Applications of Dualities in String Theory

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Contents

1 Introduction

1.1 String Theory

At present Theoretical Physics consists of two fundamental theories, the Standard Model of particle physics and General Relativity. Both theories have proven very successful in the past and there have been no experimental results that are not in agreement with one of the two models. A major unsolved problem is how the two theories can be united to one consistent theory. The most promising candidate for such a unified theory is string theory. Due to the complexity of string theory, we can only give a brief introduction to the main concepts in the following. For a more detailed description see [1, 2, 3].

The fundamental objects of string theory are one-dimensional quantized objects -the strings- which propagate in some d-dimensional background space-time. In other words, the two-dimensional worldsheet of the string Σ is embedded into space-time M_d and can be described by a map $X : \Sigma \to M_d$, where $X^M, M = 0, \ldots, (d-1)$, are space-time coordinates. The starting point for considering string theory is the two-dimensional worldsheet action, where the space-time coordinates are treated as d two-dimensional bosonic fields $X^M(\tau, \sigma)$, with τ, σ being the coordinates on the string worldsheet. As in the case of point particles, the action equals the area of the worldsheet,

$$S = \frac{1}{2\pi\alpha'}\mathrm{Vol}(\Sigma) = \frac{1}{2\pi\alpha'}\int_\Sigma d\sigma d\tau \sqrt{\mathrm{det}\partial_\alpha X^M \partial_\beta X_M}, \tag{1.1}$$

where $\alpha = 1, 2$ is the worldsheet index. The constant α' of mass dimension -2 has to be introduced to make the action dimensionless. The quantity $T = \frac{1}{2\pi\alpha'}$ is the tension of the string. The limit $\alpha' \to 0$ is the point particle limit. The above action is called the Nambu-Goto action. Classically, it is equivalent to the more convenient Polyakov action

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma d\sigma d\tau \sqrt{h} h^{\alpha\beta}\partial_\alpha X^M \partial_\beta X_M, \tag{1.2}$$

where $h^{\alpha\beta}$ is the worldsheet metric. The Polyakov action has three symmetries: it is Poincaré invariant, it is invariant under reparametrizations of the worldsheet and it is invariant under Weyl rescaling. Reparametrization and Weyl invariance can be used to set $h^{\alpha\beta} = \eta^{\alpha\beta}$, where $\eta^{\alpha\beta}$ is the flat two-dimensional Minkowski metric. In this gauge, the Polyakov action describes a free two-dimensional conformal field theory and the equation of motion for X^M is simply the free wave equation

$$(\partial_\tau^2 - \partial_\sigma^2)X^M = 0. \tag{1.3}$$

The general solution to the equation of motion is

$$X^M(z, \bar{z}) = X^M(z) + \bar{X}^M(\bar{z}), \tag{1.4}$$

where $z = \tau - \sigma$, $\bar{z} = \tau + \sigma$ and $X^M(z), \bar{X}^M(\bar{z})$ describe the "left-moving" and "right-moving" part of X^M. For closed strings, that means for strings whose worldsheet is a closed circle for constant $\tau = \tau_0$, the worldsheet theory factorizes into an independent left- and right-moving part. For open strings, which are strings whose ends do not meet at fixed τ, the left- and right-moving part are not independent. For simplicity we consider closed strings in the following.

So far, the string theory is purely classical. The quantum theory is obtained via first quantization, that means the worldsheet scalars X^M are considered as operators fulfilling the commutator relations of two-dimensional quantum mechanics. The quantized bosonic string describes ordinary quantum mechanics on the worldsheet. As usual in quantum mechanics, one can construct a vacuum state $|0\rangle$ and a Fock space of quantum mechanical states by applying the raising operator to the vacuum.

The energy-momentum tensor T of the worldsheet theory is given by the variation of the action with respect to the metric. The energy-momentum tensor splits into a left- and a right-moving part $T(z,\bar{z}) = T(z) + \bar{T}(\bar{z})$ and can be expanded in modes

$$
\begin{aligned}
T(z) &= \sum L_n z^{-n-2}, \qquad L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z), \\
\bar{T}(\bar{z}) &= \sum \bar{L}_n \bar{z}^{-n-2}, \qquad \bar{L}_n = \oint \frac{d\bar{z}}{2\pi i} \bar{z}^{n+1} T(\bar{z}),
\end{aligned} \tag{1.5}
$$

where L_n and $\bar{L}_n$ are the Virasoro generators. The Virasoro generators fulfill the conformal, or Virasoro, algebra

