MM^\dagger+O(p^2)}\,\frac{\partial M}{\partial\phi^j}\right)\ -\ (i\leftrightarrow j)\ ,\end{aligned}$$

and for non-zero p^2 the expression on the right hand side generally has no reason to vanish. In the zero momentum limit, the first factor in parentheses vanishes *provided the matrix $M^\dagger M$ is invertible.* In this case $\{\Theta_{a,i}\}(p^2 = 0)$ are integrable and result in

$$\{\Theta_a(\phi)\}^{\text{1-loop}}(p^2=0)\ =\ \mathrm{Tr}\left(Q_a^2\ \mathrm{Im}\log M(\phi)\right)\ +\ \mathrm{const} \qquad (3.18)$$

— the basic formula in the study of axions. (In the case of QCD eq. (3.18) becomes $\Theta = \mathrm{Arg}\det(M_{\text{quark}}) + \text{const.})$ Notice however that generally eq. (3.17) leads to integrable axionic couplings only for $p^2 \ll \text{mass}^2$ of the lightest charged fermion. If some of the fermions are exactly massless (*e.g.*, when the gauge symmetry is chiral), then the matrix $M^\dagger M$ is not invertible and the $O(p^2)/(M^\dagger M + O(p^2)$ factor retains some $O(1)$ matrix elements even in the zero momentum limit, and one should not expect $\{\Theta_{a,i}\}(p^2)$ to be integrable at any p^2, however small. Eq. (3.18) also yields ill defined Θ angles whenever some charged fermions are massless. This phenomenon is well known,[30] although it is usually explained in terms of the anomalous chiral symmetry of the fermions that shifts the Θ angle by an arbitrary amount and can even remove it altogether. Note that the non-integrability of $\{\Theta_{a,i}\}$ has nothing to do with supersymmetry since the phenomenon occurs in both supersymmetric and non-supersymmetric gauge theories.

At this point we would like to present a simple example of a gauge theory in which the axionic couplings are in fact non-integrable. Consider an E_6 gauge theory with two **27** families of chiral fermions and one $\overline{\mathbf{27}}$ antifamily. Let the fermion mass matrix be

$$M(\Phi) \;=\; \begin{pmatrix} 0 & 0 & \Phi^1 \\ 0 & 0 & \Phi^2 \\ \Phi^1 & \Phi^2 & 0 \end{pmatrix} \otimes I_{27} \,, \qquad (3.19)$$

where $\Phi^{1,2}$ are two complex E_6-singlet scalar fields. The first two rows or columns of this M correspond to the **27** fields and the third row/column corresponds to the $\overline{\mathbf{27}}$'s — apart from its specific Φ dependence, M is the most general mass matrix that is allowed by the gauge symmetry. In this model, there is always a massive $\mathbf{27} + \overline{\mathbf{27}}$ multiplet of fermions and a massless **27** multiplet, but the particular linear combination of the two **27** families that remains massless depends on the expectation values $\langle \Phi^{1,2} \rangle$. Applying formula (3.14) to this model yields

$$\begin{aligned} \{\partial\Theta/\partial\Phi^i\}^{\text{1-loop}}(p^2;\langle\Phi\rangle,\langle\overline{\Phi}\rangle) &= -3i\,\frac{\langle\overline{\Phi^\imath}\rangle}{|\langle\Phi^1\rangle|^2+|\langle\Phi^2\rangle|^2+O(p^2)}, \\ \{\partial\Theta/\partial\overline{\Phi^\imath}\}^{\text{1-loop}}(p^2;\langle\Phi\rangle,\langle\overline{\Phi}\rangle) &= +3i\,\frac{\langle\Phi^i\rangle}{|\langle\Phi^1\rangle|^2+|\langle\Phi^2\rangle|^2+O(p^2)}; \end{aligned} \tag{3.20}$$

although these axionic couplings have well-defined limits at zero momentum, they do not obey the integrability equations (3.7) even in that limit.

Finally, consider the effective gauge coupling in a supersymmetrized version of the same model. Given the spectrum of the theory and the fermionic masses (3.19), we have

$$\frac{1}{g^2(p^2;\langle\Phi\rangle,\langle\overline{\Phi}\rangle)} = \frac{1}{g_0^2}+\frac{33}{16\pi^2}\log\frac{O(p^2)}{\Lambda^2}-\frac{6}{16\pi^2}\log\frac{\left|\langle\Phi^1\rangle\right|^2+\left|\langle\Phi^2\rangle\right|^2+O(p^2)}{\Lambda^2}; \tag{3.21}$$

this $1/g^2$ is not the real part of any holomorphic function of Φ^1 and Φ^2. As we explained in the first half of this section, this non-holomorphicity is directly related to non-integrability of the axionic couplings (3.20); indeed, it is easy to see that eqs. (3.9) do hold for the model at hand.

The specific gauge group and the fermionic mass matrix we used in this example were rather arbitrary. It is easy to see that the same behavior occurs whenever some charged fermions are massive and some are massless, but which particular fermionic fields remain massless depends on the scalar expectation values. One can argue that this is exactly what happens in string vacua with moduli, where one has an infinite number of massive charged fermions, a finite number of massless charged fermions, and the vertices for these massless fermions are moduli-dependent. Thus we expect the string theory to lead to effective couplings that obey eqs. (1.7) and (1.8), but not eq. (3.7); consequently, $g_a^{\text{1-loop}}$

need not be given by the real parts of some holomorphic functions of the complex moduli scalars. In the next subsection we will see that this is exactly what happens in the orbifold case.

3.3. One-Loop Results for Orbifolds.

In the previous subsection we showed that in field theory the running gauge couplings $\{g_a^{-2}\}(p^2)$ and the three-field couplings $\{g_{a,i}^{-2}\}(p^2)$ are related to each other via eqs. (1.7) (at least to the one-loop order) while no such relation generally holds for the $\{\Theta_{a,i}\}$ couplings. Consequently, unbroken supersymmetry does not require the effective $\{g_a^{-2}(\phi)\}(p^2)$ to be the real parts of holomorphic functions of the complex scalar fields. In this subsection we demonstrate that exactly the same behavior occurs in string theory, at least in the orbifold case; this is the origin of non-holomorphicity found in eq. (2.19). Specifically, we are going to compute $\{g_{a,i}^{-2}\}^{\text{1-loop}}$ and $\{\Theta_{a,i}\}^{\text{1-loop}}$ for supersymmetric orbifolds and verify that eqs. (1.7) and (1.8) are obeyed (with the gauge couplings given by eq. (2.19)), but that the integrability conditions (3.7) for $\{\Theta_{a,i}\}^{\text{1-loop}}$ are not satisfied.

