SLAC–PUB–4959
UTTG–19–89
April 1989
T

On Effective Field Theories Describing (2,2) Vacua of the Heterotic String

Lance J. Dixon[⋆]

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94309

Vadim S. Kaplunovsky[†]

Stanford University, Physics Department
Stanford, CA 94305

and

University of Texas, Physics Department
Austin, TX 78712[‡]

and

Jan Louis

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94309

Submitted to *Nuclear Physics* **B**

⋆ On leave from Joseph Henry Laboratory, Princeton University, Princeton, NJ 08544.
† Supported in part by the NSF under grants #PHY–86–12280 and #PHY–86–05978 and by Robert A. Welch Foundation.
‡ Address after April 6, 1989.

ABSTRACT

Classical vacua of the heterotic string corresponding to $c = 9$, $N = (2,2)$ superconformal theories on the world sheet yield low-energy effective field theories with $N = 1$ space-time supersymmetry in four dimensions, gauge group $E_6 \otimes E_8$, several families of $\mathbf{27}$ and $\overline{\mathbf{27}}$ matter fields, and moduli fields. String theory relates matter fields to moduli; in this article we relate the kinetic terms in the effective Lagrangian for both moduli and matter fields to the $\mathbf{27}^3$ and $\overline{\mathbf{27}}^3$ Yukawa couplings. Geometrically, we recover the result (obtained previously via the type II superstring and $N = 2$ supergravity) that moduli space is a direct product of two Kähler manifolds of restricted type, spanned by the moduli related respectively to the $\mathbf{27}$ and $\overline{\mathbf{27}}$ matter fields. The holomorphic functions of the moduli generating the two restricted Kähler metrics also determine the Yukawa couplings of the matter fields. We derive explicit formulæ for the metric for the matter fields in terms of the metric for the corresponding moduli; the two metrics are not identical to each other. The precise relation between moduli and matter metrics takes a slightly different form on subspaces of the moduli space where the unbroken gauge symmetry is enhanced beyond $E_6 \otimes E_8$; this phenomenon is illustrated using the examples of (2,2) orbifolds and tensor products of minimal $N = 2$ theories.

1. Introduction

Heterotic string theory[1] — a candidate theory of all fundamental particle interactions — has a huge set of classical vacuum states, including many four-dimensional vacua whose features allow them to serve as starting points for realistic phenomenology. The first vacua of this kind having chiral fermions in four space-time dimensions were constructed from the ten-dimensional heterotic string by compactifying six of the ten dimensions into a Calabi-Yau manifold.[2] The four-dimensional physics of the Calabi-Yau vacua is characterized by $N = 1$ supersymmetry, $E_6 \otimes E_8$ gauge group and matter fields that form several $\mathbf{27}$ or $\overline{\mathbf{27}}$ families of the E_6. Subsequently, many other heterotic string vacua were constructed that share these features. In all such vacua the six dimensions which are compactified in the Calabi-Yau case are generalized to an "internal" $N = (2,2)$ superconformal theory on the world sheet which has Virasoro anomaly $c = (9,9)$.⋆ Besides Calabi-Yau compactifications, known examples of the (2,2) vacua include (2,2) orbifolds[3] and tensor products of minimal N=2 models[4] or of other exactly solvable N=2 superconformal theories.[5] It appears quite possible that *all* $(2,2)$ vacua with spacetime supersymmetry and $E_6 \otimes E_8$ gauge symmetry are compactifications on (possibly singular) Calabi-Yau manifolds[4,6]; however, the analysis in the present paper does not rely on this remarkable connection.

The (2,2) vacua are only a small subset of the classical vacua of the heterotic string. The heterotic string itself only requires $N = (0,1)$ superconformal invariance of the world-sheet theory describing internal degrees of freedom,[1] although $N = (0,2)$ is needed if a vacuum is to exhibit space-time supersymmetry.[7] The low-energy features of the (0,2) vacua, such as the unbroken gauge group and the spectrum of massless particles, vary widely from one (0,2) vacuum to another,

⋆ Actually, the internal theory replaces both the six compact dimensions and the six left-moving world-sheet fermions that are affected by imbedding of the spin connection into the gauge group.[2]

making it very likely that some of these vacua lead to viable phenomenology. However, at the present time, the phenomenological prospects for certain (2,2) vacua (or rather minor modifications of them via a Hosotani-Witten-type mechanism[8]) appear to be at least as good as for the (0,2) vacua studied to date.[9] The (2,2) vacua are much less diverse than the (0,2) vacua, and in fact they share many common features. The subject of this article is the low-energy behavior common to all the (2,2) vacua.

For a given vacuum state of the string theory, physics at energies well below the Planck scale can be described by an effective low-energy field theory. To define a field theory one needs to list all the fields and describe the effective Lagrangian; with this information, all other quantities are computable, at least in principle. In particular, we can map out all the neighboring vacua by studying flat directions in the effective potential, and for each vacuum state we can compute scattering amplitudes for various multi-particle processes. In the state-of-the-art string theory one can list all light particles that appear in the spectrum of any particular vacuum state, but one cannot directly obtain an effective Lagrangian for the low-energy limit of the theory. Instead, we shall follow the so-called S-matrix approach (see for example ref. [10]): One constructs an effective field theory that yields the same scattering amplitudes as the full string theory does in the low-energy limit.† Since the subject of this article is not a particular vacuum state of the heterotic string but the whole class of the (2,2) vacua, we shall derive some universal relations between various string amplitudes valid for all members of this class and require that the effective low-energy field theory obeys the same relations between the same amplitudes. This will impose severe constraints on the low-energy effective Lagrangian; these constraints are the main results of this article.

† In ref. [11] this approach was carried out to order $O(\alpha'^0)$ for some specific four-dimensional vacua.

The S-matrix approach can be carried out to an arbitrary order in perturbation theory. In this article we shall limit ourselves to the classical effective field theory in space time and compute all scattering amplitudes at the tree level. For the string this means that the world sheet is always a complex sphere. However, the two dimensional conformal field theories on the world sheet will be fully quantized, with no semi-classical or perturbative approximations, and all correlation functions of various world-sheet operators that appear in this article are exact. Note that while for most two-dimensional quantum field theories we do not have explicit expressions for the various exact correlators, we may still have exact Ward identities relating those correlators to each other; in this article, we shall use heavily the Ward identities of the left-moving $N = 2$ supersymmetry of the "internal" world-sheet theory.

The fields of an effective low-energy theory describing any classical (2,2) vacuum include the gravitational sector (graviton, dilaton and axion, plus superpartners), the $E_6 \otimes E_8$ gauge multiplets, and a set of chiral superfields forming the $\mathbf{27}$ or $\overline{\mathbf{27}}$ representations of E_6 — *matter fields.* Moreover, the (2,2) world-sheet supersymmetry implies that for each $\mathbf{27}$ or $\overline{\mathbf{27}}$ supermultiplet of matter fields there is an additional E_6 singlet superfield whose scalar potential is flat.[12,13,14] Consequently, vacuum expectation values of (the scalar components of) these singlets are completely unconstrained, resulting in a multi-parameter family of (2,2) vacua; for this reason these fields are called *moduli.* From the world-sheet point of view the flat potential for moduli scalars means that the associated vertex operators are exactly marginal, *i.e.* their β-functions vanish to all orders and even beyond perturbation theory.

The possible form of effective field theories describing (2,2) vacua is constrained by four-dimensional supersymmetry. $N = 1$ supergravity theories are characterized by two analytic functions of scalar fields, the superpotential W and the Kähler function K (sometimes called the Kähler potential).[15] Cubic terms

(Yukawa couplings)[16] as well as other, non-renormalizable, terms[17] in the superpotential have been calculated in many special cases. On the other hand, the Kähler function K has been much less investigated, though it is also of considerable phenomenological interest: K determines the kinetic terms in the effective Lagrangian of moduli and matter fields which are needed to obtain the physical normalization of the Yukawa couplings. Furthermore, K enters the scalar potential and thereby influences possible supersymmetry breaking mechanisms. The Kähler function K, and its relation to the superpotential, will be the focus of this article.

The Kähler function has previously been computed in some special cases. For Calabi-Yau compactifications, the moduli fields can be divided into two sets: deformations of the complex structure, which correspond to harmonic (1,2) forms on the Calabi-Yau manifold and which accompany the $\overline{\mathbf{27}}$ matter fields; and deformations of the Kähler class, which correspond to the (1,1) forms and which accompany the **27** matter fields. In the limit that the Calabi-Yau manifold is large enough to use ten-dimensional field theory, the metric for the (1,1) moduli fields reduces to the metric on the space of (1,1) forms which is controlled by the same topological constants of the Calabi-Yau manifold that determine the $\mathbf{27}^3$ Yukawa couplings.[18] Similarly, the metric for the (1,2) moduli in the field theory limit can be expressed in terms of the $\overline{\mathbf{27}}^3$ Yukawa couplings, although these Yukawa couplings are not constants.[18,19] The metrics for both moduli and matter fields arising from the untwisted sector of an orbifold can be obtained by simply truncating the ten-dimensional effective field theory.[20] The result of this procedure actually holds for orbifolds of arbitrary size; this can be verified by using the symmetries of the string generating functional for scattering amplitudes,[21] or by using Zamolodchikov's conformal-field-theoretic formula for the metric.[22,23]

A different and more general approach can be used for the moduli sector of the effective field theory. The $N = (2, 2)$ superconformal theory which defines a classical vacuum of the heterotic string also defines a classical vacuum of the type II superstring.[12,24] In the latter case the effective four-dimensional theory is $N = 2$ supersymmetric, which severely restricts the form of the effective Lagrangian for the moduli.[25−27] In particular, the moduli space is a direct product of two Kähler spaces[24,27]; in the Calabi-Yau case these two spaces are spanned by (1,1) and (1,2) moduli respectively. The two spaces are of *restricted type*, which means that they are each determined by a holomorphic function of the respective moduli. The same holomorphic function controls kinetic terms in the effective Lagrangian of the vector fields coming from the Ramond-Ramond sector of the type II superstring[27] and couplings of those vector fields to the moduli scalars. In ref. [28] it was argued that these type II couplings are the same as the $\mathbf{27}^3$ and $\overline{\mathbf{27}}^3$ Yukawa couplings in the corresponding (2,2) vacuum of the heterotic string, and hence that the cubic superpotential for matter fields is determined by the same two holomorphic functions that determine the Kähler function for the moduli. However, there are several subtleties in making precise the correspondence between the two holomorphic functions and the superpotential; for example, space-time supersymmetry is local, and in locally supersymmetric theories Yukawa couplings take a different form then in the globally supersymmetric case treated in ref. [28].

We shall show in this article that the above general results can be obtained entirely within the heterotic string, without invoking the type II superstring or $N = 2$ supergravity in space-time. We rederive the splitting of the Kähler function of the moduli into a sum of two functions, K_1 and K_2, each depending only on the moduli related to, respectively, $\mathbf{27}$ or $\overline{\mathbf{27}}$ matter fields; we also verify that K_1 and K_2 are each of restricted type. In the process we find out that the metrics for the $\mathbf{27}$ and $\overline{\mathbf{27}}$ matter fields differ from the metrics for the corresponding moduli. (For any particular (2,2) vacuum this difference can always be eliminated by a field redefinition; what we mean is that no holomorphic field redefinition would result in metrics for moduli and matter fields that are equal to each other for all vacuum expectation values of the moduli fields.) This difference between the moduli and matter metrics — which has not been obtained from the type II superstring (and probably cannot be obtained that way) — plays a key role in deriving precise relations between moduli and matter couplings. In particular, relations between $K_{1,2}$ and the superpotential which were argued for in ref. [28] now become consistent with *local* space-time supersymmetry.

The main body of this article is organized as follows: Section 2 is an overview of the light fields characteristic of the (2,2) vacua from both space-time and world-sheet points of view. Section 3 is devoted to the S-matrix approach to low energy physics. First, we relate the low-energy limits of various four-particle scattering amplitudes to the Kähler function and the superpotential of the effective field theory. Next, we use the left-moving superconformal symmetry of the world-sheet theory to relate scattering amplitudes that involve moduli scalars to amplitudes involving matter fields. Imposing these relations on the field-theoretical amplitudes we establish several constraints on the geometry of the field space; in particular, the Kähler function of the moduli fields must decompose into K_1+K_2 and the metric for matter fields obeys differential equations that can be integrated in terms of K_1 and K_2. Moreover, we derive equations that relate K_1 to the $\mathbf{27}^3$ Yukawa couplings and K_2 to the $\overline{\mathbf{27}}^3$ couplings. Solving these equations we find that all the $\mathbf{27}^3$ ($\overline{\mathbf{27}}^3$) couplings can be expressed in terms of derivatives of a single holomorphic function $\mathcal{F}_1$ ($\mathcal{F}_2$) of the appropriate moduli; Kähler functions $K_{1,2}$ of the (1,1) and (1,2) moduli spaces have restricted type and are determined by the same $\mathcal{F}_{1,2}$ that determine the Yukawa couplings. In section 4 we consider the effects on K of enlarging the unbroken gauge group beyond $E_6 \otimes E_8$. (Such extra gauge factors occur in almost all exactly solvable (2,2) vacua that have been discussed to date.) We find that the equations relating the superpotential to the Kähler function of the moduli fields remain unchanged, but the equations for the metric of the matter fields have to be modified. In section 5 we summarize our results and discuss their implications. The article also has three appendices: Appendix A contains an alternative derivation of stringy constraints on the moduli-dependence of the $\mathbf{27}^3$ and $\overline{\mathbf{27}}^3$ terms in the superpotential; Appendix B exhibits a coordinate system in which these terms can be expressed as derivatives of $\mathcal{F}_1$ and $\mathcal{F}_2$; and Appendix C gives the precise relation between the scattering of moduli scalars in the heterotic string and in the type II superstring.

2. Light Scalar Fields and their Vertex Operators

2.1. Low-Energy Effective Field Theory.

The goal of this article is to describe the low-energy behavior of the heterotic string in terms of an effective field theory. The general form of an effective Lagrangian for the light bosons is

$$\mathcal{L}_{\rm Bose} \;=\; \sqrt{-\mathcal{G}}\left\{ \frac{1}{2\kappa^2}\mathcal{R} \;+\; \frac{1}{4e^2}\left(F_{ij}^{(a)}\right)^2 \;-\; g_{A\bar{B}}\,\mathcal{D}_i\phi^A\,\mathcal{D}^i\bar{\phi}^{\bar{B}} \;-\; V(\phi,\bar{\phi}) \;+\; \cdots \right\}, \tag{2.1}$$

where $\mathcal{G}$ and $\mathcal{R}$ are the determinant and the scalar curvature of the space-time metric $\mathcal{G}_{ij}$ while $g_{A\bar{B}}$ is the metric on the space of scalar fields $\phi^A,\bar{\phi}^{\bar{B}}$. Other notations in eq. (2.1) are as follows: $V(\phi,\bar{\phi})$ is the scalar potential, $F_{ij}^{(a)}$ are gauge field strengths, $\mathcal{D}_i$ are gauge-covariant derivatives with respect to space-time coordinates x^i, and '$\cdots$' stand for the axion coupling to $F\tilde{F}$ and terms with more than two space-time derivatives. All string vacua we are interested in possess unbroken $N=1$ supersymmetry in four dimensions, so the effective low-energy theory should be consistent with $N=1$ supergravity too. Therefore all fermionic terms in the effective Lagrangian are related to the bosonic terms, and the bosonic Lagrangian (2.1) itself has to obey several constraints. First, the scalar metric $g_{A\bar{B}}(\phi,\bar{\phi})$ should be Kähler, *i.e.*, expressible in terms of a single real analytic function K of complex scalar fields ϕ^A and their hermitian conjugates $\bar{\phi}^{\bar{A}}$:

$$g_{A\bar{B}}(\phi,\bar{\phi}) \;=\; K_{,A\bar{B}} \;\equiv\; \frac{\partial^2 K(\phi,\bar{\phi})}{\partial\phi^A\,\partial\bar{\phi}^{\bar{B}}}\,. \tag{2.2}$$

Second, the scalar potential $V(\phi,\bar{\phi})$ should have a special form

$$\begin{aligned} V(\phi,\bar{\phi}) \;=\;& \exp(\kappa^2 K)\cdot\Big[g^{A\bar{B}}(W_{,A}+\kappa^2 W K_{,A})(\overline{W}_{,\bar{B}}+\kappa^2\overline{W}K_{,\bar{B}}) \;-\; 3\kappa^2\,|W|^2\Big]\\ &+\;\frac{e^2}{8}\sum_{(a)}\Big(K_{,A}\cdot Q^{(a)}\cdot\phi^A \;+\; \bar{\phi}^{\bar{A}}\cdot Q^{(a)}\cdot K_{,\bar{A}}\Big)^2\,, \end{aligned} \tag{2.3}$$

where $W(\phi)$ is a holomorphic function of ϕ and $Q^{(a)}$ are the (hermitian) generators of the gauge group. (See ref. [15, 29] for a derivation of eq. (2.3), and for the fermionic terms in the $N=1$ supergravity action.) The two terms in the potential (2.3) are often called the F-term and the D-term, after the common notations for the auxiliary fields in scalar and gauge supermultiplets which give rise to them. Finally, if the gauge coupling e^2 depends on the scalar fields, then e^{-2} should be a harmonic function, *i.e.*, the real part of a holomorphic function $f(\phi^A)$, and the imaginary part of the same $f(\phi)$ controls the coupling of axions to $F\tilde{F}$.[29] If the gauge group is a direct product of several subgroups, then there may be a separate $f(\phi)$ for each gauge coupling.*

In a general $N=1$ supergravity the Kähler function $K(\phi,\bar{\phi})$, the superpotential $W(\phi)$ and the gauge couplings $e^2 = 1/\mathrm{Re}\,f(\phi)$ are completely arbitrary and independent of each other. However, in all classical vacua of the heterotic string all gauge couplings are equal to each other[30] and are controlled by a single

---

⋆ Actually, even for a simple non-abelian gauge group one can have $f_{(a)(b)}(\phi)$ transforming as a symmetric square of the adjoint representation of the gauge group instead of a single gauge-invariant $f(\phi)$.[29]

scalar field — the four-dimensional dilaton; in space-time supersymmetric vacua the dilaton is the real part of the dilaton/axion complex field D and $e^{-2} = \mathrm{Re}D$.[31] This article is concerned with K and W; we shall show how they are related to each other in effective theories describing the (2,2) vacua.

In a generic (2,2) vacuum of the heterotic string the gauge group is $E_6 \otimes E_8$ (the E_8 component is pure gauge) and the massless scalars can be listed as follows: the dilaton/axion field D; several generations of matter fields A^α and A^μ that transform as $\mathbf{27}$ and $\overline{\mathbf{27}}$ under E_6 (in our notations we shall always label the $\mathbf{27}$ matter fields with indices taken from the beginning of the greek alphabet while indices from the middle of the greek alphabet will always refer to the $\overline{\mathbf{27}}$ matter fields); and several moduli fields M^A whose expectation values parametrize families of related (2,2) vacua. Non-generic (2,2) vacua may present us with additional gauge fields and/or additional scalars that are E_6 singlets but are not moduli.[32] (That is, expectation values of these singlets must vanish in all (2,2) vacua of the heterotic string; from the low-energy point of view, non-zero VEVs of non-moduli singlets usually lead to positive values of the scalar potential). For example, there is an extra $SU(3)$ gauge group in the case of the Z_3 orbifold,[3] and there are 224 extra scalar E_6-singlets in the case of the Calabi-Yau threefold defined as a particular quintic surface in CP^4.[4] In this article our concern is not with a particular (2,2) vacuum of the heterotic string, but with entire families of such vacua that can be continuously transformed into each other by changing expectation values of the moduli fields. In moduli spaces describing such families of (2,2) vacua, gauge groups bigger than E_6 appear only at some isolated points or on some lower-dimensional submanifolds. We shall discuss such submanifolds in section 4; in this and the following section we shall concentrate on generic neighborhoods in the moduli space. The case of massless singlets that are not moduli is more complicated; at present it is controversial whether such singlets can stay massless throughout the entire moduli space.[4,33] In this article we shall allow for existence of those singlets, but will not pay them any more attention than we must.

Ideally, we would like to survey the entire field space of an effective field theory corresponding to a family of the (2,2) vacua of the heterotic string. Unfortunately, our state-of-the-art string technology is limited to scattering amplitudes that involve a finite number of particles in the spectrum of a string vacuum state. This limits our survey to the moduli space and its infinitesimal neighborhood in the field space. For points in the moduli space the superpotential vanishes together with its first and second derivatives with respect to all massless fields; hence, using E_6 invariance, we can write

$$W \;=\; \tfrac{1}{3}\,W_{\alpha\beta\gamma}(M)\,A^\alpha A^\beta A^\gamma + \tfrac{1}{3}\,W_{\lambda\mu\nu}(M)\,A^\lambda A^\mu A^\nu + O(A^\alpha A^\mu B) + O(B^3) + \cdots\,, \tag{2.4}$$

where B stand for the E_6 singlet fields that are not moduli but nevertheless remain massless throughout the moduli space, and '$\cdots$' refer to superpotential terms that are of quartic or higher order in matter fields. The coefficients $W_{\alpha\beta\gamma}$ and $W_{\lambda\mu\nu}$ are the $\mathbf{27}^3$ and $\overline{\mathbf{27}}^3$ Yukawa couplings and are of obvious phenomenological interest. At this point we allow them to be arbitrary holomorphic functions of the moduli M^A, but later we shall see that the moduli-dependence of the Yukawa couplings is constrained. In the same spirit, we can write the Kähler function K as

$$\begin{aligned} K \;=\; & \frac{-1}{\kappa^2}\,\log(D+\overline{D}) \;+\; \hat{K}(M,\overline{M}) \\ & +\; G_{\alpha\bar\beta}(M,\overline{M})\,A^\alpha\overline{A}^{\bar\beta} \;+\; G_{\mu\bar\nu}(M,\overline{M})\,A^\mu\overline{A}^{\bar\nu} \\ & +\; O(B\overline{B}) \;+\; O(A^\alpha A^\mu) \;+\; O(\overline{A}^{\bar\alpha}\overline{A}^{\bar\mu}) \\ & +\; \text{terms involving higher powers of the matter fields.} \end{aligned} \tag{2.5}$$

Here $\hat{K}$ is the Kähler function of the moduli space, which is itself a Kähler manifold with metric $g_{A\bar B} = \hat{K}_{,A\bar B}$ (here and henceforth capital latin indices are reserved for the moduli fields). On the moduli space, the metrics for the **27** and $\overline{\mathbf{27}}$ matter fields are given by the moduli-dependent matrices $G_{\alpha\bar{\beta}}$ and $G_{\mu\bar{\nu}}$; away from the moduli space, the situation becomes more complicated and different kinds of fields start mixing with each other.

Given formulæ (2.4) and (2.5), we can write an explicit expression for the scalar potential of the matter fields:

$$\begin{aligned} V \;=\; & \frac{e^2}{2}\sum_{(a)}\Big(G_{\alpha\bar{\beta}}\,\overline{A}^{\bar{\beta}}\,Q^{(a)}\,A^{\alpha} \;+\; G_{\mu\bar{\nu}}\,\overline{A}^{\bar{\nu}}\,Q^{(a)}\,A^{\mu}\Big)^2 \qquad (2.6)\\ +\; & \frac{\exp(\kappa^2\hat{K})}{D+\overline{D}}\bigg[W_{\alpha\beta\epsilon}G^{\epsilon\bar{\eta}}\overline{W}_{\bar{\eta}\bar{\gamma}\bar{\delta}}\cdot\Big(A^{\alpha}A^{\beta}\Big)_{\overline{\mathbf{27}}}\Big(\overline{A}^{\bar{\gamma}}\overline{A}^{\bar{\delta}}\Big)_{\mathbf{27}} \\ & \qquad +\; W_{\kappa\lambda\rho}G^{\rho\bar{\sigma}}\overline{W}_{\bar{\sigma}\bar{\mu}\bar{\nu}}\cdot\Big(A^{\kappa}A^{\lambda}\Big)_{\mathbf{27}}\Big(\overline{A}^{\bar{\mu}}\overline{A}^{\bar{\nu}}\Big)_{\overline{\mathbf{27}}} \\ & \qquad +\; X_{\alpha\mu\bar{\beta}\bar{\nu}}\cdot\Big(A^{\alpha}A^{\mu}\Big)_{\mathbf{1}}\Big(\overline{A}^{\bar{\beta}}\overline{A}^{\bar{\nu}}\Big)_{\mathbf{1}} \\ & \qquad\quad +\; O(|A|^2\,|B|^2) \;+\; O(|B|^4) \;+\; \cdots\bigg] \end{aligned}$$

where '$\cdots$' stand for terms of higher than quartic order in matter fields. Here $(A^{\alpha}A^{\beta})_{\overline{\mathbf{27}}}$ denotes the part of the product of A^{α} and A^{β} that transforms as a $\overline{\mathbf{27}}$ under E_6 ($A^{\alpha}A^{\beta}$ transforms as $\mathbf{27}\times\mathbf{27} = \overline{\mathbf{27}} + \mathbf{351} + \mathbf{351}'$), *etc.* The fourth term in (2.6) (the third F-term) is controlled by Yukawa couplings of the type $BA^{\alpha}A^{\mu}$ and is absent in theories that do not have massless non-moduli singlets B. Fortunately, even when this term is present, it does not affect the scattering amplitudes that we will use for relating $\hat{K}$, $G_{\alpha\bar{\beta}}$ and $G_{\mu\bar{\nu}}$ to the Yukawa couplings $W_{\alpha\beta\gamma}$ and $W_{\lambda\mu\nu}$, so we do not need an explicit expression for the $X_{\alpha\mu\bar{\beta}\bar{\nu}}$. Note that because $D+\overline{D}\equiv 2\mathrm{Re}D = 2/e^2$, the entire scalar potential (2.6) is proportional to the gauge coupling e^2.

2.2. Vertex Operators for Moduli and Matter Fields.

This article is concerned with classical vacua of the heterotic string that can be obtained by adjoining a $c = (9,9)$, $N = (2,2)$ superconformally invariant theory on the world sheet to the four $N = (0,1)$ free world-sheet superfields that are responsible for the four-dimensional space-time and to the left-moving $SO(10) \otimes E_8$ Kac-Moody algebra that is responsible for most of the gauge group. $N = (2,2)$ superconformal theories are characterized by having two $N = 2$ super-Virasoro algebras — one left-moving, one right-moving — each generated by a Virasoro operator T_B, an abelian current J and a conjugate pair of fermionic operators $T_F^{\pm}$ of conformal weight $h = \frac{3}{2}$ and J-charges $q = \pm 1$. We shall assume that all primary Neveu-Schwarz fields of either algebra have integral J-charges; this is required for the right-moving $N = 2$ superalgebra to lead to $N = 1$ supersymmetry in space-time,[7] and for the left-moving $N = 2$ superalgebra to lead to the enlargement of the gauge group from $SO(10)$ to E_6. The left-moving superalgebra is also responsible for the existence of moduli fields and their relation to the matter fields; this algebra is going to be our main tool.