$$
[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n}, \tag{1.6}
$$

and the equivalent algebra for the right-moving part. The number c is the central charge of the conformal algebra. Each boson of the worldsheet theory contributes one to the central charge, thus the d worldsheet scalars X^M lead to $c = d$. The important point is that the physical states of the bosonic string theory are in one-to one correspondence with the primary states of the conformal algebra. Primary states satisfy

$$
\begin{aligned}
L_m|\text{phys}\rangle &= 0, \quad m > 0, \\
L_0|\text{phys}\rangle &= h|\text{phys}\rangle, \\
(L_0 - \bar{L}_0)|\text{phys}\rangle &= 0,
\end{aligned} \tag{1.7}
$$

and the equivalent equations for the right-moving part, where h is the conformal weight of the physical state. It can be shown that the theory is consistent only if $d = 26$ and $h = 1$. Thus the bosonic string lives in a 26-dimensional space-time and all physical states have conformal weight $h = 1$. The space of physical states consists of a vacuum and a discrete spectrum of infinitely many excitations with increasing mass. The masses are quantized in units of $1/\sqrt{\alpha'}$. A fundamental problem of bosonic string theory is that the vacuum state is a tachyon, that means it has a negative mass square. Another problem is that the bosonic string does not include fermions in the space-time. Thus, the theory is not a good candidate for a realistic theory.

The tachyon vanishes from the string spectrum if one considers local supersymmetry on the worldsheet. String theory with local $N = (1,1)$ worldsheet supersymmetry is called superstring theory. Its spectrum is tachyon free and the worldsheet supersymmetry leads to a spacetime theory that includes fermions. The worldsheet theory is

$$
S = -\frac{1}{2\pi} \int_\Sigma d\tau d\sigma \left(\partial^\alpha X^M \partial_\alpha X_M - i\bar{\psi}^M \gamma^\alpha \partial_\alpha \psi_M \right), \tag{1.8}
$$

where ψ are the worldsheet fermions and γ^α are the two-dimensional Dirac matrices

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{1.9}$$

The above action is in superconformal gauge, that means the local symmetries of the action are fixed such that the worldsheet metric is flat. Due to different boundary conditions on the worldsheet, the worldsheet fermions give rise to space-time fermions as well as space-time bosons. The former sector is called the R-sector (Ramond-sector) and the latter one is called the NS-sector (Neveu-Schwarz-sector) of the theory.

In addition to the energy-momentum tensor (1.5), there is a second conserved current $T_F(z,\bar{z}) = T_F(z) + \bar{T}_F(\bar{z})$ generating the $N = (1,1)$ supersymmetry. The supercurrent can be expanded as

$$T_F(z) \;=\; \sum G_r z^{-r-3/2}, \qquad G_r = \oint \frac{dz}{2\pi i} z^{r+\frac{1}{2}} T_F(z). \tag{1.10}$$

The conformal algebra (1.7) is replaced by the superconformal algebra

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m} \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s}, \\ [L_n, G_r] &= \left(\frac{n}{2} - r\right) G_{n+r}. \end{aligned} \tag{1.11}$$

The central charge is $c = 3/2d$, each worldsheet scalar contributes one and each worldsheet fermion contributes 1/2. Physical states satisfy

$$\begin{aligned} G_r|\text{phys}\rangle &= 0, \quad r > 0, \\ L_m|\text{phys}\rangle &= 0, \quad m > 0, \\ L_0|\text{phys}\rangle &= h|\text{phys}\rangle \end{aligned} \tag{1.12}$$

in the NS-sector. In the R-sector the last equation gets modfied to

$$L_0|\text{phys}\rangle \;=\; 0. \tag{1.13}$$

It can be shown that the theory is consistent for $d = 10$ and $h = 1/2$ only. In the R-sector, the above equation leads to a vacuum $|0\rangle$ with mass $m^2 = 0$, in the NS-sector, one has $m^2 = -1/2$. Thus at first sight the superstring also contains a tachyon. This inconsistency is removed by making a truncation of the theory, called the GSO-Projection.

The GSO projection does not only remove the tachyon from the spectrum, but also guarantees an $N = 1$ space-time supersymmetry of the ten-dimensional theory for each left- and right-moving part of the worldsheet theory. Also, the ten-dimensional theory after the GSO projection is anomaly-free. Having $N = 1$ supersymmetry from both the left- and the right-moving sector, one ends up with a theory in $d = 10$ with a total $N = 2$ supersymmetry. The spectrum of physical states after the GSO projection consists of massless states plus infinitely many massive excitations. There are several possibilities to do the GSO projection and obtain a consistent theory.