Our starting point is the CP-even three-particle scattering amplitude

$$\begin{aligned}
&\mathcal{A}^{\text{string}}_{\text{even}}(A^a_\mu, A^b_\nu, \phi^i) \qquad\qquad (3.22)\\
&\quad\equiv\ \delta_{ab}\,(p_1^\nu p_2^\mu - g^{\mu\nu} p_1\cdot p_2)\cdot\Big(\{g_{a,i}^{-2}\}^{\text{1-loop}}\ +\ (2\delta\Pi_a+\delta\Pi_i)^{\text{1-loop}}\{g_{a,i}^{-2}\}^{\text{tree}}\Big)\\
&\quad=\ \sum_{\text{even s}}(-)^{s_1+s_2}\int\limits_{\tau\in\Gamma} d^2\tau\, Z(\tau,\mathrm{s})\int d^2\zeta_1\int d^2\zeta_2\,\Big\langle V^0_{A^a_\mu}(\zeta_1)\,V^0_{A^b_\nu}(\zeta_2)\,V^0_{\phi^i}(0)\Big\rangle(\tau,\mathrm{s})\,,
\end{aligned}$$

where V^0_A and V^0_ϕ are the zero-picture[31] vertex operators for the gauge and scalar bosons, respectively, and $Z(\tau,\mathrm{s})\equiv\mathrm{Tr}_{s_1}\Big((-)^{s_2}q^{H-1}\bar{q}^{\bar{H}-1/2}\Big)$ are the partition functions for the even spin structures of the heterotic string. The odd (Ramond-Ramond) spin structure produces the CP-odd amplitude $\mathcal{A}^{\text{string}}_{\text{odd}}$; we will return to it later for computing the $\{\partial\Theta/\partial\phi\}$ couplings. $\mathcal{A}^{\text{string}}$ is a scattering amplitude and not a 1PI Green's function, therefore eq. (3.22) includes one-loop corrections $\delta\Pi_a$ and $\delta\Pi_i$ to the external legs of the amplitude. Fortunately, the tree-level couplings $\{g_{a,i}^{-2}\}^{\rm tree}$ vanish for all massless scalars Φ^i except the dilaton (*cf.* eq. (1.3)), so we do not have to actually compute one-string-loop corrections $\delta\Pi_a$ and $\delta\Pi_i$ (which is just as well since they diverge on shell). It is possible however that string loop corrections to scalar propagators cause mixing of the dilaton with other massless scalars; to avoid this potential problem, we henceforth limit our attention to the differences between $\frac{1}{k_a}\{f_{a,i}\}$ for different gauge couplings a; this is similar to computing only the differences between the threshold corrections $\frac{1}{k_a}\Delta_a$ in ref. [16] and in section 2 of this article.

Another peculiarity of the string-theoretical formula (3.22) is that it is valid only for on-shell momenta $p_1^2 = p_2^2 = p_3^2 = 0^\star$ and thus yields only $\{g_{a,i}^{-2}\}(p^2 = 0)$. One could with more effort compute $\{f_{a,i}\}^{\rm string}(p^2 \neq 0)$ from a four-particle amplitude such as $\mathcal{A}(A_\mu^a, A_\nu^b, \phi^i, {\rm graviton})$ instead of (3.22), but in this article we simply restrict our verification of eqs. (1.7) and (1.8) to the $p^2 = 0$ limit. Since the infrared-divergent term in eq. (1.5) is moduli-independent, we expect that $\{g_{a,i}^{-2}\}^{\rm string}(p^2 = 0)$ will be infrared-convergent, like the field-theoretical expression (3.13).

The actual evaluation of $\{\partial g_a^{-2}/\partial\phi^i\}^{\text{1-loop}}$ closely parallels the calculation of the threshold corrections in ref. [16] and their moduli-dependence in section 2 of this article. We begin with the vertex operators for the untwisted modulus ϕ^i and for the gauge bosons, which are

$$\begin{aligned} V_{\phi^i}^0(\zeta,\bar\zeta) &= \frac{v_{IJ}(\phi^i)}{2\pi}\cdot\partial X^I(\zeta)\cdot\Big(\bar\partial X^J + i(p\cdot\Psi)\Psi^J\Big)(\bar\zeta)\cdot e^{ip\cdot X(\zeta,\bar\zeta)}\,, \\ V_{A_\mu^a}^0(\zeta,\bar\zeta) &= \tfrac{1}{2\pi}\,J_a(\zeta)\cdot\big(\bar\partial X^\mu + i(p\cdot\Psi)\Psi^\mu\big)(\bar\zeta)\cdot e^{ip\cdot X(\zeta,\bar\zeta)}, \end{aligned} \tag{3.23}$$

$\star$ We interpret these mass-shell conditions as constraints on complex Euclidean momenta $p_{1,2,3}$; real Minkowski momenta that satisfy these constraints and also $p_1+p_2+p_3=0$ would be collinear, and that would cause amplitude (3.22) to vanish kinematically for transverse gauge bosons.

where

$$v_{IJ}(\phi^i) \;\equiv\; \frac{\partial}{\partial\phi^i}\,(G_{IJ}+B_{IJ}) \tag{3.24}$$

is a (c-number) matrix corresponding to a particular untwisted modulus ϕ^i and J_a are the Kac-Moody currents responsible for the gauge symmetry. Given these vertices and proceeding exactly as in ref. [16], we reduce formula (3.22) to

$$\{g_{a,i}^{-2}\} \;=\; \frac{v_{IJ}(\phi^i)}{32\pi^3}\int_\Gamma d^2\tau\,\mathcal{B}_a^{IJ}(\tau,\bar\tau) \;+\; k_a\cdot(a\text{-independent term}), \tag{3.25}$$

where

$$\mathcal{B}_a^{IJ}(\tau,\bar\tau) \;=\; |\eta(\tau)|^{-4}\cdot\sum_{\text{even s}}(-)^{s_1+s_2}\,\frac{dZ_\Psi(\mathbf{s},\bar\tau)}{2\pi i\,d\bar\tau}\times \\ \mathrm{Tr}_{s_1}\left(:\partial X^I\,\bar\partial X^J:\cdot\,Q_a^2\cdot(-)^{s_2F}q^{H-\frac{11}{12}}\bar q^{\bar H-\frac{3}{8}}\right)_{\text{int}} \tag{3.26}$$

— formulæ very similar to eqs. (2.1) and (2.2), except for a missing $1/\tau_2$ factor in eq. (3.25) (it is canceled by the extra integral $\int d^2\zeta\,=\,\tau_2$) and for the extra operator $:\partial X^I\,\bar\partial X^J:$ in the trace in eq. (3.26). The latter operator is normal ordered, so only the zero modes of the free bosons X^I and X^J contribute to its expectation value.

Next we proceed as in section 2.1 and rewrite the traces in eq. (3.26) as sums over the orbifold twist sectors:

$$\begin{aligned}
&\mathrm{Tr}_{s_1}\left(:\partial X^I\,\bar\partial X^J:\cdot\,Q_a^2\cdot(-)^{s_2F}q^{H-\frac{11}{12}}\bar q^{\bar H-\frac{3}{8}}\right)_{\text{int}} \\
&\quad=\;\frac{1}{|\mathbf{G}|}\sum_{\substack{g,h\in\mathbf{G}\\ gh=hg}}\mathrm{Tr}_{(g,s_1)}\left(:\partial X^I\,\bar\partial X^J:\cdot\,Q_a^2\cdot h\cdot(-)^{s_2F}q^{H-\frac{11}{12}}\bar q^{\bar H-\frac{3}{8}}\right)_{\text{int}} \\
&\quad=\;\frac{1}{|\mathbf{G}|}\sum_{\substack{g,h\in\mathbf{G}\\ gh=hg}}\mathrm{Tr}_{(g,s_1)}\left(Q_a^2\cdot h\cdot(-)^{s_2F}q^{H-\frac{11}{12}}\bar q^{\bar H-\frac{3}{8}}\right)_{\text{int}}\cdot\left\langle:\partial X^I\,\bar\partial X^J:\right\rangle(g,h).
\end{aligned} \tag{3.27}$$