A general multiplet of the $N = 2$ superalgebra has four components, but there are also chiral multiplets that have only two components; lower components of chiral multiplets satisfy $2h = |q|$. Of particular interest to us are chiral multiplets of the left-moving algebra whose lower components $\Psi^{\pm}$ have $h = \frac{1}{2}$ and $q = \pm 1$; upper components $\Phi^{\pm}$ of these multiplets are marginal (have $h = 1$) and neutral ($q = 0$). The singular terms in the operator product expansions of the superalgebra generators with $\Psi^{\pm}$ and $\Phi^{\pm}$ can be summarized in the following formulæ:

$$
\begin{aligned}
T_B(w)\cdot\Psi^\pm(z) &= \frac{1/2}{(w-z)^2}\,\Psi^\pm(z) \;+\; \frac{1}{w-z}\,\partial\Psi^\pm(z) \;+\;\cdots\,,\\
T_B(w)\cdot\Phi^\pm(z) &= \frac{\partial}{\partial z}\left(\frac{\Phi^\pm(z)}{w-z}\right) \;+\;\cdots\,,\\
J(w)\cdot\Psi^\pm(z) &= \frac{\pm 1}{w-z}\,\Psi^\pm(z) \;+\;\cdots\,,\\
J(w)\cdot\Phi^\pm(z) &= 0 \;+\;\cdots\,,\\
2T_F^\pm(w)\cdot\Psi^\pm(z) &= 0 \;+\;\cdots\,,\\
2T_F^\pm(w)\cdot\Phi^\pm(z) &= \frac{\partial}{\partial z}\left(\frac{\Psi^\pm(z)}{w-z}\right) \;+\;\cdots\,,\\
2T_F^\mp(w)\cdot\Psi^\pm(z) &= \frac{1}{w-z}\,\Phi^\pm(z) \;+\;\cdots\,,\\
2T_F^\mp(w)\cdot\Phi^\pm(z) &= 0 \;+\;\cdots\,,
\end{aligned}
\tag{2.7}
$$

where '$\cdots$' stand for terms that are not singular when $(w-z)\to 0$. Formulæ (2.7) disregard all right-moving quantum numbers of the operators $\Psi^\pm$ and $\Phi^\pm$, even their $\bar z$-dependence. This is justified by complete commutativity of the left-moving and right-moving superalgebras.

The operators $\Psi^\pm$ are important because they appear in vertices of the **27** and the $\overline{\mathbf{27}}$ matter fields. To be precise, matter fields that belong to decuplets of the $SO(10)\subset E_6$ have vertices of the form

$$
\begin{aligned}
\mathbf{27}\text{ fields}\quad & A_{\hat p}^{\alpha} &\longleftrightarrow\quad & i\lambda^{\hat p}(z)\cdot\Psi_\alpha^+(z,\bar z)\,, \quad \alpha=1,\ldots,N_1\,,\\
\overline{\mathbf{27}}\text{ fields}\quad & A_{\hat p}^{\mu} &\longleftrightarrow\quad & i\lambda^{\hat p}(z)\cdot\Psi_\mu^-(z,\bar z)\,, \quad \mu=1,\ldots,N_2\,,\\
\text{anti-}\mathbf{27}\text{ fields}\quad & \overline{A}_{\hat p}^{\bar\alpha} &\longleftrightarrow\quad & \left(i\lambda^{\hat p}\Psi_\alpha^+\right)^\dagger \;=\; i\lambda^{\hat p}(z)\cdot\Psi_{\bar\alpha}^-\,,\\
\text{anti-}\overline{\mathbf{27}}\text{ fields}\quad & \overline{A}_{\hat p}^{\bar\mu} &\longleftrightarrow\quad & \left(i\lambda^{\hat p}\Psi_\mu^-\right)^\dagger \;=\; i\lambda^{\hat p}(z)\cdot\Psi_{\bar\mu}^+\,,
\end{aligned}
\tag{2.8}
$$

where $\hat p$ is the $SO(10)$ vector index and $\lambda^{\hat p}(z)$ are free left-moving fermions that generate the $SO(10)$ Kac-Moody algebra — the $SO(10)$ Kac-Moody currents are $i\lambda^{\hat p}\lambda^{\hat q}$ for $p,q\in\mathbf{10}$, $p<q$. This Kac-Moody algebra is enlarged to E_6 by adding to it the left-moving current J — which generates the $U(1)$ subgroup of E_6 that

commutes with the $SO(10)$ — and also the $\mathbf{16}+\overline{\mathbf{16}}$ Ramond sector operators that are products of the $SO(10)$ spinors with the $h=\frac{3}{8}$, $q=\pm\frac{3}{2}$ operators obtained from the unit operator via spectral flow in the $N=2$, $c=9$ superalgebra.[34] Spectral flow also relates vertex operators of the $\mathbf{27}$ fields that transform as $\mathbf{16}+\mathbf{1}$ under $SO(10)$ to the Ψ^+ operators and relates vertex operators of the $\overline{\mathbf{16}}+\mathbf{1}\in\overline{\mathbf{27}}$ fields to the Ψ^-. Notice that vertex operators of both A^α and $\overline{A}^{\bar\mu}$ involve Ψ^+ since both types of fields transform as $\mathbf{27}$'s under E_6; however, the right-moving structure of operators Ψ^+_α and $\Psi^+_{\bar\mu}$ is quite different, corresponding to the fact that Ψ^+_α makes a holomorphic scalar field in space-time whereas $\Psi^+_{\bar\mu}$ makes an anti-holomorphic field — carrying a barred index. The same is true for the operators Ψ^-_μ and $\Psi^-_{\bar\alpha}$.

Space-time fields are dual to the world-sheet fields in the sense that a linear redefinition of the former results in the inverse redefinition of the latter; this accounts for the lowered indices $\alpha,\mu,\bar\alpha,\bar\mu$ in (2.8). Likewise, integrated correlation functions of vertices give Green's functions of space-time fields with the external legs truncated and therefore they also carry lowered indices. For example, the two-point functions of the matter vertices yield the Zamolodchikov metric[22] for the matter fields:

$$\begin{aligned}\left\langle\left(i\lambda^{\hat p}\Psi^+_\alpha\right)(z,\bar z)\cdot\left(i\lambda^{\hat q}\Psi^-_{\bar\beta}\right)(z',\bar z')\right\rangle &= \delta^{\hat p\hat q}G_{\alpha\bar\beta}\cdot\left|z-z'\right|^{-4},\\ \left\langle\left(i\lambda^{\hat p}\Psi^-_\mu\right)(z,\bar z)\cdot\left(i\lambda^{\hat q}\Psi^+_{\bar\nu}\right)(z',\bar z')\right\rangle &= \delta^{\hat p\hat q}G_{\mu\bar\nu}\cdot\left|z-z'\right|^{-4}.\end{aligned}\tag{2.9}$$

The λ's anticommute with $\Psi^\pm$ and $\left\langle\lambda^{\hat p}(z)\cdot\lambda^{\hat q}(z')\right\rangle=\delta^{\hat p\hat q}(z-z')^{-1}$; therefore, the matter metrics $G_{\alpha\bar\beta}$ and $G_{\mu\bar\nu}$ can be obtained from two-vertex correlators $\left\langle\Psi^+\Psi^-\right\rangle$ via

$$\begin{aligned}\left\langle\Psi^+_\alpha(z,\bar z)\cdot\Psi^-_{\bar\beta}(z',\bar z')\right\rangle &= G_{\alpha\bar\beta}\cdot(z-z')^{-1}(\bar z-\bar z')^{-2},\\ \left\langle\Psi^-_\mu(z,\bar z)\cdot\Psi^+_{\bar\nu}(z',\bar z')\right\rangle &= G_{\mu\bar\nu}\cdot(z-z')^{-1}(\bar z-\bar z')^{-2}.\end{aligned}\tag{2.10}$$

Note that the Zamolodchikov metrics for the matter fields obtained from eqs. (2.9) or (2.10) are the same $G_{\alpha\bar{\beta}}$ and $G_{\mu\bar{\nu}}$ that appear in the Lagrangian of the low-energy effective field theory via eq. (2.5): This is a necessary condition for the effective theory to reproduce the correct residues of massless poles in string scattering amplitudes. (One can see this explicitly for the four-particle amplitudes presented in the next section, by factorizing them on the graviton poles.) In the effective field theory, matrices $G_{\alpha\bar{\beta}}$ and $G_{\mu\bar{\nu}}$ explicitly depend on the moduli fields; in the string formulæ (2.10), this dependence is implicit: Each set of vacuum expectation values of the moduli fields corresponds to a specific vacuum state of the heterotic string, and the correlators in (2.10) are evaluated for that particular vacuum state.

Now consider the operators $\Phi^{\pm}$. They are marginal and neutral, which makes them vertex operators for massless scalars that are E_6 singlets.[4,13,14] Moreover, $\Phi^{\pm}$ are upper components of $N=2$ supermultiplets and hence can be added to the world-sheet Lagrangian without breaking the left-moving $N=2$ superalgebra. From the space-time point of view this means that scalar fields associated with $\Phi^{\pm}$ are not just massless singlets but moduli — fields that can have arbitrary vacuum expectation values without breaking the (2,2) structure of the vacuum and therefore without generating a potential (see refs. [13,14] for proofs of this assertion). As with the matter vertices, moduli vertices are dual to moduli fields, but since the moduli space is non-linear, this duality is local: Given a (2,2) vacuum and a corresponding point on the moduli space, moduli vertices are dual to dM^A and $d\overline{M}^{\bar{A}}$; from the differential geometry point of view this means that the moduli vertices are co-vectors on the moduli space. For a general coordinate system on the moduli space we thus have vertices for holomorphic moduli fields that are some linear combinations of the vertices Φ^+_α and Φ^-_μ, and similarly for the anti-holomorphic moduli:

$$\begin{aligned} M^A \ &\leftrightarrow\ U^\alpha_A \cdot \Phi^+_\alpha \ +\ U^\mu_A \cdot \Phi^-_\mu \,, \\ \overline{M}^{\bar{A}} \ &\leftrightarrow\ \overline{U}^{\bar{\alpha}}_{\bar{A}} \cdot \Phi^-_{\bar{\alpha}} \ +\ \overline{U}^{\bar{\mu}}_{\bar{A}} \cdot \Phi^+_{\bar{\mu}} \,. \end{aligned} \tag{2.11}$$

The U^α_A, *etc.*, are moduli-dependent matrices, but from the world-sheet point of view they are c-numbers and not operators. The two-dimensional operators Φ^+_α, *etc.*, here are fixed by the current algebra (2.7) in terms of Ψ^+_α, *etc.*, which appear in the vertex operators for the four-dimensional fields A^α and A^μ via eqs. (2.8).

For any particular point in the moduli space we can find local coordinates M^α and M^μ that trivialize the U matrices at that point; unfortunately, it is generally impossible to simultaneously trivialize the U matrices everywhere in the moduli space, or even in a finite piece of the moduli space. However, we shall prove in the next section that one can define separate sets of fields M^a and M^m such that the matrix elements U^α_m, U^μ_a, $\overline{U}^{\bar\alpha}_{\bar m}$ and $\overline{U}^{\bar\mu}_{\bar a}$ in eqs. (2.11) all vanish in a finite patch. In the case of a Calabi-Yau compactification M^a and M^m are respectively (1,1) moduli and (1,2) moduli; for the (2,2) vacua that are not obviously related to the Calabi-Yau manifolds M^a are simply the moduli related to the **27** matter fields while M^m are the moduli related to the $\overline{\mathbf{27}}$'s. To simplify the terminology, we shall refer to these two types of moduli as (1,1) moduli and (1,2) moduli regardless of whether the (2,2) vacuum under consideration has anything to do with Calabi-Yau manifolds. In a basis that distinguishes between the two types of moduli fields moduli vertex operators are given by:

$$
\begin{array}{rlcll}
\text{(1,1) moduli} & M^a & \leftrightarrow & U^\alpha_a \cdot \Phi^+_\alpha\,, & a = 1, \ldots, N_1\,, \\
\text{(1,2) moduli} & M^m & \leftrightarrow & U^\mu_m \cdot \Phi^-_\mu\,, & m = 1, \ldots, N_2\,, \\
\text{anti-(1,1) moduli} & \overline{M}^{\bar a} & \leftrightarrow & \overline{U}^{\bar\alpha}_{\bar a} \cdot \Phi^-_{\bar\alpha}\,, & \bar a = 1, \ldots, N_1\,, \\
\text{anti-(1,2) moduli} & \overline{M}^{\bar m} & \leftrightarrow & \overline{U}^{\bar\mu}_{\bar m} \cdot \Phi^+_{\bar\mu}\,, & \bar m = 1, \ldots, N_2\,.
\end{array}
\tag{2.12}
$$

In our notations we shall distinguish (1,1) and (1,2) moduli from each other by labelling the former with lower case indices taken from the beginning of the latin alphabet while reserving the middle of the alphabet for the latter; capital latin indices will refer to moduli fields of either kind, *i.e.*, an M^A can be either an M^a or an M^m.

With these conventions, all string amplitudes involving moduli fields carry U factors; in particular, the metric for the (1,1) moduli fields is given by:

$$\frac{g_{a\bar{b}}}{|z-z'|^4} \;=\; U_a^{\alpha}\overline{U}_{\bar{b}}^{\bar{\beta}} \cdot \left\langle \Phi_{\alpha}^{+}(z,\bar{z}) \cdot \Phi_{\bar{\beta}}^{-}(z',\bar{z}') \right\rangle . \tag{2.13}$$

Now, using the current algebra (2.7) we can show that

$$\begin{aligned} \left\langle \Phi_{\alpha}^{+}(z,\bar{z}) \cdot \Phi_{\bar{\beta}}^{-}(z',\bar{z}') \right\rangle &= \oint_{z} \frac{dw}{2\pi i} \left\langle 2T_F^{-}(w) \cdot \Psi_{\alpha}^{+}(z,\bar{z}) \cdot \Phi_{\bar{\beta}}^{-}(z',\bar{z}') \right\rangle \\ &= \oint_{z'} \frac{dw}{2\pi i} \left\langle \Psi_{\alpha}^{+}(z,\bar{z}) \cdot 2T_F^{-}(w) \cdot \Phi_{\bar{\beta}}^{-}(z',\bar{z}') \right\rangle \\ &= \frac{\partial}{\partial z'} \left\langle \Psi_{\alpha}^{+}(z,\bar{z}) \cdot \Psi_{\bar{\beta}}^{-}(z',\bar{z}') \right\rangle . \end{aligned} \tag{2.14}$$

Hence, in view of eq. (2.10),

$$g_{a\bar{b}} \;=\; U_a^{\alpha}\, G_{\alpha\bar{\beta}}\, \overline{U}_{\bar{b}}^{\bar{\beta}} . \tag{2.15}$$

Similarly, the metric for the (1,2) moduli is given by

$$g_{m\bar{n}} \;=\; U_m^{\mu}\, G_{\mu\bar{\nu}}\, \overline{U}_{\bar{n}}^{\bar{\nu}} . \tag{2.16}$$

Equations (2.15) and (2.16) are examples of string-derived relations between different terms in the effective Lagrangian of the light scalars — in this case, kinetic terms for the moduli and for the matter fields. To make full use of these equations one obviously needs to know the U matrices; we shall compute them in the next section.

Now consider the matrix elements of the moduli metric $g_{A\bar{B}}$ that mix the (1,1) and the (1,2) moduli:

$$
\begin{aligned}
\frac{g_{a\bar{n}}}{|z-z'|^4} &= U^\alpha_a \overline{U}^{\bar{\nu}}_{\bar{n}} \cdot \left\langle \Phi^+_\alpha(z,\bar{z}) \cdot \Phi^+_{\bar{\nu}}(z',\bar{z}') \right\rangle \\
&= U^\alpha_a \overline{U}^{\bar{\nu}}_{\bar{n}} \times \oint \frac{dw}{2\pi i} \left\langle 2T^-_F(w) \cdot \Psi^+_\alpha(z,\bar{z}) \cdot \Phi^+_{\bar{\nu}}(z',\bar{z}') \right\rangle \;=\; 0
\end{aligned}
\tag{2.17}
$$

because the operator product of $T^-_F(w)$ and $\Phi^+(z')$ has no singularity at $w \to z'$; $g_{m\bar{b}}$ also vanishes for similar reasons. But the moduli space is a Kähler manifold, so $g_{a\bar{n}} = \hat{K}_{,a\bar{n}}$ and $g_{m\bar{b}} = \hat{K}_{,m\bar{b}}$; hence, block-diagonalization of the moduli metric implies that

$$
\hat{K}(M^A, \overline{M}^{\bar{A}}) \;=\; K_1(M^a, \overline{M}^{\bar{a}}) \;+\; K_2(M^m, \overline{M}^{\bar{m}})
\tag{2.18}
$$

(up to a harmonic function that does not affect the metric).* It immediately follows from eq. (2.18) that the moduli space is a direct product of two separate spaces for the (1,1) moduli and for the (1,2) moduli, *i.e.*, the metric for the former does not depend on the latter and vice versa. Note however that this argument only proves that the moduli space is a direct product as a metric space *provided* it is a direct product as a (complex) differentiable manifold, *i.e.*, provided one can consistently define separate complex fields M^a and M^m for the two kinds of moduli, and we haven't yet proved that this is possible. For the type II superstring, the moduli space of a (2,2) vacuum has to decompose into a product of two subspaces because the effective low-energy theory has $N = 2$ space-time supersymmetry ((1,1) and (1,2) moduli scalars belong to different types of $N = 2$ supermultiplets).[24] This result applies to the heterotic string as well since both

$\star$ This argument also explains why the dilaton/axion field D appears in (2.5) all by itself: The string origin of this field differs from any others massless scalar fields in the theory, and for any vacuum state of the heterotic string there are no metric terms that mix the D with other scalars.

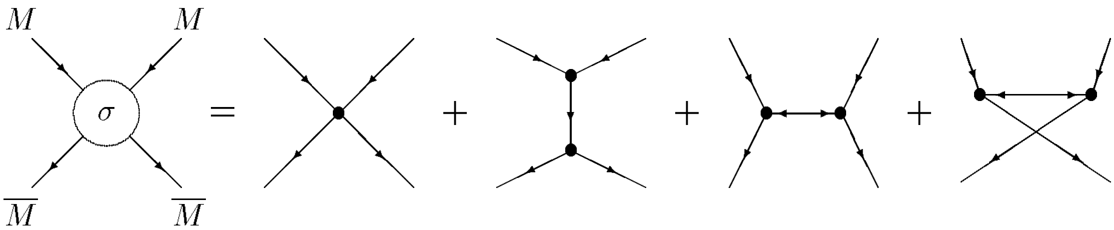
string theories yield the same moduli for the same (2,2) vacuum (see Appendix C for precise relations between moduli of the two string theories). However, one should not need the $N = 2$ four-dimensional supersymmetry to prove the moduli space decomposition for the heterotic string, and we shall give such a proof in the next section.

3. Reconstructing Geometry from String Amplitudes

3.1. Scattering Amplitudes in Field Theory.

In this section we shall follow the S-matrix program of ref. [10] that was outlined in the introduction to this article. Specifically, we shall compute four-particle scattering amplitudes that involve moduli and/or matter scalars. From the string theory point of view these amplitudes are related to each other through the current algebra (2.7). On the other hand, in field theory these amplitudes are controlled by seemingly unrelated terms in the effective Lagrangian. Specifically, four-moduli scattering amplitudes are controlled by the Riemann curvature of the moduli space, amplitudes involving two moduli and two matter fields are controlled by the moduli dependence of the metric for the matter fields, and amplitudes involving four matter fields are dominated by the gauge and Yukawa interactions.

Let us begin with the scattering of moduli. Having no potential and no gauge couplings, moduli fields interact with each other via sigma-model couplings that are present whenever the moduli space is not flat. The Feynman rules of the sigma model provide for four tree-level diagrams contributing to a four-point function:



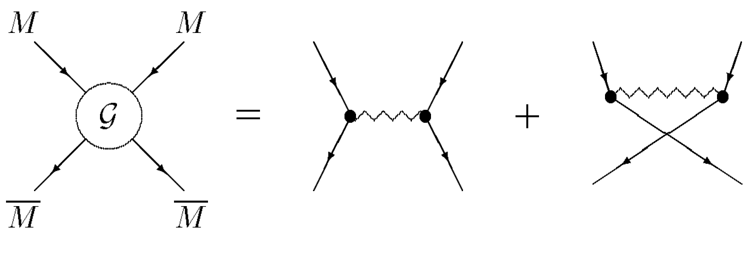
(3.1)

For a sigma model with a Kähler metric, vertices are given by the derivatives of K times the square of the total 4-momentum of the lines incoming to the vertex. (Incoming lines correspond to holomorphic fields, outgoing lines to anti-holomorphic fields.) Due to these kinematic factors, the last two diagrams in (3.1) vanish on the mass shell, while the combined effect of the first two diagrams is

$$\mathcal{A}_\sigma(M^A, M^B, \overline{M}^{\bar{C}}, \overline{M}^{\bar{D}}) \;=\; is\hat{K}_{,AB\bar{C}\bar{D}} + is\hat{K}_{,AB\bar{E}}\cdot\frac{ig^{\bar{E}F}}{s}\cdot is\hat{K}_{,F\bar{C}\bar{D}} \;\equiv\; isR_{A\bar{C}B\bar{D}}\,. \tag{3.2}$$

Here $s \equiv -(k_1 + k_2)^2$ is one of the three Mandelstam kinematic variables; the other two variables are $t \equiv -(k_1 + k_4)^2$ and $u \equiv -(k_1 + k_3)^2$. Note that in a Kähler geometry the Riemann tensor obeys $R_{A\bar{C}B\bar{D}} = R_{B\bar{C}A\bar{D}}$ in addition to the other symmetries under index permutations.

Besides sigma-model interactions, gravity also contributes to the scattering of moduli particles. A four-moduli amplitude gets contributions from the t-channel and the u-channel exchanges of a graviton:



(3.3)

which together yield

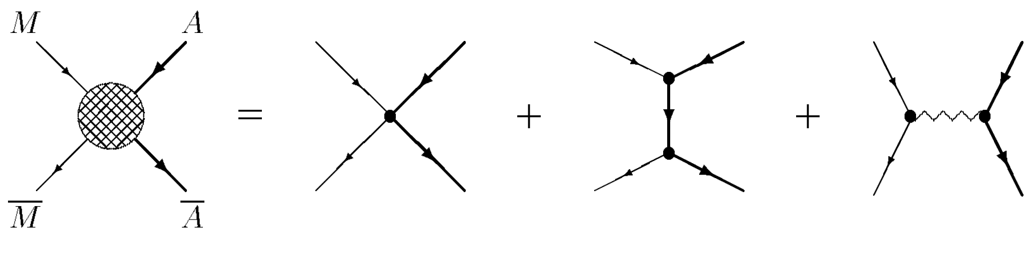
$$\mathcal{A}_{\rm grav}(M^A, M^B, \overline{M}^{\bar{C}}, \overline{M}^{\bar{D}}) \;=\; i\kappa^2 \frac{us}{t} \cdot g_{A\bar{D}}\, g_{B\bar{C}} \;+\; i\kappa^2 \frac{ts}{u} \cdot g_{A\bar{C}}\, g_{B\bar{D}} \tag{3.4}$$

(the s-channel exchange does not contribute since $g_{AB} = 0$). No other interactions present in the effective Lagrangian (2.1) contribute to the tree-level scattering of moduli scalars. Thus we can summarize the field theory amplitude for the four-moduli scattering amplitude as

$$\mathcal{A}_{\rm FT}(M^A, M^B, \overline{M}^{\bar{C}}, \overline{M}^{\bar{D}}) \;=\; is \cdot R_{A\bar{C}B\bar{D}} \;+\; i\kappa^2 \frac{us}{t} \cdot g_{A\bar{D}}\, g_{B\bar{C}} \;+\; i\kappa^2 \frac{ts}{u} \cdot g_{A\bar{C}}\, g_{B\bar{D}} \,. \tag{3.5}$$

Note that the right hand side of this formula depends solely on the geometry of the moduli space. Hence a string expression for the four-moduli scattering amplitude becomes a differential equation for $\hat{K}(M, \overline{M})$.

Next consider a scattering amplitude that involves two moduli and two matter scalars. Since the **27** and $\overline{\mathbf{27}}$ matter fields remain exactly massless for all values of the moduli, the scalar potential does not contribute to tree-level scattering amplitudes that involve only two matter scalars. Similarly, gauge interactions do not contribute to amplitudes involving only two charged particles. Thus the only interactions in (2.1) that contribute to the two-moduli two-matter amplitude are gravity and the two-derivative interactions due to moduli-dependence of the matter metric:



Therefore,

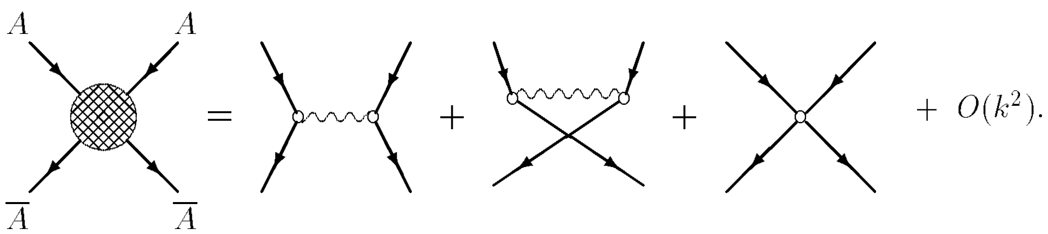
$$
\begin{aligned}
\mathcal{A}_{\mathrm{FT}}(M^A, A^\beta, \overline{A}^{\bar\gamma}, \overline{M}^{\bar D}) &= is \cdot R_{\beta\bar\gamma A\bar D} \;+\; i\kappa^2 \frac{us}{t} \cdot G_{\beta\bar\gamma}\, g_{A\bar D}\,, \\
\mathcal{A}_{\mathrm{FT}}(M^A, A^\lambda, \overline{A}^{\bar\mu}, \overline{M}^{\bar D}) &= is \cdot R_{\lambda\bar\mu A\bar D} \;+\; i\kappa^2 \frac{us}{t} \cdot G_{\lambda\bar\mu}\, g_{A\bar D}\,,
\end{aligned}
\tag{3.6}
$$

where

$$
\begin{aligned}
& R_{\beta\bar\gamma A\bar D} = K_{,A\beta\bar\gamma\bar D} - K_{,A\beta\bar\epsilon}\, G^{\bar\epsilon\eta}\, K_{,\eta\bar\gamma\bar D} = G_{\beta\bar\gamma,A\bar D} - G_{\beta\bar\epsilon,A}\, G^{\bar\epsilon\eta}\, G_{\eta\bar\gamma,\bar D} \\
\text{and}\quad & R_{\lambda\bar\mu A\bar D} = K_{,A\lambda\bar\mu\bar D} - K_{,A\lambda\bar\rho}\, G^{\bar\rho\sigma}\, K_{,\sigma\bar\mu\bar D} = G_{\lambda\bar\mu,A\bar D} - G_{\lambda\bar\rho,A}\, G^{\bar\rho\sigma}\, G_{\sigma\bar\mu,\bar D}
\end{aligned}
\tag{3.7}
$$

are mixed components of the Riemann tensor for the whole field space. Again, given a string expression for the two-moduli two-matter-fields amplitude, formulæ (3.6) become differential equations for the matter metrics $G_{\alpha\bar\beta}$ and $G_{\mu\bar\nu}$.