A consistent closed superstring theory is type IIA theory. The spectrum of closed strings is obtained by tensoring the left- and right-moving states of the NS- and the R-sector. After the GSO projection, the massless spectrum of the theory includes two spin 1/2 fermions of opposite chirality and two spin 3/2 gravitinos of opposite chirality originating from the (R, NS) and (NS, R)-sector. From the (NS, NS)-sector there are one real scalar (the dilaton), an antisymmetric rank two tensor and the graviton. The (R, R)-sector consists of a vector and an antisymmetric rank three tensor. The $N = 2$ supersymmetry is split in 16+16' supercharges of opposite chirality in ten dimensions.

The second consistent $N = 2$ space-time supersymmetric closed superstring theory is called type IIB theory. The spectrum includes two spin 1/2 fermions and two spin 3/2 gravitinos of the same chirality from the (R, NS) and (NS, R)-sector. In the (NS, NS)-sector, the spectrum is identical to the IIA spectrum, there is one real scalar (the dilaton), an antisymmetric rank two tensor and the graviton. The (R, R)-sector consists of a scalar, an antisymmetric rank two and an antisymmetric rank four tensor. The ten-dimensionsional $N = 2$ supersymmetry has 32 supercharges of the same chirality.

In addition to the type II thories, there is a third closed superstring theory which leads to a consistent space-time supersymmetric theory in ten dimensions. This is the heterotic string theory. The worldsheet theory of the heterotic string is a hybrid construction of the bosonic string and the superstring. The left-moving part of the worldsheet theory is the 26-dimensional bosonic string and the right-moving part is the ten-dimensional superstring. To obtain a resulting ten-dimensional theory, one assumes that ten left-moving bosonic fields $X^M(z)$, $M = 1, \ldots, 10$ live in the flat space-time and the remaining sixteen scalars X^I, $I = 1, \ldots, 16$, live on a compact 16-dimensional torus. The spectrum of the heterotic string is obtained by tensoring the left-moving ten plus sixteen internal scalars with the right-moving NS- and R-sector of the 10-dimensional superstring. Again, there is a tachyon in the spectrum, which gets projected out by the GSO projection. Tensoring the ten-dimensional part of the left-moving bosonic string with the right moving spectrum of the NS sector leads to the ten-dimensional dilaton, the antisymmetric two-form and the graviton. From the right-moving R-sector, there are one spin 3/2 gravitino and one spin 1/2 fermion. In addition, there are the massless particles obtained from tensoring the sixteen internal bosons with the right-moving NS and R-sector. This leads to additional vectors and spin 1/2 fermions, where the internal indices play the role of the gauge indices. Anomaly freedom of the theory requires that these vectors and fermions transform in the adjoint representation of $E_8 \times E_8$ or $SO(32)$. The anomalies of the heterotic string are explained in detail in appendix C. From the construction of the heterotic string it is clear that the theory has ten-dimensional $N = 1$ space-time supersymmetry originating in the right-moving part of the worldsheet theory.

So far we have been considering closed strings only. However, it is also possible to construct a consistent string theory from open strings. For open strings, the left- and right-moving modes of the spectrum are not independent. Apart from that, one can construct the spectrum of the open superstring in an equivalent way as described above for the closed superstring. The open superstring also has a critical dimension $d = 10$. Due to the $N = 1$ worldsheet supersymmetry (which is half of the $N = (1, 1)$ worldsheet supersymmetry of the closed string), the open superstring theory has ten-dimensional $N = 1$ supersymmetry. The massless spectrum of the open string is just the right-moving (or equivalently the left-moving) part of the fermionic string. For massive excitations the spectra are identical up to a scaling factor in the mass. The spectrum contains a tachyon from the NS-vacuum, which gets projected out by the GSO projection. This leaves a massless vector from the NS sector and a massless spin 1/2 fermion from the R sector. Open string theories however naturally contain closed strings, because closed strings are open strings in the special situation that their ends meet. Thus to obtain a complete theory, one should also consider the coupling of open strings to closed strings obtained by open strings with meeting ends. Being constructed of open strings, these closed strings are unoriented, which means the left- and right-moving worldsheet fermions are identified. This distinguishes the closed strings of type I theory from the closed strings of type II theory. The closed strings contribute some additional massles particles to the spectrum. One has the dilaton and the graviton from the NS-NS sector, one spin 1/2 fermion and a spin 3/2 gravitino from the NS-R and R-NS sector and one antisymmetric tensor of rank two from the R-R sector. The spectrum is still $N = 1$ supersymmetric. This theory is called type I theory. It can be shown that type I theory is anomaly-free only if the vector and the fermion of the open string sector transform in the adjoint representation of the gauge group SO(32).