The second equation here holds because in each separate (g,h) sector there is no correlation between the free bosons $X^{I,J}$ and other world-sheet degrees of freedom. The only sectors (g,h) in which these two bosons have zero modes (and thus the only sectors whose contributions to eq. (3.27) do not necessarily vanish) are the sectors in which X^I and X^J coordinates of the six-torus are invariant with respect to both g and h. All such sectors are $N=2$ supersymmetric and together they form an $N=2$ orbifold $\mathbf{T}^6/\mathbf{G}'$, where $\mathbf{G}'$ is the little group of X^I and X^J. Obviously, not all pairs (X^I, X^J) lead to non-trivial little groups $\mathbf{G}'$ (and the trivial case $\mathbf{G}'=1$ is just the $N=4$ supersymmetric untwisted sector that yields $\mathcal{B}_a^{IJ}=0$ as well as $\mathcal{B}_a=0$). It is easy to see that the untwisted moduli made from the (X^I, X^J) pairs that do lead to non-trivial $\mathbf{G}'$ are precisely the moduli that appear in eq. (2.19) — the same set of moduli listed in table 1 for abelian orbifolds.

At this point the problem of computing the $\{g_{a,i}^{-2}\}$ couplings for $N=1$ orbifolds has been reduced to the $N=2$ supersymmetric case in which ϕ^i is one of the moduli of the unrotated two-torus, and we can now repeat the arguments of section 2.2 and appendix A almost verbatim. This gives us the following formulæ (for the $N=2$ case)

$$\{g_{a,i}^{-2}\} \;=\; b_a \cdot \frac{v_{IJ}(\phi^i)}{8\pi} \int\limits_{\Gamma} d^2\tau\, \tilde{Z}^{IJ}(\tau,\bar\tau) \;+\; k_a \cdot (a\text{-independent term}), \qquad (3.28)$$

where

$$\tilde{Z}^{IJ}(\tau,\bar\tau) \;=\; \sum_{(p_L,p_R)\in\Gamma_{2,2}} q^{p_L^2/2}\bar{q}^{p_R^2/2}\cdot p_L^I p_R^J \qquad (3.29)$$

is the factor in $\mathcal{B}_a^{IJ}$ contributed by the zero modes of X^I and X^J. (As in the steps leading to eq. (2.10), modular invariance forces the internal trace to be a constant factor b_a in eq. (3.28).) For the $N=1$ supersymmetric orbifolds the factor b_a in eq. (3.28) becomes $b'_a|\mathbf{G}'|/|\mathbf{G}|$ for the appropriate $\mathbf{G}'$, and in eq. (3.29) $\Gamma_{2,2}$ is the invariant lattice of $\mathbf{G}'$ rather than $\mathbf{G}$.

Let us now go back to eqs. (2.9) and (2.11) and consider $\hat{Z}_{\rm torus}$ as a function of the four moduli of the fixed torus $\mathbf{T}^2$. It is a straightforward exercise to show that

$$\frac{\partial \hat{Z}_{\rm torus}(T,\overline{T},U,\overline{U})}{\partial\phi} \;=\; 2\pi\tau_2\,\tilde{Z}^{IJ}(T,\overline{T},U,\overline{U})\cdot\frac{\partial}{\partial\phi}\,(G_{IJ}+B_{IJ}) \;\equiv\; 2\pi\tau_2\,\tilde{Z}^{IJ}\cdot v_{IJ}(\phi), \tag{3.30}$$

where ϕ is any linear combination of T, U, $\overline{T}$ and $\overline{U}$. Therefore, combining eqs. (3.28) and (3.30) together, we can write[⋆]

$$\left\{\frac{\partial g_a^{-2}}{\partial\phi^i}\right\}^{\text{1-loop}} \;=\; \frac{b_a}{8\pi}\int\limits_{\Gamma}\frac{d^2\tau}{2\pi\tau_2}\,\frac{\partial\hat{Z}_{\rm torus}}{\partial\phi^i} \;=\; \frac{1}{16\pi^2}\,\frac{\partial\Delta_a}{\partial\phi^i} \tag{3.31}$$

(modulo a-independent terms), and this is precisely the formula (1.7) for supersymmetric orbifolds.

In order to confirm that eqs. (1.8) apply to supersymmetric orbifolds, we must calculate $\{\Theta_{a,i}\}^{\text{1-loop}}$ from the CP-odd part of the one-loop scattering amplitude involving two gauge bosons and one neutral scalar. This amplitude arises from the odd (Ramond-Ramond) spin structure on the world sheet torus and is therefore computed according to somewhat different rules[32] than the even amplitude (3.22). After eliminating the ghost degrees of freedom, we have

$$\begin{aligned}\mathcal{A}_{\rm odd}(A^a_\mu,A^b_\nu,\phi^i) \;&\equiv\; \frac{i\delta_{ab}}{8\pi^2}\,\epsilon^{\rho\mu\sigma\nu}p_{1\rho}p_{2\sigma}\cdot\{\partial\Theta_a/\partial\phi^i\}^{\text{1-loop}} \qquad (3.32)\\ &= \int\limits_{\tau\in\Gamma} d^2\tau\, Z'_{\rm RR}(\tau)\int d^2\zeta_1\int d^2\zeta_2\oint d\bar{\zeta}\,\left\langle \mathcal{T}_F(\bar{\zeta})\,V^0_{A^a_\mu}(\zeta_1)\,V^0_{A^b_\nu}(\zeta_2)\,V^{-1}_{\phi^i}(0)\right\rangle'_{\rm RR}\ ,\end{aligned}$$

where $\mathcal{T}_F$ is the fermionic stress tensor operator and V_ϕ^{-1} is the scalar vertex in

⋆ The second equation here is the derivative of eq. (2.10). Since the integral (2.10) does not converge uniformly, interchanging the order of differentiation and integration requires insertion of a regulator like that used in appendix B.

the (-1)-picture.[31] For an orbifold and its untwisted modulus ϕ we have

$$\begin{aligned} \mathcal{T}_F(\bar\zeta) &= g_{\mu\nu}\,\Psi^\mu(\bar\zeta)\cdot\bar\partial X^\nu(\bar\zeta) \;+\; G_{KL}\,\Psi^K(\bar\zeta)\cdot\bar\partial X^L(\bar\zeta)\,, \\ V_\phi^{-1}(\zeta,\bar\zeta) &= \frac{v_{IJ}(\phi)}{2\pi}\cdot\partial X^I(\zeta)\cdot\Psi^J(\bar\zeta)\cdot e^{ip\cdot X(\zeta,\bar\zeta)}, \end{aligned} \tag{3.33}$$

where G_{KL} is the metric for the six compact dimensions X^K and $v_{IJ}(\phi)$ is defined in eq. (3.24). The primes in eq. (3.32) refer to removal of the fermionic zero modes from the Ramond-Ramond partition function (which would otherwise vanish) into the product of vertex operators, which therefore has to supply a Ψ^μ or Ψ^I operator for every world sheet fermion that has a zero mode. In particular, the four Ψ^μ fermions always have zero modes in the Ramond-Ramond sector; the appropriate operators are contained in the gauge boson vertices (3.23) and together yield the $\epsilon^{\rho\mu\sigma\nu}p_{1\rho}p_{2\sigma}$ factor in the amplitude (3.32).