Finally, consider scattering processes that involve four matter fields and no moduli. Among the matter fields, two must transform as the **27** of E_6 while the other two transform as the $\overline{\mathbf{27}}$; there are three E_6-invariant ways to contract the gauge indices of these fields. For the sake of notational simplicity we shall not use E_6-invariant amplitudes; instead, we shall restrict our attention to the case when all four external fields belong to decuplets of the $SO(10) \subset E_6$; all other amplitudes can be reconstructed from these by the E_6 invariance. Several kinds of interactions contribute to scattering of four matter fields, but at low energies two effects dominate the amplitude: gauge scattering and contact four-scalar interactions due to quartic terms in the scalar potential (2.6):



(3.8)

Assuming the gauge group is exactly E_6 (no "accidentally" massless gauge bosons), the gauge scattering amplitudes are:

$$
\begin{aligned}
\mathcal{A}_{\rm gauge}(A^\alpha_{\hat p}, A^\beta_{\hat q}, \overline{A}^{\bar\gamma}_{\hat r}, \overline{A}^{\bar\delta}_{\hat s}) \;&=\; ie^2\,\frac{u-s}{t}\cdot G_{\alpha\bar\delta}\,G_{\beta\bar\gamma}\cdot\left(\delta^{\hat p\hat q}\delta^{\hat r\hat s}-\delta^{\hat p\hat r}\delta^{\hat q\hat s}-\tfrac{1}{3}\delta^{\hat p\hat s}\delta^{\hat q\hat r}\right)\\
&+\; ie^2\,\frac{t-s}{u}\cdot G_{\alpha\bar\gamma}\,G_{\beta\bar\delta}\cdot\left(\delta^{\hat p\hat q}\delta^{\hat r\hat s}-\delta^{\hat p\hat s}\delta^{\hat q\hat r}-\tfrac{1}{3}\delta^{\hat p\hat r}\delta^{\hat q\hat s}\right),\\
\mathcal{A}_{\rm gauge}(A^\alpha_{\hat p}, A^\lambda_{\hat q}, \overline{A}^{\bar\mu}_{\hat r}, \overline{A}^{\bar\delta}_{\hat s}) \;&=\; ie^2\,\frac{u-s}{t}\cdot G_{\alpha\bar\delta}\,G_{\lambda\bar\mu}\cdot\left(\delta^{\hat p\hat q}\delta^{\hat r\hat s}-\delta^{\hat p\hat r}\delta^{\hat q\hat s}+\tfrac{1}{3}\delta^{\hat p\hat s}\delta^{\hat q\hat r}\right),\\
\mathcal{A}_{\rm gauge}(A^\kappa_{\hat p}, A^\lambda_{\hat q}, \overline{A}^{\bar\mu}_{\hat r}, \overline{A}^{\bar\nu}_{\hat s}) \;&=\; ie^2\,\frac{u-s}{t}\cdot G_{\kappa\bar\nu}\,G_{\lambda\bar\mu}\cdot\left(\delta^{\hat p\hat q}\delta^{\hat r\hat s}-\delta^{\hat p\hat r}\delta^{\hat q\hat s}-\tfrac{1}{3}\delta^{\hat p\hat s}\delta^{\hat q\hat r}\right)\\
&+\; ie^2\,\frac{t-s}{u}\cdot G_{\kappa\bar\mu}\,G_{\lambda\bar\nu}\cdot\left(\delta^{\hat p\hat q}\delta^{\hat r\hat s}-\delta^{\hat p\hat s}\delta^{\hat q\hat r}-\tfrac{1}{3}\delta^{\hat p\hat r}\delta^{\hat q\hat s}\right),
\end{aligned}
\tag{3.9}
$$

where the factors $\frac{1}{3}$ come from the ratio between the coupling constants for the $SO(10)$ and the $U(1)$ subgroups of E_6; we normalize these couplings according to the convention that for the $SO(10)$ generators $Q^{(a)}$, $\mathrm{tr}_{\mathbf{10}}\big(Q^{(a)}Q^{(b)}\big)=2\delta^{(a)(b)}$.

Contact interactions due to the D-terms in the scalar potential (2.6) contribute to the scattering amplitudes expressions that are identical to eqs. (3.9), except that the kinematical factors $\frac{u-s}{t}$ and $\frac{t-s}{u}$ are absent. Finally, contact interactions due to the F-terms in (2.6) contribute

$$
\begin{aligned}
\mathcal{A}_F(A^\alpha_{\hat p}, A^\beta_{\hat q}, \overline{A}^{\bar\gamma}_{\hat r}, \overline{A}^{\bar\delta}_{\hat s}) \;&=\; -4i\,\frac{\exp\kappa^2\hat K}{D+\overline{D}}\cdot W_{\alpha\beta\epsilon}\,G^{\epsilon\bar\eta}\,\overline{W}_{\bar\eta\bar\gamma\bar\delta}\cdot\delta^{\hat p\hat q}\delta^{\hat r\hat s},\\
\mathcal{A}_F(A^\alpha_{\hat p}, A^\lambda_{\hat q}, \overline{A}^{\bar\mu}_{\hat r}, \overline{A}^{\bar\delta}_{\hat s}) \;&=\; -4i\,\frac{\exp\kappa^2\hat K}{D+\overline{D}}\cdot X_{\alpha\lambda\bar\mu\bar\delta}\cdot\delta^{\hat p\hat q}\delta^{\hat r\hat s},\\
\mathcal{A}_F(A^\kappa_{\hat p}, A^\lambda_{\hat q}, \overline{A}^{\bar\mu}_{\hat r}, \overline{A}^{\bar\nu}_{\hat s}) \;&=\; -4i\,\frac{\exp\kappa^2\hat K}{D+\overline{D}}\cdot W_{\kappa\lambda\rho}\,G^{\rho\bar\sigma}\,\overline{W}_{\bar\sigma\bar\mu\bar\nu}\cdot\delta^{\hat p\hat q}\delta^{\hat r\hat s}.
\end{aligned}
\tag{3.10}
$$

Note that in the low-energy limit both gauge and potential contributions behave like $O(1)$ under uniform rescaling of all 4-momenta. All other interactions, such as quartic terms in the Kähler function, or effects of moduli exchanges, or gravity, *etc.*, contribute terms that in the low-energy limit decrease like $O(k^2)$ or faster.

Hence, as far as the effective field theory is concerned, the scattering amplitudes for four matter fields are:

$$
\begin{aligned}
\frac{i}{2e^2}\,\mathcal{A}_{\rm FT}(A^{\alpha}_{\hat p}, A^{\beta}_{\hat q}, \overline{A}^{\bar\gamma}_{\hat r}, \overline{A}^{\bar\delta}_{\hat s}) \;&=\; \frac{s}{t}\cdot G_{\alpha\bar\delta}\,G_{\beta\bar\gamma}\cdot\Big(\delta^{\hat p\hat q}\delta^{\hat r\hat s}-\delta^{\hat p\hat r}\delta^{\hat q\hat s}-\tfrac{1}{3}\delta^{\hat p\hat s}\delta^{\hat q\hat r}\Big)\\
&+\;\frac{s}{u}\cdot G_{\alpha\bar\gamma}\,G_{\beta\bar\delta}\cdot\Big(\delta^{\hat p\hat q}\delta^{\hat r\hat s}-\delta^{\hat p\hat s}\delta^{\hat q\hat r}-\tfrac{1}{3}\delta^{\hat p\hat r}\delta^{\hat q\hat s}\Big)\\
&+\;\exp(\kappa^2\hat K)\cdot W_{\alpha\beta\epsilon}\,G^{\epsilon\bar\eta}\,\overline{W}_{\bar\eta\bar\gamma\bar\delta}\cdot\delta^{\hat p\hat q}\delta^{\hat r\hat s}\;+\;O(k^2)\,;\\
\frac{i}{2e^2}\,\mathcal{A}_{\rm FT}(A^{\alpha}_{\hat p}, A^{\lambda}_{\hat q}, \overline{A}^{\bar\mu}_{\hat r}, \overline{A}^{\bar\delta}_{\hat s}) \;&=\; \frac{s}{t}\cdot G_{\alpha\bar\delta}\,G_{\lambda\bar\mu}\cdot\Big(\delta^{\hat p\hat q}\delta^{\hat r\hat s}-\delta^{\hat p\hat r}\delta^{\hat q\hat s}+\tfrac{1}{3}\delta^{\hat p\hat s}\delta^{\hat q\hat r}\Big)\\
&+\;\exp(\kappa^2\hat K)\cdot X_{\alpha\lambda\bar\mu\bar\delta}\cdot\delta^{\hat p\hat q}\delta^{\hat r\hat s}\;+\;O(k^2)\,;\\
\frac{i}{2e^2}\,\mathcal{A}_{\rm FT}(A^{\kappa}_{\hat p}, A^{\lambda}_{\hat q}, \overline{A}^{\bar\mu}_{\hat r}, \overline{A}^{\bar\nu}_{\hat s}) \;&=\; \frac{s}{t}\cdot G_{\kappa\bar\nu}\,G_{\lambda\bar\mu}\cdot\Big(\delta^{\hat p\hat q}\delta^{\hat r\hat s}-\delta^{\hat p\hat r}\delta^{\hat q\hat s}-\tfrac{1}{3}\delta^{\hat p\hat s}\delta^{\hat q\hat r}\Big)\\
&+\;\frac{s}{u}\cdot G_{\kappa\bar\mu}\,G_{\lambda\bar\nu}\cdot\Big(\delta^{\hat p\hat q}\delta^{\hat r\hat s}-\delta^{\hat p\hat s}\delta^{\hat q\hat r}-\tfrac{1}{3}\delta^{\hat p\hat r}\delta^{\hat q\hat s}\Big)\\
&+\;\exp(\kappa^2\hat K)\cdot W_{\kappa\lambda\rho}\,G^{\rho\bar\sigma}\,\overline{W}_{\bar\sigma\bar\mu\bar\nu}\cdot\delta^{\hat p\hat q}\delta^{\hat r\hat s}\;+\;O(k^2)\,;
\end{aligned}
\tag{3.11}
$$

we have used $2/(D+\overline{D})=e^2$ and $s+t+u=0$ in deriving these formulæ.

3.2. String Relations Between Scattering Amplitudes.

At this point we know which scattering amplitudes we need in order to reconstruct the kinetic terms in the low-energy effective Lagrangian, so let us compute them. Let us begin with the string amplitude $\mathcal{A}(M^a, A^\beta, \overline{A}^{\bar\gamma}, \overline{M}^{\bar d})$. There is only one E_6-invariant amplitude of this kind, so without loss of generality we can choose the matter particles to belong to $SO(10)$ decuplets; thus we have

$$
\begin{aligned}
\mathcal{A}(M^a, A^{\beta}_{\hat p}, \overline{A}^{\bar\gamma}_{\hat q}, \overline{M}^{\bar d}) \;&=\; U^{\alpha}_{a}\overline{U}^{\bar\delta}_{\bar d}\times \\
&\times |J(z_1,z_2,z_3)|^2\int_{\mathbf{C}} d^2 z_4\, E(z_j,\bar z_j)\cdot\Big\langle \lambda^{\hat p}(z_2)\cdot\lambda^{\hat q}(z_3)\Big\rangle\times\\
&\times\Big\langle \Phi^{+}_{\alpha}(z_1,\bar z_1)\cdot\Psi^{+}_{\beta}(z_2,\bar z_2)\cdot\Psi^{-}_{\bar\gamma}(z_3,\bar z_3)\cdot\Phi^{-}_{\bar\delta}(z_4,\bar z_4)\Big\rangle\,.
\end{aligned}
\tag{3.12}
$$

Here J is the Jacobian for using $SL_2(\mathbf{C})$ to fix z_1, z_2, z_3 and is independent of z_4, and $E(z_j, \bar{z}_j)$ is the correlation function of the exponentials $e^{i\sqrt{2\alpha'}k_j\cdot X}$ appearing in the vertex operators evaluated at non-zero momenta k_j; an explicit expression for E in terms of $z_{ij} \equiv z_i - z_j$ is given by

$$
\begin{aligned}
E(z_j, \bar{z}_j) \; &\equiv \left\langle \prod_{j=1}^{4} \exp\bigl(i\sqrt{2\alpha'}k_j \cdot X(z_j, \bar{z}_j)\bigr) \right\rangle \\
&= \prod_{i<j} |z_{ij}|^{\alpha' k_i \cdot k_j} \;\equiv\; |z_{12}z_{34}|^{-\alpha' s/2} \; |z_{14}z_{23}|^{-\alpha' t/2} \; |z_{13}z_{24}|^{-\alpha' u/2} .
\end{aligned}
\tag{3.13}
$$

Beyond the $e^{i\sqrt{2\alpha'}k_j\cdot X}$ factors, heterotic vertex operators $\Phi^{\pm}$ and $\Psi^{\pm}$ themselves depend on the momenta. However, this dependence only affects the right-moving degrees of freedom and completely commutes with the left-moving $N = 2$ superalgebra; since it is the left-moving superalgebra that we are going to use here, we can safely ignore the momentum dependence of $\Phi^{\pm}$ and $\Psi^{\pm}$.

Now, let us apply the current algebra (2.7) to the correlator that appears in eq. (3.12). Using the operator product expansion of T_F^- with Ψ^+, we can write

$$
\left\langle \Phi_1^+ \cdot \Psi_2^+ \cdot \Psi_3^- \cdot \Phi_4^- \right\rangle \; = \oint_{z_1} \frac{dw}{2\pi i}\, \frac{w-\zeta}{z_1-\zeta} \, \left\langle 2T_F^-(w) \cdot \Psi_1^+ \cdot \Psi_2^+ \cdot \Psi_3^- \cdot \Phi_4^- \right\rangle , \tag{3.14}
$$

where ζ is an arbitrary complex number, Φ_1^+ is a short-hand notation for $\Phi_\alpha^+(z_1, \bar{z}_1)$, *etc.*, and the contour of integration circles z_1 but not the other vertices. The same contour can be reinterpreted as circling (in the opposite direction) z_2, z_3, z_4 and ∞ instead of z_1; however, the integrand has no singularity at z_3 (the operator product of $T_F^-(w)$ with Ψ_3^- is non-singular at $w \to z_3$), and the single pole at $w \to z_2$ can be cancelled by choosing $\zeta = z_2$. The integral around infinity also vanishes: since the conformal dimension of T_F is $\frac{3}{2}$, the correlator is $O(w^{-3})$ at $w \to \infty$ and the leading term in the integrand of (3.14) behaves like w^{-2}.

Hence, the only contribution to the contour integral comes from the singularity at $w \to z_4$ that yields

$$\oint_{z_4} \frac{dw}{2\pi i} \frac{w - z_2}{z_1 - z_2} \cdot 2T_F^-(w) \cdot \Phi_4^- \;=\; \frac{\partial}{\partial z_4} \left(\frac{z_4 - z_2}{z_1 - z_2} \cdot \Psi_4^- \right) , \tag{3.15}$$

and the integral on the right hand side of eq. (3.12) can be rewritten as

$$\begin{aligned} &\int_{\mathbf{C}} d^2 z_4\, E \cdot \frac{\delta^{\hat{p}\hat{q}}}{z_{23}} \cdot \frac{\partial}{\partial z_4} \left(\frac{z_{42}}{z_{12}} \left\langle \Psi_1^+ \cdot \Psi_2^+ \cdot \Psi_3^- \cdot \Psi_4^- \right\rangle \right) \\ &\qquad = \; \delta^{\hat{p}\hat{q}} \times \int_{\mathbf{C}} d^2 z_4\, E \cdot \left\langle \Psi_1^+ \cdot \Psi_2^+ \cdot \Psi_3^- \cdot \Psi_4^- \right\rangle \cdot \left(\frac{\alpha' s/4}{z_{12} z_{34}} - \frac{\alpha' t/4}{z_{14} z_{23}} \right) . \end{aligned} \tag{3.16}$$

Here the factor $\delta^{\hat{p}\hat{q}}/z_{23}$ comes from $\left\langle \lambda_2^{\hat{p}} \cdot \lambda_3^{\hat{q}} \right\rangle$, and the second expression follows from the first via integration by parts (we use eq. (3.13) and $s + t + u = 0$).

Next consider the amplitude $\mathcal{A}(A^\alpha, A^\beta, \overline{A}^{\bar\gamma}, \overline{A}^{\bar\delta})$. Choosing all four matter particles to belong to the decuplets of the $SO(10)$, we have

$$\mathcal{A}(A^\alpha_{\hat{p}}, A^\beta_{\hat{q}}, \overline{A}^{\bar\gamma}_{\hat{r}}, \overline{A}^{\bar\delta}_{\hat{s}}) \;=\; |J|^2 \int_{\mathbf{C}} d^2 z_4\, E \cdot \left\langle \Psi_1^+ \cdot \Psi_2^+ \cdot \Psi_3^- \cdot \Psi_4^- \right\rangle \cdot \left\langle \lambda_1^{\hat{p}} \cdot \lambda_2^{\hat{q}} \cdot \lambda_3^{\hat{r}} \cdot \lambda_4^{\hat{s}} \right\rangle . \tag{3.17}$$

All we need now is an explicit expression for the correlator of the gauge fermions:

$$\left\langle \lambda_1^{\hat{p}} \cdot \lambda_2^{\hat{q}} \cdot \lambda_3^{\hat{r}} \cdot \lambda_4^{\hat{s}} \right\rangle \;=\; \left(\frac{\delta^{\hat{p}\hat{q}} \delta^{\hat{r}\hat{s}}}{z_{12} z_{34}} + \frac{\delta^{\hat{p}\hat{s}} \delta^{\hat{q}\hat{r}}}{z_{14} z_{23}} - \frac{\delta^{\hat{p}\hat{r}} \delta^{\hat{q}\hat{s}}}{z_{13} z_{24}} \right) . \tag{3.18}$$

If we now compare the right hand sides of eqs. (3.16) and (3.17), it becomes apparent that

$$\mathcal{A}(M^a, A^\beta_{\hat{p}}, \overline{A}^{\bar\gamma}_{\hat{p}}, \overline{M}^{\bar{d}}) \;=\; \tfrac{\alpha'}{4}\, U^\alpha_a \overline{U}^{\bar\delta}_{\bar{d}} \times \Big(s \cdot \mathcal{A}(A^\alpha_{\hat{p}}, A^\beta_{\hat{p}}, \overline{A}^{\bar\gamma}_{\hat{q}}, \overline{A}^{\bar\delta}_{\hat{q}}) \;-\; t \cdot \mathcal{A}(A^\alpha_{\hat{p}}, A^\beta_{\hat{q}}, \overline{A}^{\bar\gamma}_{\hat{q}}, \overline{A}^{\bar\delta}_{\hat{p}}) \Big) \tag{3.19}$$

(no sum over the $SO(10)$ vector indices $\hat{p} \neq \hat{q}$). Formula (3.19) is the first of several relations between various string amplitudes that we shall use as constraints on the low-energy effective field theory.

There are other two-moduli two-matter-fields amplitudes: The two moduli fields may be of the (1,2) type rather than the (1,1) type as in eq. (3.19), and the two matter fields may be $\overline{\mathbf{27}}$'s rather than $\mathbf{27}$'s. All these amplitudes involve world-sheet correlators of the type $\langle \Phi^+ \cdot \Psi^+ \cdot \Psi^- \cdot \Phi^- \rangle$, where Φ^+ is either Φ^+_α or $\Phi^+_{\bar\mu}$, Ψ^+ is either Ψ^+_α or $\Psi^+_{\bar\mu}$, *etc.* In the arguments leading to formula (3.19) we relied on the left-moving current algebra (2.7), which is not affected by the right-moving quantum numbers that distinguish between the $\mathbf{27}$ and anti-$\overline{\mathbf{27}}$ matter fields or between (1,1) and anti-(1,2) moduli. Hence, the very same arguments (modulo permutations of particles) also yield

$$\begin{aligned}
\mathcal{A}(M^k, A^\beta_{\hat p}, \overline{A}^{\bar\gamma}_{\hat p}, \overline{M}^{\bar n}) &= \tfrac{\alpha'}{4} U^\kappa_k \overline{U}^{\bar\nu}_{\bar n} \cdot \Big(u\mathcal{A}(A^\kappa_{\hat p}, A^\beta_{\hat q}, \overline{A}^{\bar\gamma}_{\hat p}, \overline{A}^{\bar\nu}_{\hat q}) - t\mathcal{A}(A^\kappa_{\hat p}, A^\beta_{\hat q}, \overline{A}^{\bar\gamma}_{\hat q}, \overline{A}^{\bar\nu}_{\hat p}) \Big) \, ; \\
\mathcal{A}(M^a, A^\lambda_{\hat p}, \overline{A}^{\bar\mu}_{\hat p}, \overline{M}^{\bar d}) &= \tfrac{\alpha'}{4} U^\alpha_a \overline{U}^{\bar\delta}_{\bar d} \cdot \Big(u\mathcal{A}(A^\alpha_{\hat p}, A^\lambda_{\hat q}, \overline{A}^{\bar\mu}_{\hat p}, \overline{A}^{\bar\delta}_{\hat q}) - t\mathcal{A}(A^\alpha_{\hat p}, A^\lambda_{\hat q}, \overline{A}^{\bar\mu}_{\hat q}, \overline{A}^{\bar\delta}_{\hat p}) \Big) \, ; \\
\mathcal{A}(M^k, A^\lambda_{\hat p}, \overline{A}^{\bar\mu}_{\hat p}, \overline{M}^{\bar n}) &= \tfrac{\alpha'}{4} U^\kappa_k \overline{U}^{\bar\nu}_{\bar n} \cdot \Big(s\mathcal{A}(A^\kappa_{\hat p}, A^\lambda_{\hat p}, \overline{A}^{\bar\mu}_{\hat q}, \overline{A}^{\bar\nu}_{\hat q}) - t\mathcal{A}(A^\kappa_{\hat p}, A^\lambda_{\hat q}, \overline{A}^{\bar\mu}_{\hat q}, \overline{A}^{\bar\nu}_{\hat p}) \Big)
\end{aligned} \tag{3.20}$$

(again, no sum over $\hat p \neq \hat q$).

On the other hand, amplitudes that involve one (1,1) modulus and one (1,2) modulus (and two matter fields) vanish identically since they involve the vanishing correlators $\langle \Phi^+ \Psi^+ \Psi^- \Phi^+ \rangle$ and $\langle \Phi^- \Psi^+ \Psi^- \Phi^- \rangle$. These two correlators are complex conjugates of each other, so it is sufficient to verify that

$$\langle \Phi^+_1 \cdot \Psi^+_2 \cdot \Psi^-_3 \cdot \Phi^+_4 \rangle \ = \oint\limits_{z_1} \frac{dw}{2\pi i} \, \frac{w - z_2}{z_1 - z_2} \cdot \langle 2T^-_F(w) \cdot \Psi^+_1 \cdot \Psi^+_2 \cdot \Psi^-_3 \cdot \Phi^+_4 \rangle \ = \ 0: \tag{3.21}$$

The first equality here is exactly analogous to eq. (3.14), and the contour integral is evaluated in exactly the same way; however, the operator product of $T^-_F(w)$ and Φ^+_4 has no singularity when $w \to z_4$, and the integral vanishes. This argument assures the vanishing of "mixed" amplitudes $\mathcal{A}(M^a, A^\beta, \overline{A}^{\bar\gamma}, \overline{M}^{\bar n})$ and $\mathcal{A}(M^a, A^\lambda, \overline{A}^{\bar\mu}, \overline{M}^{\bar n})$ (and their complex conjugates) and completes the coverage of all four-particle amplitudes involving one modulus, one matter field, one anti-matter field and one anti-modulus.*

Now consider a four-(1,1)-moduli scattering amplitude

$$\mathcal{A}(M^a, M^b, \overline{M^c}, \overline{M^{\bar d}}) \;=\; U_a^\alpha U_b^\beta \overline{U}_{\bar c}^{\bar\gamma} \overline{U}_{\bar d}^{\bar\delta} \;\times\; |J|^2 \int\limits_{\mathbf{C}} d^2 z_4\, E \cdot \left\langle \Phi_1^+ \cdot \Phi_2^+ \cdot \Phi_3^- \cdot \Phi_4^- \right\rangle . \tag{3.22}$$

Using the current algebra (2.7) we can write Φ_1^+ as a contour integral of $2T_F^-(w)$ around Ψ_1^+ and then pull the contour of integration from the back of the complex sphere so it runs around $z_{2,3,4}$ instead of z_1. But the integral of $2T_F^-(w)$ around a $\Phi^+(z)$ vanishes while the integral around a $\Phi^-(z)$ yields a $\partial_z \Psi^-$, so we obtain

$$\left\langle \Phi_1^+ \cdot \Phi_2^+ \cdot \Phi_3^- \cdot \Phi_4^- \right\rangle \;=\; \frac{\partial}{\partial z_3} \left\langle \Psi_1^+ \cdot \Phi_2^+ \cdot \Psi_3^- \cdot \Phi_4^- \right\rangle \;+\; \frac{\partial}{\partial z_4} \left\langle \Psi_1^+ \cdot \Phi_2^+ \cdot \Phi_3^- \cdot \Psi_4^- \right\rangle . \tag{3.23}$$

Retracing the steps that led us from eq. (3.12) to eq. (3.16), we can express the correlators on the right hand side of (3.23) in terms of $\Psi^\pm$ only; this gives us

$$\left\langle \Phi_1^+ \cdot \Phi_2^+ \cdot \Phi_3^- \cdot \Phi_4^- \right\rangle \;=\; -\frac{\partial}{\partial z_3} \frac{\partial}{\partial z_4} \left(\frac{z_{34}}{z_{12}} \cdot \left\langle \Psi_1^+ \cdot \Psi_2^+ \cdot \Psi_3^- \cdot \Psi_4^- \right\rangle \right) . \tag{3.24}$$

When computing a tree-level four-particle string amplitude such as (3.22), we keep locations of three vertices fixed and integrate over the fourth. Because of $SL_2(\mathbf{C})$ invariance[35] the integrand is exactly the same regardless of which of $z_{1,2,3,4}$ is used as an integration variable; the only difference is that the Jacobian J depends on the other three vertex locations. Hence, if the integrand has the form $A \cdot \partial B / \partial z_j$, $j = 1, 2, 3, 4$, we can choose z_j to be the integration variable and integrate by parts, then keep the new integrand but integrate over z_i with $i \neq j$.

$\star$ These are the only two-moduli two-matter-fields amplitudes that we will need in this article. Among other amplitudes, $\mathcal{A}(A^\alpha, A^\lambda, \overline{M^c}, \overline{M^{\bar n}})$ and its complex conjugate obey equations similar to eqs. (3.19) and (3.20), while all the remaining two-moduli two-matter-fields amplitudes vanish identically.

This technique allows integration by parts over any of the $z_{1,2,3,4}$ regardless of which z_i is the integration variable. Thus, after we substitute (3.24) into (3.22), we can integrate by parts over both z_3 and z_4, and the integral on the right hand side of eq. (3.22) becomes

$$\int_{\mathbf{C}} d^2 z_4\, E\cdot\left\langle\Psi_1^+\cdot\Psi_2^+\cdot\Psi_3^-\cdot\Psi_4^-\right\rangle\cdot\left(\frac{(\alpha' s/4)+(\alpha' s/4)^2}{z_{12}z_{34}}\ +\ \frac{(\alpha' t/4)^2}{z_{14}z_{23}}\ -\ \frac{(\alpha' u/4)^2}{z_{13}z_{24}}\right). \tag{3.25}$$

It remains to compare (3.23) to the four-matter-fields amplitudes (3.17); this yields:

$$\begin{aligned}\mathcal{A}(M^a,M^b,\overline{M}^{\bar c},\overline{M}^{\bar d})\ &=\ U^\alpha_a U^\beta_b \overline{U}^{\bar\gamma}_{\bar c}\overline{U}^{\bar\delta}_{\bar d}\ \times \\ &\times\ \left[\left(\frac{\alpha' s}{4}+\frac{\alpha'^2 s^2}{16}\right)\mathcal{A}(A^\alpha_{\hat p},A^\beta_{\hat p},\overline{A}^{\bar\gamma}_{\hat q},\overline{A}^{\bar\delta}_{\hat q})\ +\ \frac{\alpha'^2 t^2}{16}\mathcal{A}(A^\alpha_{\hat p},A^\beta_{\hat q},\overline{A}^{\bar\gamma}_{\hat q},\overline{A}^{\bar\delta}_{\hat p})\right. \\ &\qquad\left.+\ \frac{\alpha'^2 u^2}{16}\mathcal{A}(A^\alpha_{\hat p},A^\beta_{\hat q},\overline{A}^{\bar\gamma}_{\hat p},\overline{A}^{\bar\delta}_{\hat q})\right]\end{aligned} \tag{3.26}$$

(no sum over $\hat p\neq\hat q$).

Finally, consider other four-moduli scattering amplitudes. Given two types of holomorphic moduli, there are nine types of amplitudes that involve two holomorphic and two anti-holomorphic moduli fields. Three types of amplitudes are related to world-sheet correlators $\left\langle\Phi^+\Phi^+\Phi^-\Phi^-\right\rangle$; these amplitudes, to the order in α' necessary for this article, are summarized in the following formulæ:

$$\begin{aligned}\mathcal{A}(M^a,M^b,\overline{M}^{\bar c},\overline{M}^{\bar d})\ &=\ \frac{\alpha' s}{4}\,U^\alpha_a U^\beta_b \overline{U}^{\bar\gamma}_{\bar c}\overline{U}^{\bar\delta}_{\bar d}\cdot\mathcal{A}(A^\alpha_{\hat p},A^\beta_{\hat p},\overline{A}^{\bar\gamma}_{\hat q},\overline{A}^{\bar\delta}_{\hat q})\ +\ O(\alpha'^2 k^4);\\ \mathcal{A}(M^a,M^l,\overline{M}^{\bar m},\overline{M}^{\bar d})\ &=\ \frac{\alpha' u}{4}\,U^\alpha_a U^\lambda_l \overline{U}^{\bar\mu}_{\bar m}\overline{U}^{\bar\delta}_{\bar d}\cdot\mathcal{A}(A^\alpha_{\hat p},A^\lambda_{\hat q},\overline{A}^{\bar\mu}_{\hat p},\overline{A}^{\bar\delta}_{\hat q})\ +\ O(\alpha'^2 k^4);\\ \mathcal{A}(M^k,M^l,\overline{M}^{\bar m},\overline{M}^{\bar n})\ &=\ \frac{\alpha' s}{4}\,U^\kappa_k U^\lambda_l \overline{U}^{\bar\mu}_{\bar m}\overline{U}^{\bar\nu}_{\bar n}\cdot\mathcal{A}(A^\kappa_{\hat p},A^\lambda_{\hat p},\overline{A}^{\bar\mu}_{\hat q},\overline{A}^{\bar\nu}_{\hat q})\ +\ O(\alpha'^2 k^4).\end{aligned} \tag{3.27}$$

(The first equation here is eq. (3.26). The second equation substitutes $\overline{M}^{\bar m}$, M^l, $\overline{A}^{\bar\mu}$ and A^λ for M^b, $\overline{M}^{\bar c}$, A^β and $\overline{A}^{\bar\gamma}$, respectively, and can be proven by exactly the same arguments as (3.26); we have interchanged particles 2 and 3 for future convenience. The third equation substitutes in addition $\overline{M}^{\bar{n}}$, M^k, $\overline{A}^{\bar{\nu}}$ and A^κ for M^a, $\overline{M}^{\bar{d}}$, A^α and $\overline{A}^{\bar{\delta}}$, respectively, and interchanges particles 1 and 4.)

The other six four-moduli amplitudes are related to correlators $\left\langle\Phi^+\Phi^+\Phi^+\Phi^+\right\rangle$ and $\left\langle\Phi^+\Phi^+\Phi^+\Phi^-\right\rangle$ (and their hermitian conjugates) that vanish identically. Indeed,

$$\left\langle\Phi_1^+\cdot\Phi_2^+\cdot\Phi_3^+\cdot\Phi_4^+\right\rangle \;=\; \oint\limits_{z_1}\frac{dw}{2\pi i}\left\langle 2T_F^-(w)\cdot\Psi_1^+\cdot\Phi_2^+\cdot\Phi_3^+\cdot\Phi_4^+\right\rangle \;=\; 0 \qquad (3.28)$$

since the operator product of T_F^- and Φ^+ has no singularity, and

$$\begin{aligned}\left\langle\Phi_1^+\cdot\Phi_2^+\cdot\Phi_3^+\cdot\Phi_4^-\right\rangle \;&=\; \oint\limits_{z_1}\frac{dw}{2\pi i}\,\frac{w-z_4}{z_1-z_4}\left\langle 2T_F^-(w)\cdot\Psi_1^+\cdot\Phi_2^+\cdot\Phi_3^+\cdot\Phi_4^-\right\rangle\\ &=\; \frac{1}{z_{14}}\left\langle\Psi_1^+\cdot\Phi_2^+\cdot\Phi_3^+\cdot\Psi_4^-\right\rangle \;=\; 0,\end{aligned} \qquad (3.29)$$

where the last equality is just eq. (3.21).

3.3. Riemann Tensor, Moduli Space Decomposition and Metric for the Matter Fields.

Having established several string relations between various four-particle scattering amplitudes, let us use these relations to describe the geometry of the field space. We begin with the Riemann tensor of the moduli space, which is related by formula (3.5) to the scattering amplitudes for four moduli fields. Our string formulæ for these amplitudes distinguish between the (1,1) moduli and the (1,2) moduli, so we shall compute the components of the Riemann tensor in a basis that respects this distinction. (Local bases of this kind exist for all points of the moduli space regardless of whether these bases are consistent with a global coordinate system.) In such a basis, all components of $R_{A\bar{C}B\bar{D}}$ except $R_{a\bar{c}b\bar{d}}$, $R_{k\bar{m}l\bar{n}}$

and $R_{a\bar{m}l\bar{d}}$ (and components related to these by index permutations) must vanish because of the vanishing of the corresponding string amplitudes. Moreover, $R_{a\bar{m}l\bar{d}}$ vanishes too. To see this, let us substitute the second formula in (3.11) into the second formula in (3.27); the result is

$$\begin{aligned} \mathcal{A}(M^a, M^l, \overline{M}^{\bar{m}}, \overline{M}^{\bar{d}}) &= \frac{ie^2\alpha'}{2} \cdot \frac{su}{t} \cdot \left(U_a^\alpha\, G_{\alpha\bar{\delta}}\, \overline{U}_{\bar{d}}^{\bar{\delta}}\right) \left(U_l^\lambda\, G_{\lambda\bar{\mu}}\, \overline{U}_{\bar{m}}^{\bar{\mu}}\right) \\ &= i\kappa^2 \cdot \frac{su}{t} \cdot g_{a\bar{d}}\, g_{l\bar{m}} \ . \end{aligned} \tag{3.30}$$

Here the second equation follows from the first because of eqs. (2.15) and (2.16) and because in the heterotic string theory gauge and gravitational couplings are related to each other via $\alpha' e^2 = 2\kappa^2$.[1] In view of formulæ (3.5) and (2.17), eq. (3.30) implies that $R_{a\bar{m}l\bar{d}} = 0$.

Now consider the holonomy group of the moduli space. The restricted holonomy group of a Riemannian manifold is generated by parallel transport along contractible loops in that manifold, and the associated Lie algebra is generated by components of the Riemann tensor viewed as matrices in the first two indices of R. We have just seen that for the moduli space the only non-vanishing components of the Riemann tensor are $R_{a\bar{c}b\bar{d}}$ and $R_{k\bar{m}l\bar{n}}$, so the restricted holonomy group of the moduli space must be contained in $U(N_1) \otimes U(N_2)$. But there is a theorem valid for any Riemannian geometry that says that if the (restricted) holonomy group of a manifold decomposes into a direct product of commuting subgroups, then the manifold is (locally) a direct product of several submanifolds.[36] In our case, this means that there are independent moduli spaces for the (1,1) moduli and the (1,2) moduli; both of these spaces are Kähler, and the full moduli space is their direct product (hence (2.18)*). Note that only a local basis

$\star$ Actually, the statement that the moduli space decomposes refers to its metric rather than to its Kähler function $\hat{K}$. This decomposition implies eq. (2.18) only after a Kähler transform that changes $\hat{K}$ by a harmonic function $\Lambda(M) + \overline{\Lambda}(\overline{M})$ that does not affect the metric ($\Lambda(M)$ is a holomorphic function of both M^a and M^m). In $N = 1$ supergravity, a is needed to evaluate the restricted holonomy group. Therefore, in contrast to our argument at the end of section 2, this time the possibility of parametrizing the moduli space with separate (1,1) and (1,2) moduli fields is not assumed but proved.

Having verified that the moduli space is a direct product of separate moduli spaces for the (1,1) moduli and the (1,2) moduli, we would like to study the geometry of each component. Riemann tensors $R_{a\bar{c}b\bar{d}}$ and $R_{k\bar{m}l\bar{n}}$ for the (1,1) and (1,2) moduli spaces can be obtained from formulæ (3.5), (3.27) and (3.11). After some algebra that uses eqs. (2.15), (2.16) and $\alpha' e^2 = 2\kappa^2$, we obtain

$$\begin{aligned}
\frac{1}{\kappa^2} R_{a\bar{c}b\bar{d}} \;&=\; g_{a\bar{c}}\, g_{b\bar{d}} \;+\; g_{a\bar{d}}\, g_{b\bar{c}} \;-\; \exp(\kappa^2 \hat{K}) \cdot (WUUU)_{abe}\, g^{e\bar{f}}\, (\overline{WUUU})_{\bar{f}\bar{c}\bar{d}}\,, \\
\frac{1}{\kappa^2} R_{k\bar{m}l\bar{n}} \;&=\; g_{k\bar{m}}\, g_{l\bar{n}} \;+\; g_{k\bar{n}}\, g_{l\bar{m}} \;-\; \exp(\kappa^2 \hat{K}) \cdot (WUUU)_{kli}\, g^{i\bar{j}}\, (\overline{WUUU})_{\bar{j}\bar{m}\bar{n}}\,,
\end{aligned} \tag{3.31}$$

where $(WUUU)_{abe}$ is a short-hand notation for $W_{\alpha\beta\epsilon} U^\alpha_a U^\beta_b U^\epsilon_e$, *etc.* Unfortunately, equations (3.31) cannot be solved for the moduli metrics $g_{a\bar{b}}$ and $g_{m\bar{n}}$ until we know the U matrices that appear in these equations. To compute the U matrices we need additional equations relating moduli and matter metrics; such equations are provided by string formulæ (3.19), (3.20) and $\mathcal{A}(M^a, A^\beta, \overline{A}^{\bar{\gamma}}, \overline{M}^{\bar{n}}) = \cdots = 0$ for scattering amplitudes that involve both moduli and matter fields. Combining these formulæ with eqs. (3.6) and (3.11) we arrive at the following expressions for the $R_{\beta\bar{\gamma}A\bar{D}}$ components of the Riemann tensor of the whole field space:

Kähler transform that is accompanied by rescaling the superpotential by the holomorphic factor $\exp(-\kappa^2 \Lambda(M))$ is unobservable, so through the rest of this article we shall assume that eq. (2.18) holds exactly (*i.e.*, without extra harmonic terms).

$$
\begin{aligned}
\frac{1}{\kappa^2}\,R_{\beta\bar\gamma a\bar d} &= U^\alpha_a G_{\alpha\bar\gamma}\cdot G_{\beta\bar\delta}\overline{U}^{\bar\delta}_{\bar d} \;+\; \tfrac{2}{3}\,G_{\beta\bar\gamma}\cdot g_{a\bar d} \\
&\qquad\qquad\qquad\qquad -\;\exp(\kappa^2\hat K)\cdot(WUU)_{\beta a e}\,g^{e\bar f}\,(\overline{WUU})_{\bar f\bar d\bar\gamma}\,, \\
\frac{1}{\kappa^2}\,R_{\beta\bar\gamma k\bar n} &= \tfrac{1}{3}\,G_{\beta\bar\gamma}\cdot g_{k\bar n}\,, \\
R_{\beta\bar\gamma a\bar n} &= 0, \\
R_{\beta\bar\gamma k\bar d} &= 0;
\end{aligned}
\tag{3.32}
$$

components $R_{\lambda\bar\mu A\bar D}$ obey similar equations.

In order to simplify formulæ (3.32) as equations for the matrix $\mathbf{U}\equiv\{U^\alpha_a\}$, we raise the first index of the Riemann tensor and rewrite eqs. (3.7), (3.2) and (2.15) in a matrix form:

$$
\begin{aligned}
R^{\bar\alpha}_{\ \bar\beta C\bar D} \;\equiv\; [\mathbf{R}_{C\bar D}]^{\bar\alpha}_{\ \bar\beta} &= \left[\partial_C\left(\mathbf{G}^{-1}\cdot\partial_{\bar D}\mathbf{G}\right)\right]^{\bar\alpha}_{\ \bar\beta}\,, \\
R^{\bar a}_{\ \bar b C\bar D} \;\equiv\; [\mathbf{R}_{C\bar D}]^{\bar a}_{\ \bar b} &= \left[\partial_C\left(\mathbf{g}^{-1}\cdot\partial_{\bar D}\mathbf{g}\right)\right]^{\bar a}_{\ \bar b}\,, \\
g_{a\bar b} \;\equiv\; [\mathbf{g}]_{a\bar b} &= \left[\mathbf{U}\mathbf{G}\mathbf{U}^\dagger\right]_{a\bar b}\,.
\end{aligned}
$$

In these notations, comparing eqs. (3.32) and (3.31) yields

$$
\begin{aligned}
\partial_C\left(\mathbf{U}^\dagger\mathbf{g}^{-1}\mathbf{U}\cdot\partial_{\bar D}\left(\mathbf{U}^{-1}\mathbf{g}\mathbf{U}^{\dagger^{-1}}\right)\right) \;=\; &\mathbf{U}^\dagger\,\partial_C\left(\mathbf{g}^{-1}\cdot\partial_{\bar D}\mathbf{g}\right)\,\mathbf{U}^{\dagger^{-1}} \\
&+\;\tfrac{\kappa^2}{3}\,\partial_C\partial_{\bar D}\left(K_2-K_1\right)\times\mathbf{1}\,.
\end{aligned}
\tag{3.33}
$$

This equation may look cumbersome, but it is rather easy to solve; the general solution is given by

$$
U^\alpha_a(M,\overline{M}) \;=\; V^\alpha_a(M)\cdot\exp\Bigl(\frac{\kappa^2}{6}\left(K_1-K_2\right)\Bigr), \tag{3.34}
$$

where $\mathbf{V}(M)$ is an arbitrary matrix-valued holomorphic function of the moduli fields M^a and M^m. Similarly,

$$
U^\mu_m(M,\overline{M}) \;=\; V^\mu_m(M)\cdot\exp\Bigl(\frac{\kappa^2}{6}\left(K_2-K_1\right)\Bigr). \tag{3.35}
$$

The arbitrariness of $\mathbf{V}(M)$ is not an artifact of using insufficient information to fully determine the U matrices, but a consequence of independent choices of coordinate systems for the moduli and for the matter fields. We are free to make a linear redefinition of the matter fields A^α, and the coefficients of this transformation can be moduli-dependent as long as they are holomorphic functions of the moduli fields. (Non-holomorphic field redefinitions are inconsistent with the manifestly complex Kähler geometry (2.2) we have used throughout this article.) Thus the $U_a^\alpha(M,\overline{M})$ and $U_m^\mu(M,\overline{M})$ are determined only up to holomorphic matrix-valued factors; obviously, $V_a^\alpha(M)$ and $V_m^\mu(M)$ are precisely such factors. Since apart from these factors the $\mathbf{U}$ matrices are proportional to unit matrices, there is a natural choice of matter fields $A^a \equiv (V^{-1})^a_\alpha A^\alpha$ and $A^m \equiv (V^{-1})^m_\mu A^\mu$ that eliminates the $\mathbf{V}$'s; henceforth we shall always make this choice of fields and use the same indices for both moduli and matter fields (as long as we are not discussing components of the Riemann tensor that involve both kinds of fields). With this convention, we can write explicit formulæ expressing metric matrices for the matter fields in terms of the metrics and Kähler functions for the moduli:

$$
\begin{aligned}
G_{a\bar{b}} &= g_{a\bar{b}} \cdot \exp\Big(\frac{\kappa^2}{3}\,(K_2 - K_1)\Big)\,, \\
G_{m\bar{n}} &= g_{m\bar{n}} \cdot \exp\Big(\frac{\kappa^2}{3}\,(K_1 - K_2)\Big)\,.
\end{aligned}
\tag{3.36}
$$

It is important to notice that *in contrast to $g_{a\bar{b}}$ and $g_{m\bar{n}}$, $G_{a\bar{b}}$ and $G_{m\bar{n}}$ depend on both types of moduli.*

Formulæ (3.31) now become explicit equations relating the Kähler geometry of the moduli space to the Yukawa couplings of the matter fields:

$$
\begin{aligned}
\frac{1}{\kappa^2}\,R_{a\bar{c}b\bar{d}} &= g_{a\bar{c}}\,g_{b\bar{d}} \;+\; g_{a\bar{d}}\,g_{b\bar{c}} \;-\; \exp(2\kappa^2 K_1)\cdot W_{abe}\,g^{e\bar{f}}\,\overline{W}_{\bar{f}\bar{c}\bar{d}}\,, \\
\frac{1}{\kappa^2}\,R_{k\bar{m}l\bar{n}} &= g_{k\bar{m}}\,g_{l\bar{n}} \;+\; g_{k\bar{n}}\,g_{l\bar{m}} \;-\; \exp(2\kappa^2 K_2)\cdot W_{kli}\,g^{i\bar{j}}\,\overline{W}_{\bar{j}\bar{m}\bar{n}}\,.
\end{aligned}
\tag{3.37}
$$

Note that non-trivial U factors are essential for the consistency of eqs. (3.37): Because the moduli spaces for the (1,1) and the (1,2) moduli are completely independent of one another, a consistent equation for the Riemann curvature of the (1,1) moduli space cannot involve K_2, which depends on the (1,2) moduli (and vice versa). Both eqs. (3.31) contain the factor $e^{\kappa^2\hat{K}} = e^{\kappa^2 K_1}\cdot e^{\kappa^2 K_2}$ which depends on both kinds of moduli, and it is the U factors (3.34) and (3.35) that turn it into the $e^{2\kappa^2 K_1}$ factor appearing in the first eq. (3.37) and the $e^{2\kappa^2 K_2}$ factor in the second equation. The U factors also make eqs. (3.37) invariant with respect to Kähler transforms of the two moduli spaces: $K_1 \mapsto K_1 - \Lambda_1(M^a) - \overline{\Lambda}_1(\overline{M}^{\bar{a}})$ and $K_2 \mapsto K_2 - \Lambda_2(M^m) - \overline{\Lambda}_2(\overline{M}^{\bar{m}})$. Under these transforms the U factors corresponding to $\mathbf{V}\equiv\mathbf{1}$ make the matter fields A^a and A^m rescale with factors $e^{\pm\frac{\kappa^2}{3}(\Lambda_1-\Lambda_2)}$, so the Yukawa couplings transform as $W_{abc}\mapsto e^{2\kappa^2\Lambda_1}W_{abc}$ and $W_{lmn}\mapsto e^{2\kappa^2\Lambda_2}W_{lmn}$; these are precisely the transformations that leave eqs. (3.37) invariant.

3.4. Yukawa Couplings and Metric for Moduli Fields.

There are two ways to look at eqs. (3.37): as differential equations for the Kähler functions $K_{1,2}$ in terms of the Yukawa couplings W_{abc} and W_{lmn}, or as algebraic equations for the Yukawa couplings in terms of the Kähler functions and their derivatives. From the latter point of view the fact that the moduli space is a direct product immediately implies that W_{abe} — the $\mathbf{27}^3$ Yukawa couplings — should depend only on the (1,1) moduli M^a and not on the (1,2) moduli M^m, while W_{kli} — the $\overline{\mathbf{27}}^3$ Yukawa couplings — should depend only on the (1,2) moduli.⋆ Using more direct string arguments, Distler and Greene[37] proved that this is indeed the case. Less immediate constraints imposed by eqs. (3.37) on the Yukawa couplings follow from the Bianchi identity for the Riemann tensor: $\nabla_e R_{a\bar{c}b\bar{d}} = \nabla_a R_{e\bar{c}b\bar{d}}$, where ∇ is the covariant derivative operator. Since the

⋆ Yukawa couplings we are discussing here are unnormalized cubic terms in the superpotential. Normalized Yukawa couplings also depend on the matter fields' metric, and eq. (3.36) tells us that normalized $\mathbf{27}^3$ couplings do depend on the (1,2) moduli, but this dependence is limited to a common rescaling of all $\mathbf{27}^3$ couplings. The same holds for the $\overline{\mathbf{27}}^3$ couplings and (1,1) moduli.

metric tensor is covariantly constant, and $\overline{W}$ is anti-holomorphic, eqs. (3.37) imply

$$\begin{aligned}\nabla_a\left(e^{2\kappa^2K_1}W_{bcd}\right) &= \nabla_b\left(e^{2\kappa^2K_1}W_{acd}\right),\\ \nabla_k\left(e^{2\kappa^2K_2}W_{lmn}\right) &= \nabla_l\left(e^{2\kappa^2K_2}W_{kmn}\right).\end{aligned}\tag{3.38}$$

Like the result of Distler and Greene, these equations can be obtained either from eqs. (3.37) or from direct string arguments; in Appendix A we shall give a stringy proof of eqs. (3.38).

The significance of (3.38) is as an integrability condition: In Kähler geometry, covariant derivatives with holomorphic indices commute with each other, and the integrability condition for a vector field X_a to be a gradient of a scalar is thus $\nabla_a X_b = \nabla_b X_a$. ($X_a$ is a vector field on the moduli space, not in space-time.) Similarly, if a symmetric tensor field $X_{a_1\cdots a_n}$ obeys $\nabla_b X_{a_1\cdots a_n} = \nabla_{a_1}X_{ba_2\cdots a_n}$ (*i.e.*, $\nabla_b X_{a_1\cdots a_n}$ is symmetric with respect to all its $n+1$ indices), then $X_{a_1\cdots a_n} = \nabla_{a_1}\cdots\nabla_{a_n}X$ for some scalar field X.† Once we have set $\mathbf{V}\equiv\mathbf{1}$, $W_{abc}(M^a)$ and $W_{lmn}(M^m)$ become symmetric tensors on the respective moduli spaces; hence, eqs. (3.38) are integrability conditions for having

$$\begin{aligned}\kappa^3\exp(2\kappa^2K_1)\cdot W_{abc}(M^a) &= \nabla_a\nabla_b\nabla_c\left(\exp(2\kappa^2K_1)\cdot Z_1(M^a,\overline{M}^{\bar a})\right),\\ \kappa^3\exp(2\kappa^2K_2)\cdot W_{lmn}(M^m) &= \nabla_l\nabla_m\nabla_n\left(\exp(2\kappa^2K_2)\cdot Z_2(M^m,\overline{M}^{\bar m})\right),\end{aligned}\tag{3.39}$$

where $Z_{1,2}$ are some scalar functions of the respective moduli. (The factors κ^3 and $e^{2\kappa^2K_{1,2}}$ are introduced for future convenience.) Note that the functions $Z_{1,2}$ are invariant under reparametrizations of the respective moduli spaces, but they are not holomorphic; the requirement that formulæ (3.39) should yield Yukawa couplings that are holomorphic functions of the moduli is a non-trivial constraint

† These integrability conditions are local, *i.e.*, are sufficient only on simply-connected manifolds. However, in this article we ignore all topological complications and limit ourselves to simply-connected pieces of the moduli space.

on the $Z_{1,2}$. We can write this constraint in a generally covariant form, but we do not know how to solve it in a general coordinate system. Nevertheless, we shall prove in Appendix B that one can make a holomorphic redefinition of moduli fields and a Kähler transform that together will reduce eqs. (3.39) to

$$\begin{aligned} \kappa^3 W_{abc} &= \partial_a \partial_b \partial_c \mathcal{F}_1 , \\ \kappa^3 W_{lmn} &= \partial_l \partial_m \partial_n \mathcal{F}_2 , \end{aligned} \tag{3.40}$$

where $\mathcal{F}_{1,2}$ are holomorphic functions of the appropriate moduli fields. Unlike the $Z_{1,2}$ that are non-holomorphic but invariant under field redefinitions, the $\mathcal{F}_{1,2}$ are holomorphic but are tied to a particular coordinate system, and even when there are several coordinate systems for which eqs. (3.40) hold, each system will have its own $\mathcal{F}$'s. Similarly, eqs. (3.39) are invariant under Kähler transforms provided $Z_{1,2}$ transform like the Yukawa couplings W_{abc} and W_{lmn} — $Z_{1,2} \mapsto e^{2\kappa^2 \Lambda_{1,2}} Z_{1,2}$ — but eqs. (3.40) are valid only for a particular Kähler choice of $K_{1,2}$.