For an orbifold the different (g,h) sectors have different numbers of fermionic zero modes: In an $N=1$ supersymmetric sector only the four Ψ^μ have zero modes, in an $N=2$ sector two of the six Ψ^I fermions have them too, and in the $N=4$ sector all six Ψ^I have them. Consequently, the untwisted sector does not contribute to the amplitude (3.32) since only two Ψ^I operators can be supplied by the vertices V_ϕ^{-1} and $\mathcal{T}_F$ while six are needed to soak up the zero modes. On the other hand, in an $N=1$ supersymmetric sector the fermionic zero modes pose no problem, but the lack of zero modes for the bosonic operators ∂X^I (coming from the vertex V_ϕ^{-1}) and $\bar\partial X^L$ (coming from the $\mathcal{T}_F$ vertex) proves to be just as lethal. Indeed, these two operators can only be contracted with each other, but in the absence of zero modes the correlator $\left\langle\partial X^I\cdot\bar\partial X^L\right\rangle$ vanishes. Finally, the $N=2$ sectors of a supersymmetric orbifold can contribute to the amplitude (3.32), but only if both indices of the $v_{IJ}(\phi)$ matrix lie in the unrotated complex plane. Therefore, the problem again reduces to the case of an $N=2$ supersymmetric orbifold and only the moduli of the unrotated two-torus need to be considered.

(Of course, were the situation any different, this would be an immediate violation of eq. (1.8).)

The actual calculation of the couplings $\{\Theta_{a,i}\}$ for a toroidal compactification of a six-dimensional supersymmetric theory is fairly straightforward. The correlator in eq. (3.32) becomes a product of several independent expectation values

$$\langle (p_1\cdot\Psi)\Psi^\mu(p_2\cdot\Psi)\Psi^\nu\rangle' \cdot \langle J_a(\zeta_1)\,J_b(\zeta_2)\rangle \cdot v_{IJ}\left\langle \partial X^I\,\bar\partial X^L\right\rangle \cdot G_{KL}\left\langle \Psi^K\,\Psi^J\right\rangle', \quad (3.34)$$

and only the second factor here depends on the location of the vertices on the world sheet. Since we are only looking for the differences between $\frac{1}{k_a}\{\Theta_{a,i}\}$ for different gauge couplings a, we can replace $\langle J_a J_b\rangle$ with its zero-mode part $-4\pi^2\delta_{ab}\left\langle Q_a^2\right\rangle$ just as we did earlier in this article. After that, we simply evaluate all the factors in eq. (3.34) and the Ramond-Ramond partition function $Z'_{\rm RR}$; the result is

$$\{\Theta_{a,i}\} \;=\; \frac{v_{IJ}(\phi^i)\,\epsilon^{JK}G_{KL}}{\sqrt{\det G}}\cdot\int_\Gamma d^2\tau\,\pi\mathcal{C}_a(\tau)\,\tilde{Z}^{IL}(\tau,\bar\tau)\;+\;k_a\cdot(a\text{-independent term}), \quad (3.35)$$

where $\mathcal{C}_a(\tau)$ is defined in eq. (A.9) at the end of Appendix A. In the previous section we showed that $\mathcal{C}_a$ is a constant equal to b_a, so the integral in eq. (3.35) is identical to that in eq. (3.28). The matrix factor $v_{IJ}(\phi^i)\,\epsilon^{JK}G_{KL}/\sqrt{\det G}$ in eq. (3.35) differs from $v_{IL}(\phi^i)$ that appears in eq. (3.28), but eqs. (2.12) give us the following relations for the moduli T and U and their complex conjugates:

$$\frac{v_{IJ}(\phi)\cdot\epsilon^{JK}G_{KL}}{\sqrt{\det G}} \;=\begin{cases} +iv_{IL}(\phi) & \text{if } \phi \text{ is } T \text{ or } U,\\ -iv_{IL}(\phi) & \text{if } \phi \text{ is } \overline{T} \text{ or } \overline{U}.\end{cases} \quad (3.36)$$

Therefore, eq. (3.35) becomes

$$\{\partial\Theta_a/\partial\phi\} \;=\begin{cases} +8\pi^2 i\{\partial g_a^{-2}/\partial\phi\} & \text{if } \phi \text{ is } T \text{ or } U,\\ -8\pi^2 i\{\partial g_a^{-2}/\partial\phi\} & \text{if } \phi \text{ is } \overline{T} \text{ or } \overline{U},\end{cases} \quad (3.37)$$

which are precisely the eqs. (1.8) for the theory at hand.

As with the field theory case we discussed earlier in this section, non-holomorphicity of the one-loop corrections to the gauge couplings of a supersymmetric orbifold is related to the non-integrability of the axionic couplings $\{\Theta_{a,i}\}$. Indeed, using eqs. (3.9), whose applicability to orbifolds we have just confirmed, and eq. (2.14) for the moduli-dependence of the threshold corrections, we find that for example

$$\frac{\partial}{\partial\overline{T}}\left\{\partial\Theta_a/\partial T\right\}^{\text{1-loop}} = \frac{+ib_a}{8T_2^2} \quad \text{while} \quad \frac{\partial}{\partial T}\left\{\partial\Theta_a/\partial\overline{T}\right\}^{\text{1-loop}} = \frac{-ib_a}{8T_2^2}\,. \tag{3.38}$$

The fact that supersymmetric orbifolds and renormalizable gauge theories both exhibit non-integrability of $\{\Theta_{a,i}\}$ accompanied by non-holomorphicity of the gauge couplings strongly suggests that this behavior is rather common. We believe that the non-orbifold vacua of the heterotic string that give rise to massless charged fermions should also behave in the same manner, but a direct confirmation of this conjecture will require further research.

We conclude this section with a remark that the string scattering amplitudes (3.22) and (3.32) can also be used to compute $\{f_{ab,i}\}$ for twisted scalars ϕ^i in the orbifold's spectrum as well as for different kinds of string vacua. In those cases the full analytic structure of f_{ab} cannot be deduced, but even the knowledge of its first derivative would be of interest since it is precisely $\{f_{ab,i}\}$ which enter eq. (1.4). In this article we have nothing more to say about non-orbifold vacua, but for the orbifolds it is very easy to show that $\{f_{ab,i}\} = 0$ for all twisted scalars ϕ^i. Indeed, the discrete symmetry of an orbifold forbids any scattering process involving one twisted particle plus any number of particles arising from the untwisted sector; this selection rule is independent of the number of string loops. In particular, since the massless gauge bosons always belong to the untwisted sector, for any scalar ϕ^i arising from any twisted sector of an orbifold, $\mathcal{A}(A_\mu^a, A_\nu^b, \phi^i)$ and hence $\{f_{ab,i}\}$

must vanish. Of course, this argument does not apply to blown-up orbifolds (points in the full moduli space where twisted moduli have acquired vacuum expectation values).