Now let us go back to eqs. (3.37) and treat them as differential equations for the Kähler functions $K_{1,2}$ of the two moduli spaces. *Given the Yukawa couplings, the solution to each of these equations is unique up to a holomorphic field redefinition and a Kähler transform.* To see that, let us expand K_1 into a power series in M^a and $\overline{M}^{\bar{a}}$; analyticity of K_1 assures us that this is always possible. The leading operator in the differential equation for K_1 is $\partial_a \partial_b \partial_{\bar{c}} \partial_{\bar{d}}$; hence all terms in the expansion of K_1 that are at least quadratic in both holomorphic and anti-holomorphic fields are completely determined by terms that carry lower powers of M^a and/or $\overline{M}^{\bar{a}}$. On the other hand, all terms that are purely holomorphic or purely anti-holomorphic can be arbitrarily changed by Kähler transforms and the terms that are linear in either M^a or $\overline{M}^{\bar{a}}$ are freely changeable by holomorphic redefinitions of moduli fields. In particular, for any Kähler manifold we can write its Kähler function in the form $K = \sum_a M^a \overline{M}^{\bar{a}} + O(M^2 \overline{M}^2)$ (we call this form of K "holo-normal" for reasons that will be explained in Appendix B); once we do it for the (1,1) moduli space, eq. (3.37) completely determines all terms in K_1. In Appendix B we shall see that for holo-normal $K_{1,2}$ one can always write formulæ (3.40) for the Yukawa couplings. Therefore, *both the geometry of the (1,1) moduli space and the* $\mathbf{27}^3$ *Yukawa couplings are completely determined by eq.* (3.37) *in terms of a single holomorphic function* $\mathcal{F}_1$ *of the (1,1) moduli.* Similarly, *the geometry of the (1,2) moduli space and the* $\overline{\mathbf{27}}^3$ *Yukawa couplings are determined by a holomorphic function* $\mathcal{F}_2$ *of the (1,2) moduli.*

We do not have explicit formulæ for holo-normal solutions of eqs. (3.37). However, the so-called restricted Kähler manifolds familiar from the supergravity literature[25,26,38] have Kähler functions that are not holo-normal but nevertheless obey eqs. (3.37) for Yukawa couplings given by formulæ (3.40). Explicit formulæ for these Kähler functions are given by:

$$K_{1,2} \;=\; -\kappa^{-2}\,\log Y_{1,2}\,, \tag{3.41}$$

where $$Y_1(M^a,\overline{M}^{\bar a}) \;=\; \sum_{a=1}^{N_1}\bigl(\partial_a\mathcal{F}_1 \;+\; \partial_{\bar a}\overline{\mathcal{F}}_1\bigr)\cdot(M^a+\overline{M}^{\bar a}) \;-\; 2\bigl(\mathcal{F}_1+\overline{\mathcal{F}}_1\bigr)$$

and $$Y_2(M^m,\overline{M}^{\bar m}) \;=\; \sum_{m=1}^{N_2}\bigl(\partial_m\mathcal{F}_2 \;+\; \partial_{\bar m}\overline{\mathcal{F}}_2\bigr)\cdot(M^m+\overline{M}^{\bar m}) \;-\; 2\bigl(\mathcal{F}_2+\overline{\mathcal{F}}_2\bigr)\,;$$

verifying that these are indeed solutions to eqs. (3.37) is a straightforward but very tedious exercise. Notice that formulæ (3.41) are not generally covariant; this is related to the fact that the $\mathcal{F}_{1,2}$ are only defined for some special coordinate systems and transform non-trivially when we go from one such coordinate system to another.*

⋆ Comparing $\mathcal{F}$'s describing the same manifold in different coordinate systems may be facilitated by having explicit formulæ for both $K_{1,2}$ and $Z_{1,2}$ — the covariant generators of the Yukawa couplings. An explicit formula for Z_1 can be written as:

$$Z_1 \;=\; \mathcal{F}_1 \;+\; \overline{\mathcal{F}}_1 \;-\sum_a \overline{\mathcal{F}}_{1,\bar a}\,(M^a+\overline{M}^{\bar a}) \;+\; \tfrac{1}{2}\sum_{ab}\overline{\mathcal{F}}_{1,\bar a\bar b}\,(M^a+\overline{M}^{\bar a})(M^b+\overline{M}^{\bar b})\,;$$

a similar formula holds for Z_2. Verifying that these formulæ for $Z_{1,2}$ are consistent with

In the supergravity context restricted Kähler manifolds appear as manifolds spanned by scalar fields belonging to vector multiplets of a four-dimensional $N = 2$ supergravity theory. Heterotic string vacua that are only $N = 1$ supersymmetric in four dimensions *a priori* need not have anything to do with the $N = 2$ supergravity. However, the moduli of the (2,2) vacua are special since they also appear in the same vacua of the type II superstring,[14,24] and whereas the scattering amplitudes of moduli in the two string theories are not identical, they do agree to order $O(k^2)$; a proof of this assertion is given in Appendix C. Therefore, the Kähler functions of the moduli are the same in heterotic, type IIA and type IIB superstrings; this fact was used in ref. [27] to show that K_1 and K_2 are of restricted type, *i.e.* given by eq. (3.41) for some holomorphic functions $\mathcal{F}_1$ and $\mathcal{F}_2$. In ref. [28] it was argued that in addition the $\mathbf{27}^3$ and $\overline{\mathbf{27}}^3$ Yukawa couplings are given by third derivatives of the holomorphic functions $\mathcal{F}_{1,2}$; however, such a statement is meaningless without a choice of coordinate system for both the matter fields and the moduli. Our results, eqs. (3.40) and (3.41), show that there is a coordinate system in which this statement is correct.

---

eqs. (3.39) and with Yukawa couplings given by $W_{abc} = \mathcal{F}_{1,abc}$ and $W_{lmn} = \mathcal{F}_{2,lmn}$ is another straightforward but tedious exercise.

4. (2,2) Vacua with Enhanced Gauge Symmetry

4.1. Effect of Enlarged Gauge Group on Metrics for Moduli and Matter Fields.

So far in this paper we have explicitly assumed that the gauge group is $E_6 \otimes E_8$. However, it is well known that at special points or subspaces of moduli space the low-energy gauge group is $H \otimes E_6 \otimes E_8$; in such a subspace otherwise massive gauge bosons become massless and generate H. The majority of exactly solvable $N = (2,2)$, $c = 9$ models constructed to date have this property, for example orbifolds[3] and Gepner's models.[4] In this subsection we investigate how additional massless gauge bosons affect our previous results. The heterotic string relations (3.19), (3.20), and (3.27) between various scattering amplitudes remain unaltered in the presence of H. However, extra gauge bosons do affect the four-matter-fields amplitudes on the right-hand sides of these equations, and we shall see that this modifies formulæ (3.36) for the matter-field metric. Also, many string amplitudes vanish due to conservation of charges with respect to H, which results in a splitting of the moduli space into different charge sectors, analogous to, and in addition to, the splitting into (1,1) forms and (1,2) forms. Later in this section, we shall apply these results to orbifolds and calculate the metrics of moduli and matter fields coming from both untwisted and twisted sectors of an orbifold, as functions of the untwisted moduli. Then we shall perform an analogous calculation for one of Gepner's models, or more precisely for some subspaces of the moduli space that pass through the point described by Gepner's model, and that also have enhanced gauge symmetry.

For simplicity we assume that the extra gauge group is abelian⋆ — $H = U(1)^n$. Let us choose a basis for the **27** fields A^α and $\overline{\mathbf{27}}$ fields A^μ in which

⋆ The generalization to a nonabelian group is a bit subtler because gauge representation indices on matter fields can be related via U matrices to global indices on moduli, and vice versa.

they have definite $U(1)^n$-charges, which we shall denote by $Q_\alpha^{(a)}$ and $Q_\mu^{(a)}$, $(a) = 1, \ldots, n$. The effects of extra gauge bosons on the various four-matter-fields amplitudes turn out to be very similar to the effects of the $U(1)$ gauge boson in E_6 that commutes with $SO(10)$ — both are exchanged in exactly the same $SO(10)$-singlet channels. For this reason, it proves to be very convenient to treat the extra $U(1)^n$ factors on an equal footing with the $U(1) \subset E_6$ and have all $n+1$ $U(1)$ currents canonically normalized. For the $U(1)$ in E_6, which is generated by the current J of the $N = 2$ superconformal algebra, this means that we calculate charges with respect to $J_c \equiv \frac{1}{\sqrt{3}} J$ rather than J; in particular, the $U(1)$ charge of the components of a **27** ($\overline{\mathbf{27}}$) that transform as a **10** of $SO(10)$ is now $+\frac{1}{\sqrt{3}}$ $(-\frac{1}{\sqrt{3}})$. Let us define an $(n+1)$-vector of charges for the **10** fields in a **27** or $\overline{\mathbf{27}}$, where the (0) component carries the $U(1) \subset E_6$ charge:

$$\mathbf{Q}_\alpha \equiv \{Q_\alpha^{(a)}\}_{(a)=0}^n \,, \qquad \text{where} \quad Q_\alpha^{(0)} = +\frac{1}{\sqrt{3}} \text{ for a } \mathbf{27}, \quad Q_\alpha^{(0)} = -\frac{1}{\sqrt{3}} \text{ for a } \overline{\mathbf{27}}. \tag{4.1}$$

Since we can now distinguish **27**'s from $\overline{\mathbf{27}}$'s by their charge, in this section we shall not make any further distinction between them, and shall henceforth denote both types of matter fields by Greek indices from the beginning of the alphabet, A^α, A^β, *etc.*. Similarly, both (1,1) and (1,2) moduli shall be denoted by M^a, M^b, *etc.*. All the equations derived in this section will be valid for fields of either type once the appropriate charges are substituted.

In explicit models with extra gauge symmetry, the fields $T_F^\pm$ do *not* have definite H-charges,† and therefore neither do most of the moduli M^a, since their vertices are obtained from vertices of A^α by taking the operator product with

† This assertion can be proved by *reductio ad absurdum:* Suppose $T_F^\pm$ do have definite charges $\pm q$ with respect to a Kac-moody current J' that generates part of H. Then, at $z \mapsto w$ we have $J'(z) \cdot T_F^\pm(w) = \frac{\pm q}{z-w} T_F^\pm(w) + O(1) = \frac{\mp q}{w-z} T_F^\pm(z) + O(1)$, which implies that $\frac{1}{q} J'(z)$ and $T_F^\pm(z)$ belong to the same left-moving $N = 2$ supermultiplet. But the only Kac-Moody current in the same supermultiplet with $T_F^\pm$ is J, which generates the $U(1)$ inside E_6 rather than a part of H.

$T_F^{\pm}$. In fact, those moduli that are not completely neutral under H *must* be linear combinations of fields with different H-charges, because flatness of the scalar potential for the moduli requires that the D-term for each generator of H cancels. Moduli fields that are completely neutral with respect to H span the H-preserving subspace of the moduli space, and shall be denoted by N^A. The remaining moduli are charged under H and are denoted C^d. In this section capital indices will be reserved for neutral moduli — of either (1,1) or (1,2) type — while lower-case indices will be used for both charged and neutral moduli. In the orbifold examples we shall discuss below, the enhanced gauge symmetry is $H = U(1)^2$ and is present only in the orbifold limit; the neutral and charged moduli are from the untwisted and twisted sectors respectively.

Throughout the subspace of enhanced gauge symmetry the charged fields C^d must have vanishing vacuum expectation values. Thus to study this subspace we expand the effective Lagrangian into powers of C^d and $\overline{C}^{\bar{d}}$ and retain only the lowest relevant terms in this expansion, just as we did before with respect to the matter fields. As far as the Kähler function K is concerned, this means that we confine our attention to the Kähler function for the neutral fields N^A alone, plus the N^A-dependence of the metrics for the charged moduli and for all the matter fields. H-charge conservation in the effective field theory requires the matter-field metric $G_{\alpha\bar{\beta}}$ (and all its derivatives with respect to N^A and $\overline{N}^{\bar{A}}$) to be block diagonal, mixing only fields with the same H-charge:

$$G_{\alpha\bar{\beta}} \;=\; R_{\alpha\bar{\beta}C\bar{D}} \;=\; 0 \quad \text{unless} \quad \mathbf{Q}_\alpha = \mathbf{Q}_\beta\,. \tag{4.2}$$

We would like to argue that the metric for both charged and neutral moduli, $g_{a\bar{b}}$, has the same block-diagonal structure as the matter-field metric $G_{\alpha\bar{\beta}}$ — in terms of the H-charges $\mathbf{Q}_\alpha = \mathbf{Q}_\beta$ of the associated matter fields, *not* the charges of the moduli themselves. As discussed in section 2, for any particular point in the neutral moduli space one can trivialize the U-matrices that relate $g_{a\bar{b}}$ to

$G_{\alpha\bar{\beta}}$ in eqs. (2.15); the question is again how much of the trivialization can be maintained over a finite patch of the neutral moduli space. We shall see that the block-diagonal structure of $g_{a\bar{b}}$ can indeed be maintained; in essence, this result is just a refinement of the block-diagonal structure of $g_{a\bar{b}}$ with respect to (1,1) versus (1,2) moduli, and is proved in the same way.

To proceed further we need to study how the scattering amplitudes that are related to each other by (3.19), (3.20) and (3.27) receive additional field theory contributions due to exchanges of H gauge bosons. Since our inquiry is limited to the dependence of the various metrics on the neutral moduli only, all amplitudes on the left-hand sides of (3.19), (3.20) and (3.27) will contain at least two neutral moduli, and thus will be unaffected by the presence of extra gauge bosons. On the other hand, the amplitudes on the right-hand sides — for four matter fields all transforming as the **10** of $SO(10)$ — do get contributions from H gauge boson exchange (and from the associated D-terms). In fact, the n extra $U(1)$ gauge bosons are exchanged between these matter fields precisely when the $U(1)$ gauge boson in E_6 (also an $SO(10)$ singlet) is exchanged. The only difference is a factor $Q^{(a)}_\alpha Q^{(a)}_\beta$ for the $(a)^{\rm th}$ gauge boson, replacing $Q^{(0)}_\alpha Q^{(0)}_\beta = \pm\frac{1}{3}$ for the $U(1) \subset E_6$ gauge boson. (The sign is plus if A^α and A^β are both $\mathbf{27}$'s or both $\overline{\mathbf{27}}$'s, and is minus if one is a $\mathbf{27}$ and one a $\overline{\mathbf{27}}$.) Hence the only correction needed for the three field theory amplitudes (3.11) is to make the respective replacements

$$\begin{aligned} \tfrac{1}{3} &\mapsto \mathbf{Q}_\alpha \cdot \mathbf{Q}_\beta \ ; \\ -\tfrac{1}{3} &\mapsto \mathbf{Q}_\alpha \cdot \mathbf{Q}_\lambda \ ; \\ \tfrac{1}{3} &\mapsto \mathbf{Q}_\kappa \cdot \mathbf{Q}_\lambda \ . \end{aligned} \tag{4.3}$$

The corrections (4.3) do not affect amplitudes on the right-hand side of equations (3.27), and therefore formulæ (3.31) for the Riemann tensor of the moduli space still hold. However, some of the amplitudes on the right-hand sides of eqs. (3.19) and (3.20) are affected, with the result that eqs. (3.32) for $R_{\beta\bar{\gamma}A\bar{D}}$ now become (using also (3.31))

$$R_{\beta\bar\gamma A\bar D} \;=\; -\kappa^2\,\mathbf{Q}_A\cdot\mathbf{Q}_\beta\,G_{\beta\bar\gamma}\,g_{A\bar D} \;+\; U^b_\beta\overline{U}^{\bar c}_{\bar\gamma}\cdot R_{b\bar c A\bar D}\,. \tag{4.4}$$

Here $\mathbf{Q}_A$ is the charge vector of the matter field associated with the modulus N^A (recall that N^A itself is neutral) and $R_{b\bar c A\bar D}$ is given by eq. (3.31). Charge conservation under all $n+1$ $U(1)$ factors — eq. (4.2) — when combined with formula (4.4) tells us that, unless the moduli fields M^b and M^c accompany matter fields A^β and A^γ with the same H-charges $\mathbf{Q}_\beta=\mathbf{Q}_\gamma$, we must have $R_{b\bar c A\bar D}=0$ for all neutral moduli N^A and N^D. For the case when moduli N^b and N^c are themselves neutral, this immediately implies that the restricted holonomy group of the space of H-preserving moduli decomposes into a product of commuting subgroups. Consequently,[36] the H-preserving subspace of the moduli space is locally a direct product of smaller-dimension subspaces; each of the latter subspaces is spanned by all the neutral moduli N^A that accompany matter fields with a given $U(1)^{n+1}$ charge-vector $\mathbf{Q}_A$. In terms of the Kähler function this is expressed by

$$K(N,\overline{N}) \;=\sum_q K^q(N^q,\overline{N}^{\bar q})\,, \tag{4.5}$$

where q labels the different charge sectors (not just individual fields). Note that if we ignore all but the (0) component of the charge-vector $\mathbf{Q}_\alpha$, then there are just two different charge sectors, one containing all the (1,1) moduli and the other containing all the (1,2) moduli; then eq. (4.5) just reproduces the (1,1)/(1,2) splitting (*cf.* eq. (2.18)).

At this point we can solve eq. (4.4) along the lines of our solution to eqs. (3.32). We find

$$U^\alpha_a(N,\overline{N}) \;=\; V^\alpha_a(N)\cdot\exp\left(\tfrac12\kappa^2\,\mathbf{Q}_\alpha\cdot\mathbf{Q}_q K^q\right), \tag{4.6}$$

with an implicit sum over all charge sectors q for the neutral moduli. Clearly, we can set the $\mathbf{V}$ matrices equal to $\mathbf{1}$ as before. Then the matter-field metric takes the form

$$G_{a\bar{b}} \;=\; g_{a\bar{b}} \cdot \exp\left(-\kappa^2\,\mathbf{Q}_\alpha \cdot \mathbf{Q}_q K^q\right). \tag{4.7}$$

This equation can be thought of as giving the matter-field metric in terms of the moduli metric, but it also shows that the moduli metric has the same block-diagonal structure as the matter-field metric,

$$g_{a\bar{b}} = 0 \quad \text{unless} \quad \mathbf{Q}_\alpha = \mathbf{Q}_\beta\,. \tag{4.8}$$

As we will see later in the section, this property can be an aid in solving for the moduli metric on subspaces of enhanced gauge symmetry. Notice that in the absence of extra gauge bosons formulæ (4.7) reduce to formulæ (3.36): For $n=0$, $\mathbf{Q}_{\alpha,\mu} = \pm\frac{1}{\sqrt{3}}$ and $\sum_q \mathbf{Q}_q K^q = \frac{1}{\sqrt{3}}\cdot(K_1 - K_2)$.

Next consider the effect of H-charge conservation on the Yukawa couplings. We must have

$$W_{\alpha\beta\gamma} = 0 \quad \text{unless} \quad \mathbf{Q}_\alpha + \mathbf{Q}_\beta + \mathbf{Q}_\gamma \;=\; (\pm\sqrt{3}, 0, \ldots, 0). \tag{4.9}$$

The $\pm\sqrt{3}$ in the zeroth component of this equation enforces E_6 invariance on the cubic couplings, requiring that they be either $\mathbf{27}^3$ or $\overline{\mathbf{27}}^3$ couplings (*cf.* eq. (4.1), which gives the $Q_\alpha^{(0)}$ charges); the reason for this total $U(1) \subset E_6$ charge being $\pm\sqrt{3}$ rather than zero is our convention of measuring charges of all matter fields as if they were $\mathbf{10}$'s of $SO(10)$, while in the actual $\mathbf{27}^3$ or $\overline{\mathbf{27}}^3$ couplings only two of the fields are decuplets and the third is an $SO(10)$ singlet.

From eqs. (4.9) and (4.6) we see that all effects of extra gauge bosons on the U-matrices cancel out of the products $(WUUU)_{abc}$ and $(WUUU)_{lmn}$, which allows us to proceed from eqs. (3.31) to eqs. (3.37) exactly as before, without any modifications. Naturally, the subsequent derivation of formulæ (3.40) and (3.41) also goes through exactly as before. The only difference is that eqs. (4.9) impose powerful constraints on the form of $\mathcal{F}_{1,2}$; we shall exploit these constraints later in this section. To summarize, we have just shown that in a subspace of the moduli space where an extra gauge group H appears, the metric for the moduli N^A spanning the subspace is given in terms of holomorphic functions $\mathcal{F}_{1,2}(N)$ by the same formula as before. In addition the subspace has a direct product structure; each factor in the product is spanned by all the moduli that are accompanied by matter fields with a fixed H-charge. Finally, the matter-field metric is expressed in terms of the moduli metric by a different relation than before.

4.2. Orbifold Examples.

Abelian (2,2) orbifolds provide a nice application of our results. They are constructed by twisting some six-dimensional torus by a "point group" P of $SO(6)$ rotations that are symmetries of the torus, *i.e.* that act crystallographically on the corresponding lattice. To preserve exactly $N=1$ spacetime supersymmetry, P should lie in an $SU(3)$ subgroup of $SO(6)$, and should act nontrivially on all three complex planes of the torus. To get a (2,2) orbifold of the heterotic string, each $SU(3)$ rotation of the six-dimensional torus is accompanied by the identical gauge transformation in a standard $SU(3)$ subgroup of one of the two E_8 factors, namely the $SU(3)$ appearing in the decomposition $E_8 \supset E_6 \otimes SU(3)$. Hence, the four-dimensional gauge group is enlarged beyond $E_6 \otimes E_8$ and contains also the subgroup H of the $SU(3)$ that commutes with P. If P is abelian, H can be $U(1)^2$, $SU(2)\times U(1)$ (if $P=Z_4$ or Z_6), or $SU(3)$ (if $P=Z_3$). For simplicity we restrict ourselves here to the case $H=U(1)^2$.

The orbifold model has various moduli coming both from the untwisted sector and from the twisted sectors. The untwisted (1,1) moduli correspond to changing the radii of the torus, and the untwisted (1,2) moduli to changing its complex structure, while preserving the torus's symmetry under P. The extra gauge group H is present for any choice of radius (or complex structure), so the untwisted moduli must be neutral under H and can be denoted by N^A.⋆ If $H = U(1)^2$, then it is easy to see that there are exactly three untwisted (1,1) moduli, with the following vertex operators:

$$\text{untwisted (1,1) moduli} \quad N^A \;\leftrightarrow\; \Phi_A^+ \;=\; \partial_z X^i \cdot \bar{\partial}_{\bar{z}} \overline{X^{\bar{\imath}}}, \qquad A = i = 1, 2, 3, \tag{4.10}$$

where the three complex scalar fields X^i parametrize the torus. In addition there may be up to three untwisted (1,2) moduli, depending on the choice of P, although usually there are none. Here we shall compute the N^A-dependence of the metrics for the untwisted and twisted (1,1) moduli and **27**'s while ignoring untwisted and twisted (1,2) moduli and $\overline{\mathbf{27}}$'s, and the dependence of the above metrics on the untwisted (1,2) moduli[21,40]; it is entirely straightforward to treat them too using the same approach.

In order to apply our formula (4.7) for the matter-field metric to the orbifold case, we need to know the charges of the matter fields, which in turn relies on how the $N = 2$ superconformal algebra and the extra $U(1)^2$ currents are represented in the conformal field theory. In terms of X^i and its superpartner ψ^i, one has

$$\begin{aligned} 2T_F^+ &= \sum_{i=1}^{3} \psi^i \cdot \partial_z \overline{X^{\bar{\imath}}}, \\ 2T_F^- &= \sum_{i=1}^{3} \bar{\psi}^{\bar{\imath}} \cdot \partial_z X^i, \\ J &= \sum_{i=1}^{3} J^i \;\equiv\; \sum_{i=1}^{3} \psi^i \cdot \bar{\psi}^{\bar{\imath}}. \end{aligned} \tag{4.11}$$

The extra $U(1)^2$ currents are the two linear combinations of the J^i that are orthogonal to $J = \sum J^i$, for example $\frac{1}{\sqrt{2}}(J^1 - J^2)$ and $\frac{1}{\sqrt{6}}(J^1 + J^2 - 2J^3)$.

⋆ Actually, H can be enlarged even further at special radii where the toroidal compactifications themselves lead to four-dimensional gauge groups larger than $E_8 \otimes E_8$.[39] We won't consider those special radii here.

However, it is more convenient to use the basis J^1, J^2, J^3 for the three charges; in this basis eq. (4.9) becomes

$$W_{abc} = 0 \;\; \text{unless} \;\; \mathbf{Q}_\alpha + \mathbf{Q}_\beta + \mathbf{Q}_\gamma \;=\; (1,1,1). \tag{4.12}$$

The three currents J^i can be bosonized: $J^i = i\partial H^i$, $\psi^i = e^{iH^i}$, $\bar{\psi}^{\bar{\imath}} = e^{-iH^i}$; this is useful for finding the charges of twisted matter fields.

The vertex operators for the three untwisted **27** fields that accompany the untwisted moduli in (4.10) use the $h = \frac{1}{2}$, $q = 1$ lower component fields

$$\Psi^+_A = \psi^i \cdot \bar{\partial}_{\bar{z}} \overline{X^{\imath}}, \quad A = i = 1,2,3. \tag{4.13}$$

The $U(1)^3$ charges are clearly

$$\mathbf{Q}_1 = (1,0,0), \qquad \mathbf{Q}_2 = (0,1,0), \qquad \mathbf{Q}_3 = (0,0,1). \tag{4.14}$$

Vertex operators for twisted **27**'s use the lower components of twist superfields, which have the form

$$\Psi^+_{\rm tw} \;=\; \sigma \cdot \mathbf{s} \;=\; \sigma \cdot e^{i\eta_1 H^1} e^{i\eta_2 H^2} e^{i\eta_3 H^3}\,. \tag{4.15}$$

Here σ and $\mathbf{s}$ are respectively bosonic and fermionic twist fields with dimensions $\frac{1}{2}\sum \eta_i(1-\eta_i)$ and $\frac{1}{2}\sum \eta_i^2$; the η_i are the angles (divided by 2π) through which the three complex planes are rotated in the given twisted sector, $0 \le \eta_i < 1$. To get the correct dimension $h = \frac{1}{2}$ for $\Psi^+_{\rm tw}$, one requires $\eta_1 + \eta_2 + \eta_3 = 1$. The $U(1)^3$ charges of $\Psi^+_{\rm tw}$ are easily read off the second equation in (4.15):

$$\mathbf{Q}_{\rm tw} = (\eta_1, \eta_2, \eta_3), \quad 0 \le \eta_i < 1, \quad \eta_1 + \eta_2 + \eta_3 = 1. \tag{4.16}$$

The orbifold can have many twisted sectors, each characterized by a different set of rotation angles (η_1, η_2, η_3) and hence by different $U(1)^3$ charges, but the charges of all the **27**'s in a given twisted sector are the same. Also, by applying T_F^- as given in eq. (4.11) to $\Psi^+_{\rm tw}$ given in (4.15) to obtain the vertices for the twisted moduli, one can check that those moduli are indeed charged under H.

Now we are ready to calculate the various metrics (in the orbifold limit), starting with the Kähler function for the three untwisted (1,1) moduli N^A. We begin by noting that the charges $\mathbf{Q}_\alpha$ for the three associated **27** fields are all unequal (*cf.* eq. (4.14)). Hence the moduli space is a direct product, and

$$K(N,\overline{N}) \;=\; K^1(N^1,\overline{N}^{\bar 1}) + K^2(N^2,\overline{N}^{\bar 2}) + K^3(N^3,\overline{N}^{\bar 3}). \tag{4.17}$$

The superscript on K labels the different charge sectors as in eq. (4.5); each sector happens to contain only one field in this case. The curvature equation (3.37) can now be applied to each field N^A separately. Since the only nonzero Yukawa coupling involving at least two untwisted **27**'s is W_{123} (as can be seen just from H-charge conservation, eq. (4.12)), the term in (3.37) involving W_{AAe} vanishes, leaving

$$R_{A\bar A A\bar A} \;=\; 2\kappa^2\, g_{A\bar A}\, g_{A\bar A}\,, \quad A=1,2,3. \tag{4.18}$$

Thus each untwisted modulus spans a one-dimensional Kähler manifold of constant Riemannian curvature; that space is $SU(1,1)/U(1)$[41] whose Kähler function is given by

$$K^A \;=\; -\kappa^{-2}\log\bigl(N^A + \overline{N}^{\bar A}\bigr). \tag{4.19}$$

(up to field redefinitions and Kähler transforms). The full untwisted moduli space is $(SU(1,1)/U(1))^3$, with metric

$$g_{A\bar B} \;=\; \kappa^{-2}\,\delta_{A\bar B}\cdot(N^A+\overline{N}^{\bar A})^{-2}\,, \qquad A,B=1,2,3. \tag{4.20}$$

This space is a restricted Kähler manifold — the total untwisted Kähler function

(4.5) can be written as

$$K(N,\overline{N}) \;=\; -\kappa^{-2}\log Y(N,\overline{N})\,,\ \text{where}\ Y(N,\overline{N}) \;=\; \prod_{A=1}^{3}(N^A+\overline{N}^{\bar{A}}) \qquad (4.21)$$

is derived from the holomorphic function[27] $\mathcal{F}(N) = N^1N^2N^3$.[⋆] The result (4.21) has previously been obtained via several different approaches — by truncating the $N=1$ supergravity Lagrangian in ten dimensions,[20] by using the symmetries of the orbifold S-matrix generating functional,[21] by direct computation,[23] and by combining the $N=2$ supergravity/type II approach with a Peccei-Quinn symmetry.[27] From eq. (3.40) we see that the one nonvanishing untwisted $\mathbf{27}^3$ term in the superpotential must be a constant, $W_{123} = \kappa^{-3}\partial_1\partial_2\partial_3\mathcal{F} = \kappa^{-3}$. This is an example of the strong consistency constraints on the solutions of (3.37) when the matter fields carry charge under an extra gauge symmetry.

The metric $g_{d\bar{e}}(N,\overline{N})$ for the twisted (1,1) moduli C^d can also be obtained by integrating (3.37). There are two types of twisted moduli to consider. The first type is accompanied by a **27** field with charge $\mathbf{Q}_\delta = (\eta_1,\eta_2,\eta_3)$ with all three $\eta_i > 0$. In this case eqs. (4.16) and (4.12) show that $W_{Adx} = 0$ for any choice of (1,1) index x. Thus eq. (3.37) again simplifies, to

$$R_{d\bar{e}A\bar{B}} \;=\; \kappa^2\, g_{A\bar{B}}\, g_{d\bar{e}}\,, \qquad (A,B=1,2,3;\quad d,e \text{ are twisted indices}). \qquad (4.22)$$

Using formula (4.20), eq. (4.22) can be integrated to give

$$g_{d\bar{e}} \;=\; \kappa^{-2}\,\delta_{d\bar{e}}\cdot\prod_{A=1}^{3}(N^A+\overline{N}^{\bar{A}})^{-1}. \qquad (4.23)$$

⋆ The constant-curvature metric on $\big(SU(1,1)/U(1)\big)^3$ is often written as $g_{A\bar{B}} \;=\; \kappa^{-2}\,\delta_{A\bar{B}}\cdot(1-N^A\overline{N}^{\bar{A}})^{-2}$ (no sum on A), which differs from formula (4.20) by a holomorphic redefinition of untwisted moduli fields N^A. The Kähler function that generates this metric — $K = -\kappa^{-2}\log\prod_A(1-N^A\overline{N}^{\bar{A}})$ — does not *appear* to be of the restricted type. We chose to define fields N^A in a way that leads to formula (4.21) precisely to show that $\big(SU(1,1)/U(1)\big)^3$ is a restricted Kähler manifold.

Another way to get this result is by expanding $\mathcal{F}$ in powers of C^d[42]:

$$\mathcal{F} \;=\; N^1N^2N^3 \;+\; \tfrac{1}{4}\sum_d (C^d)^2 \;+\; \cdots. \tag{4.24}$$

The key point is that because $W_{Add'} = 0$, the $O((C^d)^2)$ term in $\mathcal{F}$ must be independent of N^A. (It can then be put into the above form by a linear transformation of the C's.) From this $\mathcal{F}$ one obtains

$$K(N,C) \;=\; -\,\kappa^{-2}\log Y(N,C), \qquad Y(N,C) \;=\; Y(N) \,+\, C^d\overline{C}^{\bar{d}} \,+\, \cdots, \tag{4.25}$$

which in turn gives eq. (4.23). One can also easily calculate the $O(C^2\overline{C}^2)$ terms in $K(N,C)$ in this way, using formulæ (3.37) and known expressions[43] for the Yukawa couplings of three twisted **27** fields, though we will not do so here.

The second type of twisted moduli is accompanied by a **27** field where one of the η's in (4.16) vanishes, say $\eta_3 = 0$, so that the **27** charges are $\mathbf{Q}_\delta = (\eta_1, 1-\eta_1, 0)$. Now there can be nonzero Yukawa couplings of the form $W_{3dd'}$, where $\mathbf{Q}_3 = (0,0,1)$ and $\mathbf{Q}_{\delta'} = (1-\eta_1, \eta_1, 0)$. The $W_{3dd'}$ will complicate eq. (4.22) when $A, B = 3$. However, all twisted moduli C^d of this type actually preserve one of the two extra $U(1)$'s, namely the $U(1)$ generated by $J^1 + J^2 - 2J^3$: C^d is a linear conbination of two fields with charge vectors $(-\eta_1, \eta_1, 0)$ and $(1-\eta_1, -1+\eta_1, 0)$, which are both orthogonal to $J^1 + J^2 - 2J^3$. Furthermore, under this extra $U(1)$ the charge $(+1)$ of the **27** fields accompanying N^1, N^2 and C^d differs from the charge (-2) of the **27** accompanying N^3. Thus we can use the extra $U(1)$ preserved by the C^d to argue that the Kähler function $K(N^1, N^2, C^d)$ is independent of N^3. In particular, $g_{d\bar{e}}$ is independent of N^3, so we need only study $R_{d\bar{e}A\bar{B}}$ for $A, B = 1, 2$. But for $A, B = 1, 2$ the Yukawa couplings $W_{Add'}$ *do* vanish, so eq. (4.22) holds, and integration of it with respect to N^1 and N^2 gives

$$g_{d\bar{e}} \;=\; \kappa^{-2}\,\delta_{d\bar{e}}\cdot\prod_{A=1}^{2}(N^A + \overline{N}^{\bar{A}})^{-1} \tag{4.26}$$

in place of formula (4.23).

Finally, we use eq. (4.7) to compute the matter metrics. For the three untwisted **27** fields, with $U(1)^3$ charges given by (4.14), and using also the Kähler functions from (4.19), we get

$$G_{A\bar{B}} \;=\; g_{A\bar{B}}\cdot\exp(-\kappa^2 K^A) \;=\; \kappa^{-2}\,\delta_{A\bar{B}}\cdot(N^A+\overline{N}^{\bar{A}})^{-1}\,. \qquad A,B=1,2,3 \quad (4.27)$$

This result agrees with ref. [20] — the latter result for $K(N^A, A^A)$ is obtained by truncating ten-dimensional supergravity and is valid to all orders in the **27** fields; our result only gives the $O(A^2)$ terms. For the twisted **27** fields, the charges are given by (4.16) and the twisted moduli metrics by eqs. (4.23) and (4.26), yielding

$$\begin{aligned} G_{d\bar{e}} \;&=\; g_{d\bar{e}}\cdot\prod_{A=1}^{3}(N^A+\overline{N}^{\bar{A}})^{\eta_A} \\ &=\; \kappa^{-2}\,\delta_{d\bar{e}}\times\begin{cases}\prod_{A=1}^{3}(N^A+\overline{N}^{\bar{A}})^{\eta_A-1}, & \text{if all } \eta_A>0\,;\\ \prod_{A=1}^{2}(N^A+\overline{N}^{\bar{A}})^{\eta_A-1}, & \text{if } \eta_3=0.\end{cases} \end{aligned} \qquad (4.28)$$

4.3. Tensor Product Examples.

Another application of our results is to the exactly solvable versions of Calabi-Yau compactifications discussed by Gepner,[4] which are constructed by taking the tensor product of a number of minimal $c<3$, $N=2$ superconformal theories. Each component theory has its own $U(1)$ current J^i, and so the tensor product theory has an extra $U(1)^n$ gauge symmetry if there are $n+1$ components — the sum of all $n+1$ currents, $J\equiv\sum_i J^i$, generates the $U(1)$ contained in E_6, not an extra $U(1)$ factor. (In some cases there may be a few additional gauge bosons.) The tensor product theory describes a particular point in the moduli space of a Calabi-Yau manifold; at that point the four-dimensional gauge symmetry is enhanced. Some of the additional gauge bosons may remain massless on subspaces of the moduli space that pass through the tensor product point (and have positive dimensions); the geometry of such subspaces can be studied along the same lines as the orbifold examples just discussed.

Vertex operators $\Psi^{\pm}$ of the product theory are made of products of world-sheet fields of each $c_i < 3$ component. For each component field the $U(1)$ charge q_i and the conformal weight h_i obey $2h_i \geq |q_i|$, assuming the conventional $N=2$ normalization of the currents, $J^i(z)J^j(w) \sim \frac{c_i}{3}\delta^{ij}\cdot(z-w)^{-2}$. Hence, the only way to assemble $\Psi^{\pm}$ with $h \equiv \sum_i h_i = \frac{1}{2}$ and $q \equiv \sum_i q_i = \pm 1$ is to have $q_i = \pm 2h_i$ for each i; that is, $\Psi^{\pm}$ are products of lower members of (anti) chiral multiplets of each of the $n+1$ left-moving $N=2$ superalgebras. The $U(1)^{n+1}$ charge vectors of matter fields generated by these $\Psi^{\pm}$ can be written as $\mathbf{Q} = (q_1, \ldots, q_{n+1})$. To obtain moduli vertices $\Phi^{\pm}$ we act with $T_F^{\mp} \equiv \sum_i T_{Fi}^{\mp}$ on $\Psi^{\pm}$; as a result, moduli are linear combinations of fields with charge vectors

$$(q_1, \ldots, q_i \mp 1, \ldots, q_{n+1}) \quad \text{for all } i \text{ such that } q_i \neq 0. \tag{4.29}$$

The sign here is '$-$' for the (1,1) moduli and '$+$' for the (1,2) moduli; the reason why we do not apply $T_{Fi}^{\mp}$ to $\Psi^{\pm}$ when $q_i = 0$ is that in this case the i^{th} factor in $\Psi^{\pm}$ is the unit operator. Since $\sum_i q_i = \pm 1$, eq. (4.29) verifies that all moduli are neutral with respect to $J \equiv \sum_i J^i$ which generates the $U(1) \subset E_6$; in addition, whenever some $q_i = 0$, the modulus is also neutral with respect to the currents $J^i - \frac{c_i}{9}J$, which are orthogonal to J and hence generate abelian gauge factors outside of E_6.

If a modulus N^A is to be completely neutral with respect to the entire $H = U(1)^n$, the charge vector $\mathbf{Q}_\alpha$ of the accompanying matter field should have the form $(0, \ldots, \pm 1, \ldots, 0)$ (*cf.* eqs. (4.14) for untwisted matter fields in the orbifold case). Alas, in any $N=2$ world-sheet theory all chiral primary fields have $|q| = 2h \leq c/3$; hence, if all components have $c_i < 3$ then no modulus field is totally neutral and *the full $U(1)^n \otimes E_6 \otimes E_8$ enhanced gauge symmetry exists only at the tensor product point* in the moduli space. On the other hand, subgroups of H generated by currents $\{J^i - \frac{c_i}{9}J\}_{i\in I}$ with $|I| < n$ are preserved by the moduli accompanying matter fields that have $q_i = 0$ for all $i \in I$. Note however that all **27** ($\overline{\mathbf{27}}$) fields of this kind have the same charges $-c_i/9$ ($+c_i/9$) with respect to all unbroken subgroups. Therefore, eq. (4.5) does *not* require that the space of neutral moduli factorize into a direct product (except into (1,1) and (1,2) moduli, if the neutral moduli include moduli of both kinds). This behavior is quite different from the orbifold examples of the previous subsection.

As a particular example let us consider the tensor product of five copies of the $k = 3$, $c = \frac{9}{5}$ element of the $c < 3$ discrete series[4]; this theory leads to 101 $\overline{\mathbf{27}}$ matter fields and one **27** field. The **27** field has $U(1)^5$ charge vector $\mathbf{Q} = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$, so the one (1,1) modulus of this (2,2) vacuum breaks all four $U(1)$ factors outside $E_6 \otimes E_8$. Note that a large expectation value of this modulus turns this model of Gepner into a large-radius Calabi-Yau vacuum (with the internal manifold being a quintic surface in CP^4) which indeed does not have an enlarged gauge group. On the other hand, if one of the 101 (1,2) moduli is given an expectation value, it will preserve an extra $U(1)$ group in 20 of the 101 cases, an extra $U(1)^2$ in 60 of the cases, and an extra $U(1)^3$ in 20 of the cases; the full $H = U(1)^4$ exists only at the tensor product point. This behavior follows from the $U(1)^5$ charges of the matter fields which are summarized in the following table:

$$\begin{aligned}
\text{one } \mathbf{27} \text{ field has charge } \mathbf{Q} &= \left(\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}\right); \\
\text{one } \overline{\mathbf{27}} \text{ field has charge } \mathbf{Q} &= \left(\tfrac{-1}{5}, \tfrac{-1}{5}, \tfrac{-1}{5}, \tfrac{-1}{5}, \tfrac{-1}{5}\right); \\
20\ \overline{\mathbf{27}} \text{ fields have charge } \mathbf{Q} &= \left(\tfrac{-2}{5}, \tfrac{-1}{5}, \tfrac{-1}{5}, \tfrac{-1}{5}, 0\right); \\
30\ \overline{\mathbf{27}} \text{ fields have charge } \mathbf{Q} &= \left(\tfrac{-2}{5}. \tfrac{-2}{5}, \tfrac{-1}{5}, 0, 0\right); \\
30\ \overline{\mathbf{27}} \text{ fields have charge } \mathbf{Q} &= \left(\tfrac{-3}{5}, \tfrac{-1}{5}, \tfrac{-1}{5}, 0, 0\right); \\
20\ \overline{\mathbf{27}} \text{ fields have charge } \mathbf{Q} &= \left(\tfrac{-3}{5}, \tfrac{-2}{5}, 0, 0, 0\right);
\end{aligned} \tag{4.30}$$

all charges here are given modulo permutations of the five components. We would also like to know the maximal set of moduli preserving a given subgroup of $U(1)^4$. For example, while there are 20 moduli each preserving three extra gauge factors (the moduli accompanying $\overline{\mathbf{27}}$ fields in the last row of table (4.30)), only two moduli preserve the particular $U(1)^3$ subgroup generated by $J^1 - \frac{1}{5}J$, $J^2 - \frac{1}{5}J$ and $J^3 - \frac{1}{5}J$; the charges of the two accompanying $\overline{\mathbf{27}}$ fields are

$$\mathbf{Q}_1 = (0,0,0,\tfrac{-3}{5},\tfrac{-2}{5}), \qquad \mathbf{Q}_2 = (0,0,0,\tfrac{-2}{5},\tfrac{-3}{5}). \tag{4.31}$$

Similarly, the $U(1)^2$ subgroup generated by *e.g.* $J^1 - \frac{1}{5}J$ and $J^2 - \frac{1}{5}J$ persists on a subspace of twelve (complex) dimensions, while a forty-dimensional subspace preserves the single $U(1)$ generated by $J^1 - \frac{1}{5}J$.

Since the only (1,1) modulus does not preserve the enhanced gauge symmetry, let us concentrate on the (1,2) moduli and the $\overline{\mathbf{27}}$ matter fields. At the tensor product point, the $U(1)^4 \otimes E_6$ charge conservation restricts the $\overline{\mathbf{27}}^3$ Yukawa couplings in a manner similar to eq. (4.12) for the $\mathbf{27}^3$ couplings in the orbifold case:

$$W_{abc} = 0 \ \text{ unless } \ \mathbf{Q}_\alpha + \mathbf{Q}_\beta + \mathbf{Q}_\gamma \ = \ (\tfrac{-3}{5},\tfrac{-3}{5},\tfrac{-3}{5},\tfrac{-3}{5},\tfrac{-3}{5}). \tag{4.32}$$

(Equation (4.32) follows from (4.9) after correcting for the non-canonical normalization of the currents J^i.) The first three components (plus the sum of the last two) of this constraint remain valid on the two-dimensional subspace that preserves the $U(1)^3$ subgroup discussed above. This information suffices to show that $W_{ABC} = W_{ABx} = 0$, where A, B, C are indices for the two $U(1)^3$-neutral fields N^A, with $\overline{\mathbf{27}}$ charges given by eq. (4.31), and x is the index for any of the 99 remaining (1,2) moduli C^x. The vanishing of these Yukawa couplings means that eq. (3.37) for the N^A becomes

$$R_{A\bar{C}B\bar{D}} \ = \ \kappa^2 (g_{A\bar{C}}\, g_{B\bar{D}} + g_{A\bar{D}}\, g_{B\bar{C}})\,, \tag{4.33}$$

with $A, B, C, D = 1, 2$. The neutral moduli space is thus a constant-Riemannian-curvature Kähler manifold. Such manifolds are coset spaces; the m-dimensional manifold whose Riemann tensor is given by (4.33) is $SU(m,1)/(SU(m)\otimes U(1))$,[41] with metric

$$\kappa^2\, g_{A\bar{B}} \;=\; \frac{\delta_{A\bar{B}}}{1-N^C\overline{N}_C} \;+\; \frac{\overline{N}_A N_{\bar{B}}}{(1-N^C\overline{N}_C)^2}\,, \qquad (4.34)$$

$A, B, C = 1, 2, \ldots, m$.* In the present case $m = 2$.

The metric for the 99 charged (1,2) moduli, as a function of N^1 and N^2, is block-diagonal according to the $U(1)^3$ charges of the $\overline{\mathbf{27}}$'s, as discussed in subsection 4.1; there are 44 blocks altogether, including the neutral block. Of the 43 charged blocks, 19 contain fields C^x for which $W_{Axz} = 0$ for all fields N^A and C^z (these 19 blocks happen to contain 63 of the 99 fields). The metric for these fields obeys the same eq. (4.22) as the metric for the twisted moduli in the orbifold example, and the solution here is

$$g_{x\bar{y}} \;=\; \kappa^{-2}\,\delta_{x\bar{y}}\cdot(1-N^C\overline{N}_C)^{-1}\;. \qquad (4.35)$$

The metric for the remaining 36 moduli (contained in 24 blocks) is not so easily found, because the relevant Yukawa couplings are not forbidden by H-charge conservation. As usual, the matter-field metrics are given in terms of the moduli metrics by eq. (4.7).

Similar analysis can be applied to the twelve-dimensional subspace spanned by the (1,2) moduli that preserve the $U(1)^2$ group generated by $J^1-\frac{1}{5}J$ and $J^2-\frac{1}{5}J$. As in the previous case, Yukawa couplings of the form W_{ABC} and W_{ABx} all vanish, which again means that the neutral subspace — now $SU(12,1)/(SU(12)\otimes U(1))$ — has a constant-curvature metric given by eq. (4.34), with $m = 12$. The remaining 89 (1,2) moduli split into 14 blocks according to the $U(1)^2$ charges of the matter fields; unfortunately, none of those blocks can be handled as easily as the twisted orbifold moduli.

⋆ The space $SU(m,1)/(SU(m)\otimes U(1))$ is a restricted Kähler manifold[25]: The holomorphic function $\mathcal{F}(N) = -\frac{1}{4}(1+\sum(N^C)^2)$ leads to a Kähler function $K(N,\overline{N}) = -\kappa^{-2}\log(1-N^C\overline{N}_C)$ and metric (4.34).

5. Conclusions

We would like to conclude this article with a discussion of two particularly important classes of (2,2) vacua of the heterotic string: the Calabi-Yau vacua, and the vacua in which E_6 is broken via the Hosotani-Witten mechanism. First, however, let us briefly summarize our results. This paper focused on the subclass of classical four-dimensional vacua of the heterotic string that exhibit $N = 1$ space-time supersymmetry and $E_6 \otimes E_8$ gauge group. We derived various identities among tree-level string scattering amplitudes of the moduli and the $\mathbf{27}$ and $\overline{\mathbf{27}}$ matter fields using the left-moving $N = 2$ superconformal algebra. By demanding that the low-energy effective field theory gives rise to the same S-matrix identities, we were able to relate various pieces of the effective action to each other. We confirmed that the metric for (1,1) and (1,2) moduli is locally a direct product of two restricted Kähler manifolds, as can also be understood from type II superstring arguments.[24,27] Each factor is completely determined by the $\mathbf{27}^3$ and $\overline{\mathbf{27}}^3$ terms in the superpotential for the corresponding matter fields. We also determined the metric for the matter fields, which is needed to establish the precise relation between the $\mathbf{27}^3$ and $\overline{\mathbf{27}}^3$ terms and the moduli metric, as well as to properly normalize the physical Yukawa couplings. Unlike the moduli metric, the metric for the matter fields depends on both (1,1) and (1,2) moduli. Finally, we showed how one can often solve exactly for the moduli and matter-field metrics on subspaces of the moduli space where the four-dimensional gauge group is enhanced beyond $E_6 \otimes E_8$.

Despite our repeated use of the terms "(1,1) moduli" and "(1,2) moduli", in the foregoing we never assumed that the four-dimensional (2,2) vacua described Calabi-Yau compactifications of the ten-dimensional heterotic string. On the other hand, Calabi-Yau compactifications certainly provide a large class of (2,2) vacua, and recent work[4,6] suggests that they may exhaust the (2,2) vacua with spacetime supersymmetry and $E_6 \otimes E_8$ gauge symmetry. It is therefore of interest to see what implications the general (2,2) results have for the Calabi-Yau case. First of all, one of the (1,1) moduli of a Calabi-Yau manifold represents its overall radius.† In the large-radius limit where this (1,1) modulus has a large expectation value, the world-sheet theory describing the Calabi-Yau vacuum is a weakly coupled sigma model. In this region of the moduli space (often called the "field-theory limit" as well as the large-radius limit), a Kaluza-Klein treatment of the ten-dimensional effective field theory for the heterotic string provides a good approximation to the four-dimensional effective theory. Expressions for the Yukawa couplings and Kähler functions in this approximation are known, and can be checked against the relations between them described in this article. These relations can then be used to extend some of the large-radius Calabi-Yau results to smaller radius, as has also been pointed out in ref. [28].

In the infinite radius limit, the $\mathbf{27}^3$ Yukawa coupling are purely topological, *i.e.* they do not depend on any of the moduli, and the (1,1) moduli space appears to be a restricted Kähler manifold with a cubic $\mathcal{F}_1 = \frac{1}{6} W_{abc} M^a M^b M^c$.[18,19] For Calabi-Yau manifolds of finite size, (some of) the Yukawa couplings receive corrections that are non-perturbative in the overall radius[12]; thus $\mathcal{F}_1$ is no longer purely cubic, and the metric for the (1,1) moduli receives non-perturbative corrections too. (Actually, the corrections are required to be non-perturbative — *i.e.*, exponentially suppressed in the large radius limit — only when the manifold is smooth and its curvature is everywhere small compared to $1/\alpha'$.) For Calabi-Yau compactification on manifolds that are not large, the field theory limit no longer can be expected to provide a good approximation to $\mathcal{F}_1$; nevertheless, even in this case $\mathcal{F}_1$ — whatever it might be — determines both the metric for the (1,1) moduli and the $\mathbf{27}^3$ Yukawa couplings[28]: eqs. (3.40) and (3.41) are valid for all

† We do not know at present how to identify this modulus in an arbitrary (2,2) vacuum, although for many particular exactly solvable (2,2) vacua it is readily identified; such an identification in the general case would be an important step in establishing that all (2,2) vacua are in fact Calabi-Yau compactifications.[44]

(2,2) vacua. Moreover, since $\mathcal{F}_1(M^a)$ is a holomorphic function, knowledge of its large-radius limit provides some constraints on its overall behavior.

For the (1,2) moduli the situation is quite different. Neither $\overline{\mathbf{27}}^3$ Yukawa couplings nor the metric for the (1,2) moduli depends on any of the (1,1) moduli, including the overall radius of the manifold. Therefore, all results obtained in the field theory limit for the $\overline{\mathbf{27}}^3$ couplings and for the (1,2) moduli metric[19] are exactly valid for all sizes of the manifold.[28] The precise correspondence between the (1,2) moduli metric obtained in ref. [19] and the metric found in this paper can be established by introducing homogeneous coordinates on the (1,2) moduli space as explained in ref. [45].

The difference in the large-radius behaviors of the (1,1) and (1,2) moduli metrics is due to the identification of the overall radius of a manifold as one of its (1,1) moduli. (The approximate Peccei-Quinn symmetry for the radius mode also plays a rôle, constraining the form of the radius-dependence of the (1,1) moduli metric.) In fact, equations (3.37) for the curvature of the moduli space in terms of the Yukawa couplings are completely symmetric under simultaneous interchange of **27**'s and $\overline{\mathbf{27}}$'s, and of (1,1) and (1,2) moduli. This symmetry is related to the ambiguity in the relative sign of the left-moving $U(1)$ current $J(z)$ and the right-moving current $\bar{J}(\bar{z})$: if we change the sign of $J(z)$, then $\Psi^+_\alpha \leftrightarrow \Psi^-_\mu$ and $\Phi^+_\alpha \leftrightarrow \Phi^-_\mu$. (Similarly, if we change the sign of $\bar{J}(\bar{z})$ while keeping the sign of $J(z)$, then $\Psi^+_\alpha \leftrightarrow \Psi^+_{\bar{\mu}}$, *etc.*) This ambiguity makes it difficult to identify, at a generic point in the moduli space of a putative Calabi-Yau compactification, which modes are the (1,1) moduli and which the (1,2) moduli. It is conceivable that in some cases both assignments could be "correct"; *i.e.* that the same (2,2) vacuum could be interpreted as compactification on a Calabi-Yau manifold $\mathcal{M}$ with Hodge numbers $h_{1,1} = N_1$, $h_{1,2} = N_2$, and also on another manifold $\mathcal{M}'$ with $h'_{1,1} = N_2$, $h'_{1,2} = N_1$. At present we have no convincing examples of this phenomenon, however.