4. Conclusion

The main result of this article is formula (2.19), expressing the moduli-dependence of threshold corrections Δ_a for $N = 1$ supersymmetric orbifold vacua of the heterotic string. We conclude this article by comparing our result with other calculations[7,8,27,33,34] of the same quantities in related four-dimensional $N = 1$ supersymmetric vacua.* Reference [7] considered the four-dimensional effective field theory obtained by truncating the ten-dimensional effective theory, and found that a moduli-dependent one-loop contribution to f_a arose from the ten-dimensional Green-Schwarz anomaly cancelling term. The truncation procedure used (previously outlined in ref. [3]) corresponds roughly to the untwisted sector of an orbifold compactification. However, in an actual four-dimensional string vacuum, such as an orbifold, truncation is not legitimate at the loop-level because it omits the contributions of an infinite number of states which can propagate in the loops (both twisted states and winding states in the orbifold case).

Another approach to determine the moduli-dependence of $f_a(T_i)$ was to use the classical Peccei-Quinn symmetries[3,5,8] of the moduli T_i for a Calabi-Yau manifold or an orbifold, which have the form $\mathrm{Re}\, T_i \to \mathrm{Re}\, T_i + \mathrm{const}$, together with the requirement that $f_a(T_i)$ be holomorphic. The resulting f_a found was linear,[7,8]

$$f_a^{\text{1-loop}}(T_i) \;=\; \tfrac{-i}{16\pi^2} A_a^i\, T_i, \qquad\qquad \text{or} \quad \Delta_a \;=\; A_a^i \,\mathrm{Im}\, T_i, \tag{4.1}$$

but the constants A_a^i could not be determined from the symmetry considerations.

⋆ While this paper was typed, we received several preprints[35] that appear to overlap with refs. [27,34].

Later it was realized that the Peccei-Quinn symmetries are spoiled by world-sheet instantons,[36] although the instanton effects are exponentially suppressed in the large radius limit, $\mathrm{Im}\, T \to \infty$, where T denotes the breathing mode of the internal manifold (the overall radius).[†] Thus it comes as no surprise that our result (2.19) for an orbifold vacuum reduces to formula (4.1) in the large-radius limit. Indeed, the leading term in $\log(|\eta(T_i)|^4\, \mathrm{Im}\, T_i)$ in the $\mathrm{Im}\, T_i \to \infty$ limit is $-(\pi/3)\ \mathrm{Im}\, T_i$, so this term in eq. (2.19) yields eq. (4.1) and fixes the constants to be $A_a^i = (\pi/3) b_a^i |\mathbf{G}^i|/|\mathbf{G}|$ for the orbifold case. The next-to-leading term, $\log(\mathrm{Im}\, T_i)$, violates the holomorphicity assumption for f_a, although it is consistent with the Peccei-Quinn symmetries. The moduli-independent contributions are obviously consistent with both properties. Finally, the terms generated by Taylor expanding the $\prod_{n=1}^{\infty}(1-q^n)$ factor in $\eta(T)$ are consistent with holomorphicity but violate the Peccei-Quinn symmetries; these exponentially-suppressed terms represent world-sheet instanton contributions.

Reference [33] considered the large-radius limit of the threshold corrections in more detail, and showed that in some cases — *e.g.* the hidden E_8 left intact in some Calabi-Yau compactifications — the constant of proportionality A_a for $f_a(T)$ in eq. (4.1) could be related to the β function coefficient $b_a^{(N=1)}$. It was recognized that this relation did not have to hold in more complicated compactifications, for example if Wilson lines broke the hidden E_8, or for orbifolds — in the latter case the lack of any such relation is confirmed by the large-radius limit of formula (2.19).

Finally, references [27] and [34] combined the form of a potential gluino condensate with duality invariance in order to constrain the possible form of f_a for orbifolds, using earlier results of ref. [37]. Under the assumption of a large-radius

† In the approximation considered in refs. [7,8], T was the only modulus present in the massless spectrum.

behavior similar to eq. (4.1) it was found that

$$\Delta_a \;=\; -b_a^{(N=1)} \cdot \log\left(|\eta(T)|^4\,\mathrm{Im}\,T\right), \tag{4.2}$$

which bears a striking resemblance to eq. (2.19). However, the result (4.2) has two serious problems: (1) The overall breathing mode T appears instead of the T_i that correspond to individual complex planes. (2) The β function coefficient appearing as a prefactor, $b_a^{(N=1)}$, is that computed from the $N=1$ supersymmetric massless spectrum, and not the b_a^i computed from the auxiliary $N=2$ theories defined above. In particular, both papers are in contradiction with the $\mathbf{Z_3}$ and $\mathbf{Z_7}$ orbifold examples, for which Δ_a should not depend on any of the untwisted moduli. Both papers employ as part of their analysis an effective supergravity Lagrangian describing gaugino condensation; it is possible that the problem arises at this stage.* Both papers attempt to explain the origin of the non-holomorphicity of the $f_a(T,\bar{T})$ deduced from (4.2). In ref. [27] the superconformal compensator field for supergravity plays a role in the explanation. However, the explanation cannot account for the more complicated non-holomorphic behavior of eq. (2.19); furthermore, we have shown that similar behavior occurs in theories not coupled to gravity. Reference [37] proposes a redefinition of M_{GUT} similar to eq. (2.16) in order to explain away the apparent non-holomorphicity of eq. (4.2). We have already explained in section 2.3 that such a redefinition is not possible for $N=1$ orbifolds, but even if it were possible, it would not help: Supersymmetry is concerned with holomorphicity of the whole f_a, and the $\log M_{\mathrm{GUT}}^2$ term in eq. (1.5) must be considered along with the threshold corrections Δ_a when M_{GUT} is moduli-dependent.

What can be said about higher-loop threshold corrections? In reference [8] the Peccei-Quinn symmetry for the dilaton/axion field S, plus the dilaton's role

$\star$ Some subtleties in the effective supergravity approach will be discussed in ref. [12].

as the string loop expansion parameter, were combined with a holomorphicity requirement on f_a in order to argue that all higher-loop corrections to f_a vanish. A similar conjecture was made in ref. [18] based on the relationship of f_a with the anomaly cancelling term. Serious doubt is cast on these arguments by the nonholomorphicity of f_a found in this paper, even at the one-loop level. Also, because of the chiral anomaly, the Peccei-Quinn symmetry for S is equivalent to an R symmetry[4] and one must be rather careful about the exact form of the symmetry beyond one-loop; this is related to the "multiplet of anomalies" discussed in refs. [38].

This paper also explained why the threshold corrections Δ_a do not have to be the real parts of holomorphic functions of the moduli, in contrast to naive expectations based on supersymmetry. The explanation is important for understanding the fermionic terms in the effective supergravity Lagrangian with bosonic terms (1.1). The functions f_{ab} appear in several places in the Lagrangian,[2] but the form in which they appear is either $f_{ab,i}$ or $\mathrm{Re}\, f_{ab}$ — the only place $\mathrm{Im}\, f_{ab}$ appears undifferentiated is where it multiplies the total derivative $F^a_{\mu\nu}\tilde{F}^{b\mu\nu}$ in (1.1). We showed in section 3 how the effective derivatives $\{g^{-2}_{ab,i}\}$ and $\{\Theta_{ab,i}\}$ could be well-defined at the quantum level, even if the angles $\Theta_a = -8\pi^2\, \mathrm{Im}\, f_a$ were ill-defined. Therefore, despite the nonholomorphicity of f_{ab} and the ambiguity in its imaginary part, loop corrections to the fermionic terms in the effective supergravity Lagrangian containing f_a are completely unambiguous, and using eqs. (1.7) and (1.8) they can all be expressed in terms of Δ_a and its derivatives.