The (2,2) vacua we have been discussing are not the most phenomenologically promising heterotic string vacua; one generally prefers a smaller four-dimensional gauge group than E_6. One way to achieve this is to give Planck-scale expectation values to certain components of $\mathbf{27}$'s and $\overline{\mathbf{27}}$'s in a (2,2) vacuum, in a way that is consistent with the classical string equations of motion and spacetime supersymmetry, *i.e.* that retains a (0,2) superconformal symmetry.[32,46] (This has the Calabi-Yau interpretation of deforming the vector bundle that describes the embedding of the spin connection into the gauge group.[32]) Because the left-moving $N = 2$ algebra is broken in these vacua, the techniques used in this article limit one to studying the (0,2) vacua perturbatively in the $\mathbf{27}$ and $\overline{\mathbf{27}}$ expectation values, and only up to fourth order with the amplitudes explicitly discussed here. (There are of course also $N = 2$ Ward identities relating higher-point scattering amplitudes for moduli and matter fields, which we have not examined in detail but which might prove useful in this context.)

Another mechanism of gauge symmetry breaking, which is discrete rather than continuous, and which is more amenable to our analysis, is the Hosotani-Witten mechanism.[8] Here a Calabi-Yau manifold $\mathcal{M}$ is modded out by a discrete symmetry group $\mathcal{H}$, and a set of Wilson lines are chosen for the quotient manifold $\mathcal{M}/\mathcal{H}$ — homomorphisms from $\mathcal{H}$ into $E_6 \otimes E_8$. The new four-dimensional gauge group is the subgroup of $E_6 \otimes E_8$ that commutes with this image of $\mathcal{H}$. Note that $\mathcal{H}$ does not have to act freely on $\mathcal{M}$— in this case $\mathcal{M}/\mathcal{H}$ is singular, and the "Wilson line" description is not entirely accurate, but we will use it anyway. The same procedure may be applied directly to a (2,2)-superconformal world-sheet theory with a discrete symmetry (or to a connected family of such (2,2) theories), regardless of whether or not there is a Calabi-Yau interpretation of the corresponding string vacuum. From the point of view of the conformal field theory, the procedure can be described as an orbifold twist[3] by $\mathcal{H}$. (For a very clear, detailed discussion of Wilson lines from the conformal-field-theory perspective, see ref. [47].)

As pointed out in ref. [47], the (twisted) conformal field theory describing Wilson lines on $\mathcal{M}/\mathcal{H}$ still possesses $N = (2,2)$ supersymmetry, although the charge quantization of the left-moving superalgebra may be spoiled — twisted states may have fractional charges with respect to $J(z)$. The spectrum of the twisted theory consists of an untwisted sector containing the $\mathcal{H}$-invariant states of the original theory, plus various twisted sectors. However, in most cases the twisted sectors *do not contain any moduli* — by which we mean fields of type $\Phi^{\pm}$, as in eq. (2.12). (The only twisted sectors in which moduli can appear are those where the twist element $h \in \mathcal{H}$ has fixed points and is not accompanied by a Wilson line.) Only the moduli preserve the left-moving $N = 2$ algebra when given an expectation value; hence the (2,2) Wilson-line vacua are parametrized solely by expectation values for the $\mathcal{H}$-invariant moduli of the untwisted theory. By analogy with the orbifold examples of section 4.2, we shall denote these untwisted moduli by N^a.

We are interested in the Kähler function $K(N^a)$ for the Wilson-line theory, plus the N^a-dependence of the metrics for the various matter fields — any massless supermultiplets transforming nontrivially under the surviving subgroup of E_6. Even though the twisted sectors do not typically contain moduli, they may still contain matter fields. (For compactifications on sufficiently large Calabi-Yau manifolds this can only happen for twists with fixed pionts.) Because the twisted matter fields are not related to any moduli (their vertices are not constructed from fields of the type $\Psi^{\pm}$), we cannot say anything in general about their metric. Henceforth we concentrate on the untwisted moduli and matter fields.

In general, fields in the untwisted sector of a twisted conformal field theory (here, the Wilson-line theory on $\mathcal{M}/\mathcal{H}$) have exactly the same scattering amplitudes at string tree-level as they do in the original, untwisted theory (here, the theory on $\mathcal{M}$). Furthermore, it would appear that the vertex operators for the untwisted moduli and matter fields are related to each other exactly as before, using the $N = 2$ algebra and the free fermions $\lambda^{\hat{p}}$ for the **10** components of the **27**'s and $\overline{\mathbf{27}}$'s (eqs. (2.12), (2.8)), *etc.* Then we could apply all of our previous results. The only subtlety is that the vertex operator relations are no longer invariant under $\mathcal{H}$, because the Wilson lines break E_6. In other words, some $\mathcal{H}$-invariant matter fields are related to $\mathcal{H}$-noninvariant moduli, and $\mathcal{H}$-invariant moduli are generally related to both $\mathcal{H}$-invariant and $\mathcal{H}$-noninvariant matter fields.[30]

For example, we can still calculate the Kähler function $K(N^a)$ for the $\mathcal{H}$-invariant moduli in terms of Yukawa couplings using eqs. (3.37); we just have to remember that the Yukawa couplings $W_{abc}(N)$ and $W_{lmn}(N)$ that appear in these equations are for **27**'s and $\overline{\mathbf{27}}$'s in the untwisted theory, and not all of their 27 E_6 components will survive the $\mathcal{H}$ projection. (The corresponding cubic superpotential couplings of the surviving E_6 components may in fact vanish by gauge invariance.) Once we have determined $K(N)$,* and hence the metric for the $\mathcal{H}$-invariant moduli N, we can use eqs. (3.36) to find the metric for the $\mathcal{H}$-invariant matter fields that are related to the N. The remaining matter fields are related to $\mathcal{H}$-noninvariant moduli, and so to calculate the metric for these matter fields one first calculates the metric for $\mathcal{H}$-noninvariant moduli as a function of the N. As with $K(N)$, this can be done in terms of the $\mathbf{27}^3$ and $\overline{\mathbf{27}}^3$ couplings in the untwisted theory using eqs. (3.37); the only difference is that more of these Yukawa couplings come into play.

Thus our general results for (2,2) vacua with $E_6 \otimes E_8$ gauge symmetry also provide useful information about metrics in the more realistic vacua with Wilson lines. Greater use of this information can be made in specific models by taking advantage of various discrete and enhanced gauge symmetries; work along these lines is in progress.

$\star$ A potential additional (technical) subtlety is that it may not be possible to solve (3.37) using the restricted Kähler ansatz (3.41) for $K(N)$, basically because there is no guarantee that the fields M^a appearing in (3.41) have to be eigenstates of the action of $\mathcal{H}$.

Acknowledgements: We would like to acknowledge fruitful discussions with Philip Candelas, Mirjam Cvetič, Sergio Ferrara and Andy Strominger, and to thank Michael Peskin for a careful reading of the manuscript.

APPENDIX A

This appendix contains an alternative derivation of eqs. (3.38) and of the independence of the $\mathbf{27}^3$ superpotential terms from the (1,2) moduli and of the $\overline{\mathbf{27}}^3$ terms from the (1,1) moduli.[37] This time we do not make any use of the Riemann tensor; instead, we consider scattering amplitudes $\mathcal{A}(M^{A_1},\ldots,M^{A_n},A^\beta,A^\gamma,A^\delta)$ and $\mathcal{A}(M^{A_1},\ldots,M^{A_n},A^\lambda,A^\mu,A^\nu)$ that involve three matter superfields and several moduli superfields, all of the same chirality. In terms of ordinary fields, two of the $n+3$ fields involved are fermions while the other $n+1$ fields are scalars; by supersymmetry, it does not matter which two fields are fermionic. Naïvely, these amplitudes are proportional to derivatives of the Yukawa couplings; however, in general coordinate systems, the correct field-theoretical expressions are:

$$\begin{aligned}\mathcal{A}_{\rm FT}(M^{A_1},\ldots,M^{A_n},A^b,A^c,A^d) &= ie^{\kappa^2(K_2-3K_1)/2}\,\nabla_{A_1}\cdots\nabla_{A_n}\left(e^{2\kappa^2K_1}\,W_{bcd}\right),\\ \mathcal{A}_{\rm FT}(M^{A_1},\ldots,M^{A_n},A^l,A^m,A^n) &= ie^{\kappa^2(K_1-3K_2)/2}\,\nabla_{A_1}\cdots\nabla_{A_n}\left(e^{2\kappa^2K_2}\,W_{lmn}\right).\end{aligned}\tag{A.1}$$

We shall derive eqs. (A.1) later in this appendix. Before we do that, we shall show that in string theory the amplitudes on the left hand side of eqs. (A.1) either vanish or are totally symmetric in all their $n+3$ indices.

Let us start with the case of three $\mathbf{27}$ matter fields and n (1,1) moduli. The heterotic string amplitude for this process is given by

$$\mathcal{A}(A^{\alpha_1},A^{\alpha_2},A^{\alpha_3},M^{a_4},\ldots,M^{a_{n+3}}) = U^{\alpha_4}_{a_4}\cdots U^{\alpha_{n+3}}_{a_{n+3}}\ \times \tag{A.2}$$

$$\times\,|J(z_{1,2,3})|^2\int_{\mathbf{C}^n} d^2z_4\cdots d^2z_{n+3}\ E(z_i,\bar z_i)\cdot\left\langle(i\lambda\Psi^+)_1\cdot(i\lambda\Psi^+)_2\cdot\Psi_3^{--}\cdot\Phi_4^+\cdot\cdots\cdot\Phi_{n+3}^+\right\rangle,$$

where $(i\lambda\Psi^+)_1$ is a short hand notation for $i\lambda(z_1)\Psi^+_{\alpha_1}(z_1,\bar z_1)$, *etc.* The operator Ψ^{--}_3 here — *i.e.*, $\Psi^{--}_{\alpha_3}(z_3,\bar z_3)$ — is a vertex operator for the $SO(10)$ singlet field inside the **27** multiplet A^{α_3}.† This operator has conformal weight $h=1$ and the $U(1)$ charge $q=-2$ and thus is a lower component of a chiral multiplet of the left-moving $N=2$ superalgebra. Hence, the operator product of $\Psi^{--}(z)$ with $T^-_F(w)$ is non-singular when $w\to z$, while the operator product expansion of $T^+_F(w)\cdot\Psi^{--}(z)$ starts with a simple pole $O((w-z)^{-1})$. Therefore,

$$\begin{aligned}
&\langle\lambda_1\cdot\lambda_2\rangle\times\left\langle\Psi^+_1\cdot\Psi^+_2\cdot\Psi^{--}_3\cdot\Phi^+_4\cdot\dots\cdot\Phi^+_{n+3}\right\rangle\\
&\quad=\ \frac{1}{z_{12}}\oint_{z_4}\frac{dw}{2\pi i}\,\frac{w-z_2}{z_{42}}\cdot\left\langle\Psi^+_1\cdot\Psi^+_2\cdot\Psi^{--}_3\cdot 2T^-_F(w)\cdot\Psi^+_4\cdot\Phi^+_5\cdot\dots\cdot\Phi^+_{n+3}\right\rangle\\
&\quad=\ \frac{-1}{z_{42}}\oint_{z_1}\frac{dw}{2\pi i}\,\frac{w-z_2}{z_{12}}\cdot\left\langle 2T^-_F(w)\cdot\Psi^+_1\cdot\Psi^+_2\cdot\Psi^{--}_3\cdot\Psi^+_4\cdot\Phi^+_5\cdot\dots\cdot\Phi^+_{n+3}\right\rangle\\
&\quad=\ \langle\lambda_2\cdot\lambda_4\rangle\times\left\langle\Phi^+_1\cdot\Psi^+_2\cdot\Psi^{--}_3\cdot\Psi^+_4\cdot\Phi^+_5\cdot\dots\cdot\Phi^+_{n+3}\right\rangle,
\end{aligned}\tag{A.3}$$

which means that we can interchange vertex operators $i\lambda\Psi^+$ and Φ^+ in the correlator in (A.2). Since the E factor in (A.2) and the integral itself (including the Jacobian $|J|^2$) are symmetric under permutation $z_1\leftrightarrow z_4$, we obtain

$$\begin{aligned}
&U^{\alpha_1}_{a_1}\cdot\mathcal{A}(A^{\alpha_1},A^{\alpha_2},A^{\alpha_3},M^{a_4},M^{a_5},\dots,M^{a_{n+3}})\\
&\qquad=\ U^{\alpha_4}_{a_4}\cdot\mathcal{A}(M^{a_1},A^{\alpha_2},A^{\alpha_3},A^{\alpha_4},M^{a_5},\dots,M^{a_{n+3}}).
\end{aligned}\tag{A.4}$$

Now, if we define moduli and matter fields such that $V^\alpha_a=\delta^\alpha_a$, then the $\mathbf{U}$ matrix is proportional to the unit matrix and eq. (A.4) implies that the amplitude $\mathcal{A}(A^{a_1},A^{a_2},A^{a_3},M^{a_4},\dots,M^{a_{n+3}})$ is totally symmetric with respect to its $n+3$ indices $a_1,\dots,a_{n+3}$. Similarly, the amplitude $\mathcal{A}(A^{m_1},A^{m_2},A^{m_3},M^{m_4},\dots,M^{m_{n+3}})$

† Two of the $n+3$ vertex operators here correspond to space-time fermions and thus have right-moving quantum numbers very different from the $n+1$ scalar vertices. However, the left-moving quantum numbers do not distinguish between different members of the same space-time supermultiplet. Since it is the left-moving quantum numbers that are relevant to our arguments, our notations do not indicate which fields and vertices are fermionic.

is also totally symmetric in $m_1, \ldots, m_{n+3}$. In view of eqs. (A.1), this symmetry immediately implies eqs. (3.38).

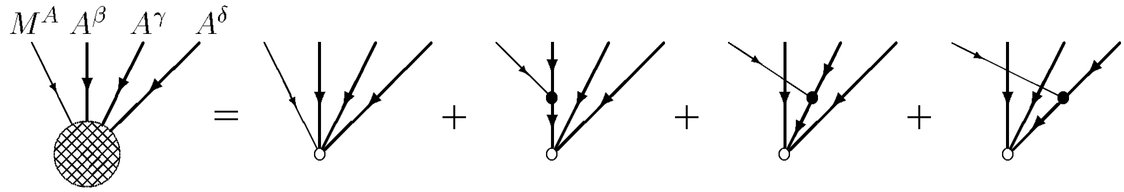
Next consider the string amplitude that involves three **27** fields and $n > 0$ (1,2) moduli. In this case we have

$$\begin{aligned}\mathcal{A}(A^\alpha, A^\beta, A^\gamma, M^{m_1}, \ldots, M^{m_n}) \ &\propto \ \left\langle (i\lambda\Psi^+)_1 \cdot (i\lambda\Psi^+)_2 \cdot \Psi_3^{--} \cdot \Phi_4^- \cdot \cdots \cdot \Phi_{n+3}^- \right\rangle \\ &= \oint_{z_4} \frac{dw}{2\pi i} \frac{w - z_3}{z_{12}\, z_{43}} \cdot \left\langle \Psi_1^+ \cdot \Psi_2^+ \cdot \Psi_3^{--} \cdot 2T_F^+(w) \cdot \Psi_4^- \cdot \Phi_5^- \cdot \cdots \cdot \Phi_{n+3}^- \right\rangle \ = \ 0,\end{aligned} \tag{A.5}$$

because operator products of T_F^+ with Φ^- or with Ψ^+ are non-singular, and the single pole in the operator product expansion of $T_F^+(w) \cdot \Psi_3^{--}$ is cancelled by the factor $w - z_3$ in the numerator. The same argument shows that the string amplitude involving three $\overline{\mathbf{27}}$ matter fields and $n > 0$ (1,1) moduli fields vanishes too. Finally, the amplitudes that involve three matter fields and both (1,1) and (1,2) moduli involve correlators of the types $\left\langle \Psi^+\Psi^+\Psi^{--} \cdot \Phi^+ \cdots \Phi^+ \cdot \Phi^- \cdots \Phi^- \right\rangle$ and $\left\langle \Psi^-\Psi^-\Psi^{++} \cdot \Phi^- \cdots \Phi^- \cdot \Phi^+ \cdots \Phi^+ \right\rangle$. Arguments similar to (A.5) show that these correlators are total world-sheet derivatives. Consequently, the amplitudes involving these correlators vanish at zero momentum. Therefore, at zero momentum, all string amplitudes involving three matter fields and several moduli fields of the same chirality vanish, unless all moduli accompanying three **27** fields are of the (1,1) type or all moduli accompanying three $\overline{\mathbf{27}}$ fields are of the (1,2) type. This is the theorem of Distler and Greene.[37]

Now let us go back to the field theory and derive eqs. (A.1). First consider amplitudes $\mathcal{A}(M^{A_1}, \ldots, M^{A_n}, A^\beta, A^\gamma, A^\delta)$ in rigidly supersymmetric field theory (no gravity). For $n = 0$, this amplitude is just the Yukawa coupling: $\mathcal{A}(A^\beta, A^\gamma, A^\delta) = iW_{\beta\gamma\delta}$. For $n = 1$, four Feynman diagrams contribute to

$\mathcal{A}(M, A, A, A)$:



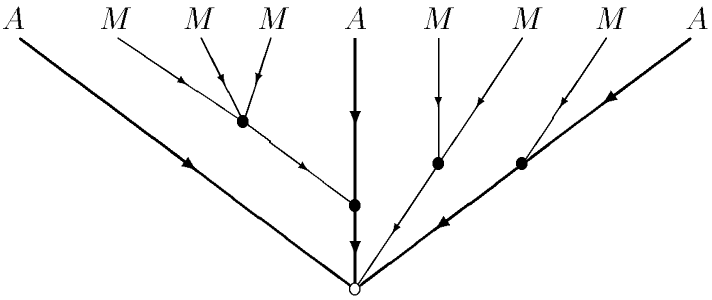
(A.6)

The contribution of the first diagram is momentum independent; for the other three three diagrams, the kinematical factors associated with the sigma-model vertex ($\bullet$) and with the matter-field propagator connecting it to the superpotential vertex ($\circ$) cancel each other, leaving only a momentum independent factor -1. Together the four diagrams yield:

$$\begin{aligned}\mathcal{A}(M^A, A^\beta, A^\gamma, A^\delta) &= iW_{\beta\gamma\delta,A} - i\Gamma^{\ \alpha}_{A\beta} W_{\alpha\gamma\delta} - i\Gamma^{\ \alpha}_{A\gamma} W_{\beta\alpha\delta} - i\Gamma^{\ \alpha}_{A\delta} W_{\beta\gamma\alpha} \\ &\equiv i\nabla_A W_{\beta\gamma\delta}\,, \end{aligned} \tag{A.7}$$

where $\Gamma^{\ \alpha}_{A\beta} = G_{\beta\bar\gamma,A} G^{\bar\gamma\alpha}$.

For $n > 1$ the number of Feynman diagrams grows rapidly. However, all tree diagrams contributing to $\mathcal{A}(M, \ldots, M, A, A, A)$ have the following structure:



(A.8)

There is one superpotential vertex ($\circ$) — the root of the tree, there are zero to n sigma-model vertices ($\bullet$), and all lines are directed downward, from the external fields (top ends of tree branches) toward the root. Each sigma-model vertex ($\bullet$) has several incoming lines and one outgoing, and the kinematic $-ik^2$

factor of the vertex precisely cancels (apart from the minus sign) the $-i/k^2$ factor of the outgoing propagator. Therefore, all amplitudes $\mathcal{A}(M,\ldots,M,A,A,A)$ are independent of the particle momenta.

To compute the moduli dependence of the amplitudes we use recursion: All Feynman diagrams of the $(n+1)$-moduli amplitude can be obtained from n-moduli diagrams by either attaching the $(n+1)^{\rm st}$ external modulus line to an existing vertex or by inserting a new sigma-model vertex (to which the new external line is attached) into an existing internal or external line (*cf.* diagrams (A.6)). As far as non-kinematic factors are concerned, attaching a new modulus line to an existing vertex calls for taking a derivative of that vertex with respect to M^A: $g_{C\bar{D},B_1\cdots B_m} \mapsto g_{C\bar{D},B_1\cdots B_m A}$, $G_{\alpha\bar{\beta},B_1\cdots B_m} \mapsto G_{\alpha\bar{\beta},B_1\cdots B_m A}$ and $W_{\alpha\beta\gamma,B_1\cdots B_m} \mapsto W_{\alpha\beta\gamma,B_1\cdots B_m A}$. The same happens when an internal line is split: $G^{\alpha\bar{\beta}} \mapsto -G^{\alpha\bar{\gamma}}G_{\bar{\gamma}\delta,A}G^{\delta\bar{\beta}} = \partial_A G^{\alpha\bar{\beta}}$ and ditto for a modulus propagator $g^{C\bar{D}}$. Inserting a new vertex into an external line yields a factor $-G_{\alpha\bar{\gamma},A}G^{\bar{\gamma}\beta} \equiv -\Gamma^{\ \beta}_{A\alpha}$, or $-\Gamma^{\ D}_{AC}$ in case of an external modulus line. Therefore, in rigidly supersymmetric field theory we have

$$\begin{aligned}\mathcal{A}(M^{A_1},\ldots,M^{A_{n+1}},A^\beta,A^\gamma,A^\delta) &= \nabla_{A_{n+1}}\mathcal{A}(M^{A_1},\ldots,M^{A_n},A^\beta,A^\gamma,A^\delta)\\ &= \cdots = i\nabla_{A_{n+1}}\cdots\nabla_{A_1}W_{\beta\gamma\delta}\,.\end{aligned} \tag{A.9}$$

Now consider gravitational corrections to eq. (A.9). Gravity itself does not contribute to scattering amplitudes $\mathcal{A}(M,\ldots,M,A,A,A)$ because it is impossible to draw a tree-level Feynman diagram for such an amplitude that contains a graviton (or gravitino) propagator. However, in a consistent supergravity theory, the Yukawa couplings are $e^{\kappa^2\hat{K}/2}W_{\beta\gamma\delta}$ rather than $W_{\beta\gamma\delta}$, and the sigma-model vertices involving fermion lines are slightly different from their purely bosonic counterparts: A vertex with two fermionic lines α and $\bar{\beta}$ yields $e^{-\kappa^2\hat{K}/4}\partial_A\cdots\partial_B\big(e^{\kappa^2\hat{K}/4}G_{\alpha\bar{\beta}}\big)$ instead of $\partial_A\cdots\partial_B G_{\alpha\bar{\beta}}$ (see ref. [15]). Two of the fields in $(M,\ldots,M,A,A,A)$ are fermions; hence each tree-level Feynman diagram contributing to this amplitude has two continuous fermionic lines connecting external fermions to the Yukawa vertex at the root of the tree ($\circ$ on diagram (A.8)). Therefore, the combined effect of gravitational corrections is to replace eq. (A.9) with

$$\mathcal{A}(M^{A_1},\ldots,M^{A_n},A^\beta,A^\gamma,A^\delta) \ = \ ie^{-\kappa^2\hat{K}/2}\,\nabla_{A_n}\cdots\nabla_{A_1}\left(e^{\kappa^2\hat{K}}W_{\beta\gamma\delta}\right). \quad \text{(A.10)}$$

Note that under a Kähler transform $\hat{K} \mapsto K-\Lambda-\overline{\Lambda}$, $W \mapsto e^{\kappa^2\Lambda}W$, the amplitude (A.10) changes its phase by $\kappa^2\mathrm{Im}\Lambda$; this behavior is common to all supergravity amplitudes that involve two fermionic partners of holomorphic scalars.

In coordinates corresponding to $\mathbf{V}\equiv\mathbf{1}$, W_{abc} is just $W_{\alpha\beta\gamma}\big|_{\alpha=a,\beta=b,\gamma=c}$; however, covariant derivatives ∇_A act differently on W_{abc} and on $W_{\alpha\beta\gamma}$ since the metrics for the moduli and for the matter fields are different. Specifically, given eqs. (3.36) relating those metrics, we have $\Gamma^{\ \beta}_{A\gamma} = \Gamma^{\ b}_{Ac}\big|_{b=\beta,c=\gamma} + \delta^\beta_\gamma\cdot\frac{\kappa^2}{3}\partial(K_2-K_1)/\partial M^A$, and for any tensor $X_{A_1\cdots A_n\beta\gamma\delta}$ that has three matter indices (and an arbitrary number of moduli indices), $\nabla_{A_{n+1}}(X_{A_1\cdots A_n\beta\gamma\delta}) = e^{\kappa^2(K_2-K_1)}\nabla_{A_{n+1}}\big(e^{\kappa^2(K_1-K_2)}X_{A_1\cdots A_nbcd}\big)$. Therefore, eq. (A.10) becomes the first equation in (A.1); the second equation (A.1) is completely analogous to the first.

We would like to conclude this appendix with a comment that there are two ways to use the result of ref. [37] about string amplitudes involving **27** matter fields and (1,2) moduli to prove that the $\mathbf{27}^3$ superpotential terms do not depend on the (1,2) moduli. (We do it because ref. [37] contains only the string arguments for vanishing of mixed amplitudes and treats the implication of this vanishing for the superpotential as if it was obvious.) One way is to use eqs. (A.1); notice however that these equations rely on formulæ (3.36) which relate moduli and matter metrics to each other: Without the U factors, there would be no eqs. (A.1). The other possibility is to use eq. (A.10) which is based on nothing but general $N=1$ supergravity. If we write the Kähler function of the entire field space in holo-normal form, then at the origin of the coordinate system the right hand side of eq. (A.10) reduces to ordinary derivatives of $W_{\alpha\beta\gamma}$ (*cf.* Appendix B). Since the $\mathbf{27}^3$ Yukawa couplings must be holomorphic function of all moduli, the theorem of ref. [37] implies that these couplings indeed depend only on the (1,1) moduli. Of course once this result is obtained, it remains valid for all coordinate systems on the moduli spaces, not just those that allow for a holo-normal form of the Kähler function.

APPENDIX B

In this appendix we construct coordinate systems on the two moduli spaces for which the Yukawa couplings can be written in the form (3.40). As a first step, let us show that for any Kähler manifold one can perform a combination of a Kähler transform and a holomorphic coordinate transform that will put the Kähler function into a holo-normal form $K = \sum_i \varphi^i \bar{\varphi}^i + O(\varphi^2 \bar{\varphi}^2)$. Let us choose an arbitrary point on the manifold and let $\{\phi^i\}$ be some complex coordinates that are regular in its neighborhood; without loss of generality we may assume that all ϕ^i vanish at the chosen point. $K(\phi, \bar{\phi})$ is an analytic function, so we can expand it into a power series in ϕ^i and $\bar{\phi}^{\bar{\imath}}$. Segregating terms that are at most linear in either holomorphic or anti-holomorphic coordinates, we write

$$K(\phi,\bar{\phi}) \;=\; \Lambda(\phi) + \overline{\Lambda}(\bar{\phi}) + g_{i\bar{\jmath}}(0)\cdot\phi^i\bar{\phi}^{\bar{\jmath}} + C_{\bar{\jmath}}(\phi)\cdot\bar{\phi}^{\bar{\jmath}} + \overline{C}_i(\bar{\phi})\cdot\phi^i + O(\phi^2\bar{\phi}^2), \tag{B.1}$$

where $\Lambda(\phi)$ sums all purely holomorphic terms in K and $C_{\bar{\imath}}(\phi) \cdot \bar{\phi}^{\bar{\jmath}}$ sums all terms that are linear in $\bar{\phi}$ but carry second or higher powers of the holomorphic coordinates ϕ^i. Now let us perform a Kähler transform $K(\phi, \bar{\phi}) \mapsto K'(\phi, \bar{\phi}) = K + \Lambda(\phi) + \overline{\Lambda}(\bar{\phi})$ and define new holomorphic coordinates $\varphi^i \equiv \phi^i + g^{i\bar{\jmath}}(0) \cdot C_{\bar{\jmath}}(\phi)$ (this transform is clearly non-degenerate since $C_{\bar{\jmath}}(\phi) = O(\phi^2)$); then we have

$$K'(\varphi,\bar{\varphi}) \;=\; g_{i\bar{\jmath}}(0)\cdot\varphi^i\bar{\varphi}^{\bar{\jmath}} \;+\; O(\varphi^2\bar{\varphi}^2)\,. \tag{B.2}$$

This expression for K' can now be put into the desired form by a linear redefinition of coordinates $\{\varphi^i\}$ that will turn the matrix $g_{i\bar{\jmath}}$ into a unit matrix.

In the coordinates $\{\varphi^i\}$, $K'_{,i_1\cdots i_n}(0)=0$ as well as $g_{i\bar{\jmath},k_1\cdots k_n}(0)=0$ (and ditto for the anti-holomorphic derivatives of K' and $g_{i\bar{\jmath}}$). It follows that at the point of expansion the Kristoffel symbols $\Gamma^{\,i}_{jk}$ vanish together with all their holomorphic derivatives (and $\Gamma^{\,\bar{\imath}}_{\bar{\jmath}\bar{k}}$ vanish with all their anti-holomorphic derivatives). This is the strongest normality requirement achievable in a general Kähler geometry by means of a holomorphic coordinate transform; for lack of a better term, we call coordinate systems obeying this requirement "holo-normal". We also call Kähler functions K "holo-normal" when they are expressed in terms of holo-normal coordinates and have no harmonic terms. Note that a coordinate system for a Kähler manifold can only be holonormal at some isolated points on the manifold; unless the manifold is flat, no coordinate system is holo-normal everywhere.

Now consider the (1,1) moduli space. Let us write its Kähler function K_1 in a holo-normal form. Then at the point $M=\vec{0}$, $K_{,a_1\cdots a_n}=0$ and $\Gamma^{\,a}_{bc}$ vanish with all their holomorphic derivatives; therefore, taking holomorphic covariant derivatives of the first eq. (3.39) yields for any $n\geq 0$:

$$\nabla_{a_1}\cdots\nabla_{a_n}\nabla_b\nabla_c\nabla_d Z_1 \;=\; \kappa^3\,\partial_{a_1}\cdots\partial_{a_n}W_{bcd}\quad \text{at } M=\vec{0}. \tag{B.3}$$

If we now define a manifestly holomorphic function

$$\mathcal{F}_1(M^a)\;\equiv\;\sum_{n=3}^{\infty}\frac{1}{n!}\,\nabla_{a_1}\cdots\nabla_{a_n}\,Z_1\big|_{M=\vec{0}}\times M^{a_1}\cdots M^{a_n}, \tag{B.4}$$

then $W_{bcd,a_1\cdots a_n}=\kappa^{-3}\mathcal{F}_{1,bcda_1\cdots a_n}$ at $M=\vec{0}$, for any $n\geq 0$. Since the $\mathbf{27}^3$ Yukawa couplings are holomorphic functions of the (1,1) moduli, it follows that W_{bcd} and $\mathcal{F}_1$ obey the first eq. (3.40) throughout the (1,1) moduli space (despite the fact that the coordinate system we used to construct $\mathcal{F}_1$ is holo-normal only at one point). Similarly, if we write K_2 in a holo-normal form, then we can construct an $\mathcal{F}_2$ that obeys the second eq. (3.40) throughout the (1,2) moduli space.

APPENDIX C

In this appendix we discuss moduli of the (2,2) vacua of the type II superstring and compare them to moduli of the same vacua of the heterotic string. First, a point of terminology: for the type II superstring we shall restrict the term moduli to scalars coming from the Neveu-Schwarz (NS) sectors of both left-moving and right-moving world-sheet theories; Ramond-Ramond scalars may be moduli in the sense of having a flat potential, but we shall not discuss them here. With this restriction in mind, $N=(2,2)$ superconformal world-sheet theories lead to essentially the same moduli scalars for the type II superstring as for the heterotic string[12,24]; the purpose of this appendix is to make this correspondence precise.

At zero space-time momentum, type II vertex operators of massless scalars are world-sheet fields $\Phi(z,\bar{z})$ of conformal dimension $(h,\bar{h})=(1,1)$ that are upper components of both left-moving and right-moving $N=(1,1)$ supermultiplets. For (2,2) vacua with integral $U(1)$ charges q and $\bar{q}$, such upper-component world-sheet fields are the same Φ^{+}_{α}, Φ^{-}_{μ}, $\Phi^{-}_{\bar{\alpha}}$ and $\Phi^{+}_{\bar{\mu}}$ that appear in moduli vertices of the heterotic string. However, at non-zero momenta there is a difference: From the left-moving point of view, the type II moduli vertices are

$$V^{\pm}(z,\bar{z}) \;=\; \exp\bigl(i\sqrt{2\alpha'}k\cdot X(z,\bar{z})\bigr)\times\left(\Phi^{\pm}(z,\bar{z})\;+\;i\sqrt{\tfrac{\alpha'}{2}}\,k\cdot\psi(z)\times\Psi^{\pm}\right), \quad \text{(C.1)}$$

where ψ^{i}, $i=0,1,2,3$ are right-moving superpartners of the bosonic fields X^{i} responsible for the four space-time coordinates, while the heterotic moduli vertices are simply $e^{i\sqrt{2\alpha'}k\cdot X}\Phi^{\pm}$. From the right-moving point of view, both type II and heterotic vertex operators for massless scalars have a structure that mirrors eq. (C.1). However, throughout this article we were able to ignore this right-moving structure of the heterotic vertex operators $\Phi^{\pm}$ and $\Psi^{\pm}$ because it completely commutes with the left-moving superalgebra (2.7). Since it is the left-moving quantum numbers of the type II moduli vertices that distinguish them from their analogues for the heterotic string, we shall continue to ignore the right-moving quantum numbers of $\Phi^{\pm}$ and $\Psi^{\pm}$ in this appendix, except for the last paragraph.

Now consider correlators of type II moduli vertices (C.1). Using the fact that $\psi^i(z)$ are free left-moving world-sheet fermions and hence $\left\langle \psi_1^i \cdot \psi_2^j \right\rangle = \delta^{ij}/z_{12}$, we conclude from (C.1) that

$$\begin{aligned}
\left\langle V_1^{\pm} \cdots V_n^{\pm} \right\rangle \;=\; E \times \Bigg\{ & \left\langle \Phi_1^{\pm} \cdots \Phi_n^{\pm} \right\rangle \;+\; \tfrac{1}{2} \sum_{i \neq j} \frac{\alpha' k_i \cdot k_j}{2 z_{ij}} \left\langle \Phi_1^{\pm} \cdots \Psi_i^{\pm} \cdots \Psi_j^{\pm} \cdots \Phi_n^{\pm} \right\rangle \\
& + \tfrac{1}{8} \sum_{\substack{\text{distinct} \\ i,j,k,l}} \frac{\alpha'^2 (k_i \cdot k_j)(k_k \cdot k_l)}{4 z_{ij}\, z_{kl}} \times \\
& \qquad \times \left\langle \Phi_1^{\pm} \cdots \Psi_i^{\pm} \cdots \Psi_j^{\pm} \cdots \Psi_k^{\pm} \cdots \Psi_l^{\pm} \cdots \Phi_n^{\pm} \right\rangle \\
& + \cdots \Bigg\},
\end{aligned} \tag{C.2}$$

where E was defined in formula (3.13) and $\left\langle \Phi_1^{\pm} \cdots \Psi_i^{\pm} \cdots \Psi_j^{\pm} \cdots \Phi_n^{\pm} \right\rangle$ is actually $-\left\langle \Phi_1^{\pm} \cdots \Psi_j^{\pm} \cdots \Psi_i^{\pm} \cdots \Phi_n^{\pm} \right\rangle$ when $i > j$, *etc.* All correlators on the right hand side appear in heterotic string formulæ for scattering of moduli and matter fields (*cf.* eqs. (3.12), (3.17) and (3.22)). Therefore, type II amplitudes for moduli scattering are related to the heterotic amplitudes via

$$\begin{aligned}
\mathcal{A}_{II}(M^{a_1}, \ldots, M^{a_n}) \;=\; & \mathcal{A}_H(M^{a_1}, \ldots, M^{a_n}) \\
& + \sum_{i \neq j} \frac{\alpha'(k_i \cdot k_j)}{4}\, U_{a_i}^{\alpha_i} U_{a_j}^{\alpha_j} \cdot \mathcal{A}_H(M^{a_1}, \ldots, A_{\hat{p}}^{\alpha_i}, \ldots, A_{\hat{p}}^{\alpha_j}, \ldots, M^{a_n}) \\
& + \sum_{\substack{\text{distinct} \\ i,j,k,l}} \frac{\alpha'^2 (k_i \cdot k_j)(k_k \cdot k_l)}{32}\, U_{a_i}^{\alpha_i} U_{a_j}^{\alpha_j} U_{a_k}^{\alpha_k} U_{a_l}^{\alpha_l} \times \\
& \qquad \times \mathcal{A}_H(M^{a_1}, \ldots, A_{\hat{p}}^{\alpha_i}, \ldots, A_{\hat{p}}^{\alpha_j}, \ldots, A_{\hat{q}}^{\alpha_k}, \ldots, A_{\hat{q}}^{\alpha_l}, \ldots, M^{a_n}) \\
& + \cdots
\end{aligned} \tag{C.3}$$

(as usual, no sum over the $SO(10)$ vector indices $\hat{p} \neq \hat{q}$), where by abuse of notations we ignore the difference between holomorphic and anti-holomorphic fields as well as the difference between (1,1) and (1,2) moduli and **27** and $\overline{\mathbf{27}}$ matter fields.

The low-energy behavior of various amplitudes in (C.3) may be obtained from the Feynman rules of the effective four-dimensional field theory. All tree-level diagrams that involve at most two external matter fields scale as $O(k^2)$ under uniform rescaling of all particle momenta; this happens because all vertices in those diagrams yield $O(k^2)$ factors. (The total number of vertices in a tree diagram exceeds the number of internal lines by one.) Consequently, amplitudes $\mathcal{A}(M,\ldots,M)$ and $\mathcal{A}(A,A,M,\ldots,M)$ behave like $O(k^2)$ (*cf.* eqs. (3.5) and (3.6)). Diagrams with four external matter lines may contain an $O(k^0)$ scalar potential vertex or two $O(k^1)$ gauge vertices; hence, the leading behaviour of amplitudes $\mathcal{A}(A,A,A,A,M,\ldots,M)$ is $O(k^0)$ (*cf.* eq. (3.11)). Feynman diagrams with more than four external matter lines can be analyzed in the same way. Leaving details as an exercise to the reader, we can state the general result as follows: The leading low-energy behavior of scattering amplitudes involving $2m > 0$ matter scalars and an arbitrary number of moduli scalars is $O(k^{4-2m})$. On the other hand, all heterotic amplitudes involving $2m > 0$ matter scalars appear in eq. (C.3) multiplied by factors of the order k^{2m}. Therefore,

$$\mathcal{A}_{II}(M^{a_1},\ldots,M^{a_n}) \ = \ \mathcal{A}_H(M^{a_1},\ldots,M^{a_n}) \ + O(k^4), \tag{C.4}$$

while the leading behavior of the two amplitudes themselves is $O(k^2)$. The result (C.4) is well-known to a few people — it is implicit in ref. 24 for example — but it does not seem to have been explicitly presented in the literature.

At the tree level of the superstring theory, scattering of particles coming from the Neveu-Schwarz sector cannot involve Ramond particles as intermediate states. Therefore, tree-level scattering of the type II moduli — which come from the NS–NS sector — does not depend on interactions that involve fermions or Ramond-Ramond bosons. The only couplings that do affect the type II amplitudes $\mathcal{A}_{II}(M,\ldots,M)$ are couplings of moduli fields to each other and to gravity. The same is true for moduli scattering in the heterotic case, albeit for a different reason. Hence, the implication of eq. (C.4) for the low-energy effective Lagrangian is that *all two-derivative interactions between moduli scalars are the same for the heterotic string and the type II superstring compactified on the same (2,2) vacuum;*[24] on the other hand, *higher-derivative interactions of moduli may differ.* In particular, the geometry of the moduli space in both string theories is the same direct product of the same two restricted Kähler manifolds⋆ (the geometry is determined by the two-derivative kinetic terms in the effective Lagrangian). However, higher-derivative interactions of moduli need not respect this direct product structure and may depend on both kinds of moduli.

Usually the $O(k^4)$ difference between heterotic and type II amplitudes for moduli scattering does not vanish. For example, compare the heterotic four-(1,1)-moduli amplitude (3.26) to its type II counterpart:

$$\mathcal{A}_{II}(M^a, M^b, \overline{M^{\bar{c}}}, \overline{M^{\bar{d}}}) \;=\; \frac{\alpha' s}{4}\, U^{\alpha}_{a} U^{\beta}_{b} \overline{U}^{\bar{\gamma}}_{\bar{c}} \overline{U}^{\bar{\delta}}_{\bar{d}} \cdot \mathcal{A}_H(A^{\alpha}_{\hat{p}}, A^{\beta}_{\hat{p}}, \overline{A}^{\bar{\gamma}}_{\hat{q}}, \overline{A}^{\bar{\delta}}_{\hat{q}}). \tag{C.5}$$

(This formula can be derived by substituting eqs. (3.26) and (3.19) into (C.3).) On the other hand, when a heterotic amplitude vanishes exactly, its type II counterpart often does so too. For example, the heterotic amplitude $\mathcal{A}_H(M^a, M^b, \overline{M}^{\bar{m}}, \overline{M}^{\bar{n}})$ vanishes exactly as an integral of the world-sheet correlator $\langle \Phi^+\Phi^+\Phi^+\Phi^+ \rangle$, which vanishes by eq. (3.28). In the type II case we have the correlator $\langle V^+V^+V^+V^+ \rangle$, which is equal to $E \times \langle \Phi^+\Phi^+\Phi^+\Phi^+ \rangle$ because all other terms in (C.2) vanish by the $U(1)$ charge conservation (one cannot have Ψ^+'s without an equal number of

⋆ In the type IIA case, the (1,1) moduli, the dilaton/axion and their Ramond-Ramond superpartners together span a quaternionic manifold that is not Kähler; however, the (1,1) moduli themselves span a restricted Kähler manifold.[24,27] In the type IIB case, the same happens to the (1,2) moduli instead of the (1,1) moduli.

Ψ^-'s). Hence, $\mathcal{A}_{II}(M^a, M^b, \overline{M}^{\bar{m}}, \overline{M}^{\bar{n}}) = \mathcal{A}_H(M^a, M^b, \overline{M}^{\bar{m}}, \overline{M}^{\bar{n}}) = 0$. Similarly, the type II amplitude $\mathcal{A}_{II}(M^a, M^b, \overline{M}^{\bar{m}}, \overline{M}^{\bar{d}})$ vanishes because

$$\begin{aligned} \left\langle V_1^+ \cdot V_2^+ \cdot V_3^+ \cdot V_4^- \right\rangle \; &= \; E \cdot \Big\{ \left\langle \Phi_1^+ \cdot \Phi_2^+ \cdot \Phi_3^+ \cdot \Phi_4^- \right\rangle - \tfrac{\alpha' s}{4 z_{34}} \left\langle \Phi_1^+ \cdot \Phi_2^+ \cdot \Psi_3^+ \cdot \Psi_4^- \right\rangle \\ &\qquad - \tfrac{\alpha' u}{4 z_{24}} \left\langle \Phi_1^+ \cdot \Psi_2^+ \cdot \Phi_3^+ \cdot \Psi_4^- \right\rangle - \tfrac{\alpha' t}{4 z_{14}} \left\langle \Psi_1^+ \cdot \Phi_2^+ \cdot \Phi_3^+ \cdot \Psi_4^- \right\rangle \Big\} \\ &= \; E \cdot \{ 0 \; - \; 0 \; - \; 0 \; - \; 0 \} \; = \; 0; \end{aligned} \tag{C.6}$$

the second equality here follows from eqs. (3.29) and (3.21). It is easy to generalize eqs. (3.28), (3.29) and (C.6) to the case of an arbitrary number of moduli vertices:

$$\begin{aligned} \left\langle V_1^+ \cdots V_n^+ \right\rangle \; &= \; E \times \left\langle \Phi_1^+ \cdots \Phi_n^+ \right\rangle \; = 0, \\ \left\langle V_1^+ \cdots V_{n-1}^+ \cdot V_n^- \right\rangle \; &= \; E \times \left\langle \Phi_1^+ \cdots \Phi_{n-1}^+ \cdot \Phi_n^- \right\rangle \; + \; 0 \; = \; 0, \end{aligned} \tag{C.7}$$

and all type II or heterotic amplitudes involving these correlators (or their complex conjugates) vanish identically.

We derived eqs. (C.7) using only the left-moving $N = 2$ superalgebra (2.7). Exactly analogous arguments based on the right-moving $N = 2$ superalgebra show that for all n:

$$\left\langle \hat{V}_1^+ \cdots \hat{V}_n^+ \right\rangle = \left\langle \hat{V}_1^+ \cdots \hat{V}_{n-1}^+ \cdot \hat{V}_n^- \right\rangle = \left\langle \hat{V}_1^- \cdots \hat{V}_{n-1}^- \cdot \hat{V}_n^+ \right\rangle = \left\langle \hat{V}_1^- \cdots \hat{V}_n^- \right\rangle \; = \; 0, \tag{C.8}$$

where $\hat{V}^\pm$ are the right-moving analogues of the $V^\pm$. From the right-moving point of view, massless scalar vertices are of the type $\hat{V}^+$ or $\hat{V}^-$ according to whether the space-time field is holomorphic or anti-holomorphic. Therefore, *for all scattering processes that involve arbitrary numbers of massless scalar fields but no fields of other kinds, the (on-shell) amplitudes vanish identically unless at least two of the fields are holomorphic and at least two are anti-holomorphic.* This result is valid not just for the type II superstring, but for the heterotic string as well, and the scalar fields involved may be moduli, **27** or $\overline{\mathbf{27}}$ matter fields or non-moduli singlets; in fact, it is valid for all space-time supersymmetric (0,2) vacua of the heterotic string for all massless scalar fields the theory may possess. From the effective-field-theory point of view this rule is a consequence of space-time supersymmetry; what eqs. (C.8) really tell us is that higher-derivative interactions which follow from the string theory do behave in the same way.

REFERENCES

1. D. Gross, J. Harvey, E. Martinec and R. Rohm, *Nucl. Phys.* **B256** (1985), 253, *Nucl. Phys.* **B267** (1986), 75.

2. P. Candelas, G. Horowitz, A. Strominger and E. Witten, *Nucl. Phys.* **B258** (1985), 46.

3. L. Dixon, J. Harvey, C. Vafa and E. Witten, *Nucl. Phys.* **B261** (1985), 620, *Nucl. Phys.* **B274** (1986), 285.

4. D. Gepner, *Phys. Lett.* **199B** (1987), 370, *Nucl. Phys.* **B296** (1988), 757.

5. Y. Kazama and H. Suzuki, University of Tokyo preprints UT-Komaba 88-8, 88-12.

6. E. Martinec, University of Chicago preprint EFI-88-76 and ENS-preprint;
   C. Vafa and N. Warner, Harvard preprint HUTP-88/A037;
   W. Lerche, C. Vafa and N. Warner, Harvard preprint HUTP-88/A065;
   B. Greene, C. Vafa and N. Warner, Harvard preprint HUTP-88/A047.

7. C. Hull and E. Witten, *Phys. Lett.* **160B** (1985), 398;
   A. Sen, *Nucl. Phys.* **B278** (1986), 289;
   T. Banks, L. Dixon, D. Friedan and E. Martinec, *Nucl. Phys.* **B299** (1988), 613.

8. Y. Hosotani, *Phys. Lett.* **126B** (1983), 309;
   E. Witten, *Phys. Lett.* **126B** (1984), 351.

9. For a review see G. Ross, CERN preprint TH–5109/88 and references therein.

10. D. Gross and J. Sloan, *Nucl. Phys.* **B291** (1987), 41.

11. J. Lauer, D. Lüst and S. Theisen, *Nucl. Phys.* **B304** (1988), 236.

12. M. Dine, N. Seiberg, X. G. Wen and E. Witten, *Nucl. Phys.* **B278** (1986), 769, *Nucl. Phys.* **B289** (1987), 319.

13. L. Dixon, in *Proceedings of the 1987 ICTP Summer Workshop in High Energy Physics and Cosmology*, Trieste, Italy, edited by G. Furlan, J. C. Pati, D. W. Sciama, E. Sezgin and Q. Shafi.

14. M. Dine and N. Seiberg, *Nucl. Phys.* **B293** (1987), 253.

15. For a review see J. Bagger, in *Proceedings of the 1984 NATO Advanced Summer Institute*, Bonn, edited by K. Dietz, R. Flume, G. von Gehlen and V. Rittenberg.

16. For a recent review see B. Greene, Harvard preprint HUTMP-88/B233 and references therein.

17. M. Cvetič, *Phys. Rev. Lett.* **59** (1987), 1795, 2829;
M. Dine and C. Lee, *Phys. Lett.* **203B** (1988), 371.

18. A. Strominger, *Phys. Rev. Lett.* **55** (1985), 2547;
A. Strominger and E. Witten, *Comm. Math. Phys.* **101** (1985), 341.

19. P. Candelas, P. Green and T. Hübsch, University of Texas preprints UTTG-24-88, 28-88 and 10-89.

20. E. Witten, *Phys. Lett* **155B** (1985), 151;
C. Burgess, A. Font and F. Quevedo, *Nucl. Phys.* **B272** (1986), 661;
S. Ferrara, C. Kounnas and M. Porrati, *Phys. Lett.* **181B** (1986), 263.

21. M. Cvetič, J. Louis and B. Ovrut, *Phys. Lett.* **B206** (1988), 227;
S. Cecotti, S. Ferrara and L. Girardello, *Nucl. Phys.* **B308** (1988), 436;
M. Cvetič, J. Molera and B. Ovrut, University of Pennsylvania preprint UPR-0367T.

22. A. Zamolodchikov, *JETP Letters* **43** (1986), 731;
A. Polyakov, *"Gauge Fields and Strings"* (in *Contemporary Concepts in Physics*, v. 3, Harwood Academic Pub., 1987).

23. D. Kutasov, Weizmann Institute preprint WIS-88/55/Oct-PH;
M. Cvetič, J. Louis and B. Ovrut, University of Pennsylvania preprint UPR-0380T.

24. N. Seiberg, *Nucl. Phys.* **B303** (1988), 286.

25. B. deWit and A. van Proeyen, *Nucl. Phys.* **B245** (1984), 89.

26. E. Cremmer, C. Kounnas, A. van Proeyen, J. Derendinger, S. Ferrara, B. de Wit and L. Girardello, *Nucl. Phys.* **B250** (1985), 385.

27. S. Cecotti, S. Ferrara and L. Girardello, *Phys. Lett.* **213B** (1988), 443 and CERN preprint TH-5080.

28. M. Dine, P. Huet and N. Seiberg, Princeton preprint IASSNS-HEP-88/54.

29. E. Cremmer, S. Ferrara, L. Girardello and A. van Proeyen, *Nucl. Phys.* **B212** (1983), 413.

30. E. Witten, *Nucl. Phys.* **B258** (1985), 75.

31. M. Dine, R. Rohm, N. Seiberg and E. Witten, *Phys. Lett.* **156B** (1986), 55.

32. E. Witten, *Nucl. Phys.* **B268** (1986), 79.

33. M. Dine and N. Seiberg, *Nucl. Phys.* **B306** (1988), 137.

34. N. Seiberg and A. Schwimmer, *Phys. Lett.* **184B** (1987), 191.

35. A. Belavin, A. Polyakov and A. Zamolodchikov, *Nucl. Phys.* **B241** (1984), 333.

36. A. Lichnerowicz, *Global Theory of Connections and Holonomy Groups*, Noordhoff International Publishing, Leyden, The Netherlands, 1976, p. 117ff.

37. J. Distler and B. Greene, *Nucl. Phys.* **B309** (1988), 295.

38. J. Derendinger and S. Ferrara, in *Proceedings of the Trieste Spring School 1984*, edited by B. de Wit, P. Fayet and P. van Nieuwenhuizen.

39. K.S. Narain, *Phys. Lett.* **169B** (1986), 41.

40. S. Ferrara and M. Porrati, *Phys. Lett.* **216B** (1989), 289.

41. E. Cremmer and A. van Proeyen, *Class. Quant. Grav.* **2** (1985), 445.

42. S. Ferrara, private communication.

43. L. Dixon, D. Friedan, E. Martinec and S. Shenker, *Nucl. Phys.* **B282** (1987), 13;
S. Hamidi and C. Vafa, *Nucl. Phys.* **B279** (1987), 465.

44. N. Seiberg, private communication.

45. S. Ferrara and A. Strominger, CERN preprint TH-5291/89.

46. J. Distler, *Phys. Lett.* **188B** (1987), 431;
J. Distler and B. Greene, *Nucl. Phys.* **B304** (1988), 1.

47. G. Athanasiu, J. Atick, M. Dine and W. Fischler, *Phys. Lett.* **214B** (1988), 55.