As mentioned in the introduction, the moduli-dependence of f_a can become particularly important if the running coupling $g_a(p^2)$ becomes strong, both because it has an $O(1)$ effect on the mass scale generated by dimensional transmutation, and because it appears in the supergravity Lagrangian along with gaugino bilinear operators that can condense at that mass scale. As a result, a potential may develop that fixes the moduli to specific values.[34,12] Thus the Δ_a we have computed in this paper could help generate through nonperturbative effects a mass for the moduli (which remain massless in perturbation theory), and they could also help fix those parameters of the low-energy theory (Yukawa couplings, *etc.*) that depend on the moduli expectation values.

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APPENDIX A

In this appendix we simplify $\mathcal{B}_a(\tau,\bar\tau)$ of eq. (2.2) for toroidal compactifications of six-dimensional N=1 supersymmetric theories. This calculation essentially repeats the analysis performed in ref. [18].

Any compactification on $\mathbf{T}^2$ of a supersymmetric six-dimensional heterotic string vacuum has an internal SCFT that splits into two noninteracting pieces, with $(c,\bar c)=(2,3)$ and $(c,\bar c)=(20,6)$.[39] The $\bar c=3$ piece is represented by two coordinates X^i for $\mathbf{T}^2$, plus their right-moving fermionic partners Ψ^i whose spin structure is summed over together with that for Ψ^μ. For a given spin structure, the partition function for this $\bar c=3$ piece is therefore

$$Z_\Psi(\mathbf{s},\bar\tau)\cdot|\eta|^{-4}\hat Z_{\rm torus}(\tau,\bar\tau)\,, \tag{A.1}$$

where

$$\hat Z_{\rm torus}(\tau,\bar\tau)\ \equiv \sum_{(p_L,p_R)\in\Gamma_{2,2}} q^{p_L^2/2}\bar q^{p_R^2/2} \qquad \text{(A.2)\&(2.9)}$$

gives the contribution of the X^i zero modes.

Unlike the $\bar c=3$ SCFT, the $\bar c=6$ SCFT is not completely determined; however, it has an $N=4$ superconformal symmetry which contains in particular an $SU(2)$ Kac-Moody algebra at level 1. The spin structure sum couples only[40] to the free boson H that provides a Frenkel-Kac construction of that algebra.* This information will allow us to rewrite the sum over even spin structures in eq. (2.2) in terms of the odd spin structure, *i.e.* as an index in the Ramond sector. Identities relating sums over even spin structures to the odd spin structure have been studied extensively in the literature,[22,18] so we will be very brief in the following.

First consider the spin-structure-dependent piece of the partition function. It is proportional to

$$\sum_{\text{even } \mathbf{s}} (-)^{s_1+s_2} Z_\Psi^2(\mathbf{s},\bar\tau)\cdot Z_{SU(2)}(\mathbf{s},r,\bar\tau)\,, \tag{A.3}$$

where both factors can be written in terms of the characters χ_0 and χ_1 for the two $SU(2)$ level 1 representations, with isospin 0 and $\frac{1}{2}$ respectively:

$$\begin{aligned} Z_\Psi^2(0,0) &= \bar\eta^{-2}\sum_{m_1,m_2\in\mathbf{Z}} \bar q^{\frac{1}{2}(m_1^2+m_2^2)} = \bar\chi_0^2+\bar\chi_1^2\,,\\ Z_\Psi^2(0,1) &= \bar\eta^{-2}\sum_{m_1,m_2\in\mathbf{Z}} (-)^{m_1+m_2}\bar q^{\frac{1}{2}(m_1^2+m_2^2)} = \bar\chi_0^2-\bar\chi_1^2\,,\\ Z_\Psi^2(1,0) &= \bar\eta^{-2}\sum_{m_1,m_2\in\mathbf{Z}} \bar q^{\frac{1}{2}((m_1+\frac{1}{2})^2+(m_2+\frac{1}{2})^2)} = 2\bar\chi_0\bar\chi_1\,, \end{aligned} \tag{A.4}$$

and

$$\begin{aligned} Z_{SU(2)}(0,s_2,r) &= (-)^{s_2 r}\cdot\bar\eta^{-1}\sum_{n\in\mathbf{Z}}\bar q^{(n+\frac{1}{2}r)^2} = (-)^{s_2 r}\bar\chi_r\,,\\ Z_{SU(2)}(1,s_2,r) &= (-)^{s_2 r}\cdot\bar\eta^{-1}\sum_{n\in\mathbf{Z}}\bar q^{(n+\frac{1}{2}(1-r))^2} = (-)^{s_2 r}\bar\chi_{1-r}\,. \end{aligned} \tag{A.5}$$

(The four-real-fermion partition functions Z_Ψ^2 can be expressed in terms of the

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⋆ The same right-moving structure holds for any $N=2$ supersymmetric four-dimensional vacuum[39]; however, the zero-modes and left-moving parts of the two X^i (if they exist) will generally couple to the $\bar c = 6$ system.

$SU(2)$ characters because $SO(4) = SU(2) \otimes SU(2)$.) Here $r = 0, 1$ accounts for the two different types of spectral flow orbits[41,22] that can appear in the $N = 4$ superconformal theory. The vanishing of the partition function is due to

$$(\chi_0^2 + \chi_1^2) \cdot \chi_r - (\chi_0^2 - \chi_1^2) \cdot (-)^r \chi_r - (2\chi_0\chi_1) \cdot \chi_{1-r} \;=\; 0 \qquad \text{for } r = 0, 1. \quad \text{(A.6)}$$

To calculate $\mathcal{B}_a(\tau, \bar{\tau})$ we must replace $Z_\Psi^2(\mathbf{s}, \bar{\tau})$ in eq. (A.3) with

$$Z_\Psi(\mathbf{s}, \bar{\tau}) \cdot \frac{d}{d\bar{\tau}} Z_\Psi(\mathbf{s}, \bar{\tau}) \;=\; \tfrac{1}{2} \cdot \frac{d}{d\bar{\tau}} Z_\Psi^2(\mathbf{s}, \bar{\tau}).$$

The spin-structure-dependent piece is now proportional to the complex conjugate of

$$\begin{aligned}(\dot{\chi}_0\chi_0 + \dot{\chi}_1\chi_1) \cdot \chi_r \;-\; (\dot{\chi}_0\chi_0 - \dot{\chi}_1\chi_1) \cdot (-)^r \chi_r \;-\; (\dot{\chi}_0\chi_1 + \dot{\chi}_1\chi_0) \cdot \chi_{1-r} \qquad &\text{(A.7)} \\ = \; (\dot{\chi}_1\chi_0 - \dot{\chi}_0\chi_1) \cdot (-)^r \chi_{1-r} \qquad \text{for } r = 0, 1, &\end{aligned}$$

where a dot denotes $d/d\tau$. The identity $\dot{\chi}_1\chi_0 - \dot{\chi}_0\chi_1 = \frac{1}{2}\pi i \eta^4$ gives a factor of $\bar{\eta}^4$ in $\mathcal{B}_a$ that cancels the $\bar{\eta}^{-4}$ from the right-moving oscillator excitations of the bosons X^μ and $X, \overline{X}$. The remaining factor of $(-)^r \chi_{1-r}$ allows us to interpret the $\bar{c} = 6$ part of the result as a trace in the Ramond sector of the superconformal theory, with the operator $(-)^r \equiv (-)^F$ inserted. That is,

$$\mathcal{B}_a(\tau, \bar{\tau}) \;=\; \hat{Z}_{\text{torus}}(\tau, \bar{\tau}) \cdot \mathcal{C}_a(\tau, \bar{\tau}) \,, \qquad \text{(A.8)}$$

where

$$\mathcal{C}_a(\tau, \bar{\tau}) \;\equiv\; \eta(q)^{-4} \cdot \text{Tr}_R \left\{ \tfrac{1}{2} (-)^F \cdot Q_a^2 \cdot q^{H - \frac{5}{6}} \bar{q}^{\bar{H} - \frac{1}{4}} \right\}_{(c, \bar{c}) = (20, 6)} . \qquad \text{(A.9)}$$

Massive fermions are not chiral; this is just as true in six space-time dimensions as in four. Since the operator $(-)^{F_{\text{int}}}$ determines the chirality of a space-time fermion, only massless fermions contribute to the trace in eq. (A.9). Since all massless fermions have $\bar{h}_{\rm int} = 1/4$, their contribution does not depend on $\bar{q}$. Therefore, $\mathcal{C}_a$ does not depend on $\bar{\tau}$ and is a holomorphic function of τ. To make this argument rigorous, consider the zero mode $\bar{G}_0$ of one of the four world-sheet supersymmetry generators $\bar{G}^A(\bar{z})$ for the $\bar{c} = 6$, $N = 4$ SCFT. For every state $|R\rangle$ in the Ramond sector of the Hilbert space of that SCFT, either $\bar{G}_0\,|R\rangle = 0$, or the states $\bar{G}_0\,|R\rangle$ and $|R\rangle$ have opposite values of $(-)^{F_{\rm int}}$. Hence, only the states annihilated by the $\bar{G}_0$ contribute to the trace in eq. (A.9). But $\bar{G}_0^2 = \bar{H}_{(20,6)} - \frac{1}{4}$, so all states that contribute to the trace in eq. (A.9) yield contributions that do not depend on $\bar{\tau}$ and $\mathcal{C}_a(\tau)$ is a holomorphic function.⋆

APPENDIX B

This appendix contains the calculation of the integral (2.10). We start by Poisson resumming the sum on $m_{1,2}$ in eq. (2.13). We then reinterpret the resulting sum on four integers $n^{1,2}$ and $l^{1,2}$ (the latter replace $m_{1,2}$) as a sum over all integral two-by-two matrices; this sum can be written as

$$\tau_2 \cdot \hat{Z}_{\rm torus}(\tau, T, U) \;=\; \sum_{A \in {\rm Mat}(2\times 2, \mathbf{Z})} e^{-2\pi i T\,\det(A)} \cdot T_2 \exp\left(\frac{-\pi T_2}{\tau_2 U_2} \left| (1, U) A \begin{pmatrix} \tau \\ 1 \end{pmatrix} \right|^2 \right). \tag{B.1}$$

Now consider two matrices A and A' related to each other by a unimodular factor — $A' = A \cdot V$, with $V \equiv \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbf{Z})$. The contributions of these two matrices to the sum (B.1) are related by the modular transformation $\tau' = \frac{a\tau + b}{c\tau + d}$. The integral we are seeking is the integral of $\tau_2 \hat{Z}_{\rm torus} - \tau_2$ with the modular invariant measure $d^2\tau/\tau_2^2$, so instead of integrating the contribution of the matrix

⋆ If only physical states contributed to $\mathcal{C}_a$, it would be a constant rather than a power series in q. Terms with positive powers of q are contributed by non-physical states with $\bar{h} = 0$ and $h =$ positive integer.

A' over the fundamental domain Γ we can integrate the contribution of A over $V\Gamma$ — the image of Γ under the $PSL(2,\mathbf{Z})$ modular transform associated with $V \in SL(2,\mathbf{Z})$. Our strategy is therefore to partition the set of all matrices A into orbits of the group $SL(2,\mathbf{Z})$, pick a representative element A_0 in each orbit and integrate its contribution over the union of $V\Gamma$ for all $V \in SL(2,\mathbf{Z})$ that yield distinct $A \equiv A_0 V$.

The group $SL(2,\mathbf{Z})$ has three types of orbits in the space $GL(2,\mathbf{Z})$:

1) The zero orbit, consisting of a single matrix $A = 0$.

2) Non-degenerate orbits, consisting of matrices with non-zero determinants; for these orbits, $V' \neq V''$ implies $A_0 V' \neq A_0 V''$. Consequently, we integrate the contribution of a representative matrix A_0 over the union of $V\Gamma$ for all $V \in SL(2,\mathbf{Z})$; this union is $2 \times \{\tau \in \mathcal{C} : \tau_2 > 0\}$ — the double cover of the upper half plane. We choose the representative non-degenerate matrices A_0 to have form

$$A_0 \;=\; \begin{pmatrix} k & j \\ 0 & p \end{pmatrix} \;, \text{ with } k > j \geq 0,\;\, p \neq 0; \tag{B.2}$$

there is a unique matrix of this form in every non-degenerate orbit.

3) Degenerate orbits, consisting of (non-zero) matrices with zero determinants. All matrices of this kind can be written in the form

$$A \;=\; \begin{pmatrix} j \\ p \end{pmatrix} \times (c,d). \tag{B.3}$$

This decomposition becomes unique up to an overall sign of j, p, c and d if we require c and d to be mutually prime. All matrices in the same degenerate orbit have the same values of j and p (modulo overall sign); on the other hand, (c,d) runs over all pairs of mutually prime integers. We choose representative matrices to have $(c,d) = (0,1)$; with this representation, $A_0 V' = A_0 V'' \Leftrightarrow$

$V' = \binom{1\,m}{0\,1} \cdot V'' \Leftrightarrow \tau' = \tau'' + m$ for some integer m. Consequently, we will integrate the contribution of a representative degenerate matrix not over the double cover of the upper half plane, but over the half-band $\{\tau \in \mathcal{C} : \tau_2 > 0, |\tau_1| < \frac{1}{2}\}$; to account for double-covering we sum over all $(j,p) \neq (0,0)$ even though (j,p) and $(-j,-p)$ label the same orbit of $SL(2,\mathbf{Z})$.

Before we proceed with the orbit by orbit integration of the series (B.1), we should verify that it is safe to interchange the order of summation and integration. The convergence of the series is not uniform with $\tau \in \Gamma$, but the only matrices whose contributions to (B.1) do not decrease exponentially in the $\tau_2 \to \infty$ limit are those of the form $\binom{0\,j}{0\,p}$. Therefore, the remainder of the series converges uniformly and can be integrated term by term or in any convenient combination, but the contributions of the matrices with zeros in the first column have to be summed together before the integration. Note that with the single exception of $A = 0$, these are precisely the matrices we choose to represent the degenerate orbits.

The contribution of the zero orbit to the integral (2.10) can be easily evaluated to yield:

$$I_1 \;=\; \int_\Gamma \frac{d^2\tau}{\tau_2^2}\, T_2 \;=\; \frac{\pi}{3} T_2 \,. \tag{B.4}$$

The contributions of the non-degenerate orbits total

$$I_2 \;=\; 2 \sum_{\substack{0 \le j < k \\ p \neq 0}} T_2 e^{-2\pi i T \cdot kp} \times \int_{-\infty}^{+\infty} d\tau_1 \int_0^{+\infty} \frac{d\tau_2}{\tau_2^2} \exp\left(-\tfrac{\pi T_2}{\tau_2 U_2} \cdot |k\tau + j + pU|^2 \right). \tag{B.5}$$

After evaluating a gaussian integral over τ_1, the sum on j becomes trivial and the two terms labelled by (k,p) and $(k,-p)$ become equal up to a τ_2-independent factor $e^{-4\pi T_2 kp}$. Therefore,

$$\begin{aligned} I_2 &= \sum_{k,p>0} 2\left(e^{-2\pi ikpT-4\pi kpT_2}+e^{2\pi ikpT}\right)\times\int_0^\infty d\tau_2\sqrt{\frac{T_2U_2}{\tau_2^3}}\cdot e^{-\pi T_2(k\tau_2-pU_2)^2/U_2\tau_2} \\ &= \sum_{k,p>0} 2\left(e^{-2\pi ikp\overline{T}}+e^{2\pi ikpT}\right)\times\frac{1}{p} \\ &= -2\sum_{k>0}\log\left(1-\bar{q}_T^k\right)\ -2\sum_{k>0}\log\left(1-q_T^k\right)\ , \end{aligned} \tag{B.6}$$

where $q_T \equiv e^{2\pi iT}$ and the integral is evaluated via the variable substitution $\tau_2 = \frac{pU_2}{2k}(y+\sqrt{2+y^2})^2$. Together with the contribution of the zero orbit we have

$$I_1+I_2 \;=\; -4\,\mathrm{Re}\log\left(q_T^{1/24}\prod_{k=1}^{\infty}(1-q_T^k)\right)\;\equiv\;-4\,\mathrm{Re}\log\eta(T)\,. \tag{B.7}$$

Now consider the degenerate orbits which together yield

$$I_3 \;=\; \int_{-1/2}^{+1/2}d\tau_1\int_0^{+\infty}\frac{d\tau_2}{\tau_2^2}\left[T_2\sum_{j,p}{}'\exp\left(-\tfrac{\pi T_2}{\tau_2U_2}\left|j+Up\right|^2\right)\;-\;\tau_2\cdot\theta(\tau\in\Gamma)\right]\ , \tag{B.8}$$

where $\theta(\tau\in\Gamma)$ is defined to be one when $\tau\in\Gamma$ and zero otherwise; the last term accounts for the subtraction $\hat{Z}_{\rm torus}-1$ in the integral (2.10) and is effectively integrated over the fundamental domain Γ rather than the whole half-band. As we mentioned above, we should compute the infinite sum over $(j,p)\neq(0,0)$ before computing the integral (or at least the integral over $\tau\in\Gamma$ that corresponds to matrices $\binom{0\ j}{0\ p}$ themselves rather than other members of their orbits). However, we can interchange the order of summation and integration if we first multiply every term in the integrand of (B.8) by a regulating factor that makes the sum uniformly convergent with respect to some finite measure. Using the regulator $(1-e^{-N/\tau_2})$, which we will eventually remove by taking $N\to\infty$, we obtain

$$I_3 \;=\; \lim_{N\to\infty}\left[\frac{U_2}{\pi}\sum_{j,p}{}'\left(\frac{1}{|j+Up|^2}-\frac{1}{|j+Up|^2+\frac{NU_2}{\pi T_2}}\right)\;-\int_\Gamma d^2\tau\,\frac{1-e^{-N/\tau_2}}{\tau_2}\right]. \tag{B.9}$$

The latter integral can be evaluated by substituting $\tau_2 = N/x$, multiplying the integrand by x^ϵ and taking the limit $\epsilon\to+0$; the result is $\log N+\gamma_E+1+\log(2/3\sqrt{3})$, where γ_E is the Euler-Mascheroni constant.

To evaluate the sum over $(j,p)\neq(0,0)$, we sum on j first and make use of the formula

$$\sum_{j=-\infty}^{+\infty}\frac{1}{(j+B)^2+C^2} \;=\; \frac{i\pi}{2C}\left[\cot\pi(B+iC)\;-\;\cot\pi(B-iC)\right]\;\xrightarrow[C\to+\infty]{}\;\frac{\pi}{C}\,.$$

After some regrouping of terms, we arrive at

$$\begin{aligned}&\frac{U_2}{\pi}\sum_{j,p}{}'\left(\frac{1}{|j+Up|^2}-\frac{1}{|j+Up|^2+\frac{NU_2}{\pi T_2}}\right) && \text{(B.10)}\\ &=\;\tfrac{\pi}{3}U_2\;+\sum_{p>0}\frac{2}{p}\,\frac{q_U^p}{1-q_U^p}\;+\sum_{p>0}\frac{2}{p}\,\frac{\bar q_U^p}{1-\bar q_U^p}\;+\sum_{p>0}\left(\frac{2}{p}-\frac{2}{\sqrt{p^2+(N/\pi T_2U_2)}}\right)\,,\end{aligned}$$

where $q_U\equiv e^{2\pi iU}$ and the first term on the right hand side comes from summing over $j\neq0$ for $p=0$; notice that all three series on the right hand side are convergent. We find it convenient to resum the first two series using

$$\sum_{p>0}\frac{1}{p}\frac{q^p}{1-q^p}\;=\sum_{p,n>0}\frac{q^{pn}}{p}\;=\;-\sum_{n>0}\log(1-q^n)\,.$$

As to the last series, in the large N limit it becomes

$$\sum_{p>0}\left(\frac{2}{p}-\frac{2}{\sqrt{p^2+(N/\pi T_2U_2)}}\right)$$

$$\approx\ 2\sum_{p>0}\left(\frac{1}{p}-\log\frac{p+1}{p}\right)\ +\int_1^\infty dp\left(\frac{2}{p}-\frac{2}{\sqrt{p^2+(N/\pi T_2U_2)}}\right)\ +\ O(1/\sqrt{N})$$

$$\xrightarrow[N\to\infty]{}\ 2\gamma_E\ +\ \log\frac{N}{4\pi T_2U_2}\,.$$

Substituting the last two formulæ into (B.10), then (B.9), we obtain

$$I_3\ =\ -4\,\mathrm{Re}\log\eta(U)\ -\ \log(T_2U_2)\ +\ \left(\gamma_E-1-\log\tfrac{8\pi}{3\sqrt{3}}\right)\,. \qquad \text{(B.11)}$$

Combining this result with eq. (B.7), we finally achieve the goal of this appendix: The explicit expression for the threshold correction as a function of toroidal moduli is

$$\Delta_a(T,\overline{T},U,\overline{U})\ =\ -b_a\cdot\log\left[\frac{8\pi e^{1-\gamma_E}}{3\sqrt{3}}\cdot T_2\left|\eta(T)\right|^4\cdot U_2\left|\eta(U)\right|^4\right]\,. \qquad \text{(B.12)\&(2.14)}$$

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