These five theories, the IIA, type IIB, the heterotic $E_8 \times E_8$, the heterotic SO(16) and the type I theory are the five consistent superstring theories.

At this point, we make some remarks about the scattering amplitudes of string theory. Scattering amplitudes of strings are given by the correlators of the conformal fields χ corresponding to the primary states $|\text{phys}\rangle$ defined in eqn. (1.12) of the superconformal algebra. The correlation functions are derived using the path integral

$$\langle \chi_1 \dots \chi_n \rangle = \sum_{\text{topologies}} \int_\Sigma \mathcal{D}X\mathcal{D}\psi\, \chi_1 \dots \chi_n e^{-S}, \tag{1.14}$$

where one takes into account the summation over all possible topologies of the string worldsheet, i.e. the contributions of tree and loop diagrams. The action S in the path integral is the two-dimensional worldsheet action. One can show that the last equation of (1.12), $L_0|\text{phys}\rangle = h|\text{phys}\rangle$, leads to the mass-shell condition $p^2 = -m^2$ for each conformal field χ_i in the scattering amplitude. This means in particular that in string theory, in contrast to quantum field theory, it is not possible to derive off-shell correlation functions.

For low energies $\alpha' \to 0$, the mass gap between the massless modes and the massive excitation becomes large and the massless modes of the string give a good approximate descritpion of the theory. From now on, we consider the massless modes of the superstring theories only. One can construct a field theory whose action reproduces at classical level the scattering amplitudes of string theory. Such a field theory is called the low energy effective field theory. For the superstring theories described above, the low energy effective theories are classical supergravity field theories [1, 3]. The low energy effective theories of the IIA/IIB string theories with $N = 2$ supersymmetry are called type IIA and type IIB supergravity. The low energy effective action of the heterotic string is ten-dimensional $N = 1$ supergravity. The low energy effective action of type I theory also has $N = 1$ supersymmetry and is called type I supergravity. In all cases, the expectation value of the dilaton ϕ is related to the coupling constant g of the low energy effective action, $g \sim e^\phi$.

1.2 Calabi-Yau Compactifications

To obtain string theories which can in principle describe realistic scenarios, it is necessary to compactify the ten-dimensional theories to four dimensions. This means the strings propagate in a ten-dimensional space-time which is, in the simplest case, a direct product of flat four-dimensional Minkowski space and some small six-dimensional compact manifold, $M_{10} = \mathbb{R}^{(1,3)} \times M_6$ [1]. Due to the product structure of the space-time, the ten-dimensional metric decomposes as [1, 3]

$$g_{MN} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & g_{ab} \end{pmatrix}, \tag{1.15}$$

where $\eta_{\mu\nu}$, $\mu = 0, \ldots, 3$, is the Minkowski metric on $\mathbb{R}^{(1,3)}$ and g_{ab}, $a = 4, \ldots, 9$, is the "internal" metric on the six-dimensional Calabi-Yau manifold. One can study compactifications either from the point of view of the conformal worldsheet theory or from the point of view of the low energy effective action.

From the low energy effective action point of view, preserving four-dimensional supersymmetry means that the expectation values of the supersymmetry variations of the fermions have to vanish. Vanishing of the variation of the spin 3/2 gravitino requires the existence of a covariantly constant spinor η,

$$D_M \eta = 0. \tag{1.16}$$

With the decomposition $\eta = \eta_4 \eta_6$, this equation splits into a four-dimensional part $D_\mu \eta_4 = 0$, and an internal six-dimensional part, $D_a \eta_6 = 0$. In the four-dimensional Minkowski space, every constant spinor is covariantly constant. In the internal manifold however, the above equation is a real constraint. It implies that

$$[D_a, D_b]\eta_6 = 0 \quad \rightarrow R_{abcd}\Gamma^{cd}\eta_6 = 0. \tag{1.17}$$

This restricts the holonomy group of the internal manifold to $SU(3)$ or a subgroup thereof. As explained in appendix A, six-dimensional manifolds with $SU(3)$ holonomy are Calabi-Yau manifolds. For the heterotic string and the type I string, there are further conditions originating from the variation of the spin 1/2 fermions charged under the gauge groups. The variation of these gauginos, most conveniently expressed in complex coordinates of the Calabi-Yau manifold, requires the existence of stable holomorphic vector bundles, that means the potential F of the gauge field has to satisfy

$$F_{ab} = F_{\bar{a}\bar{b}} = 0, \quad g^{a\bar{b}} F_{a\bar{b}} = 0 \tag{1.18}$$

in the Calabi-Yau manifold. This is not the only condition for the gauge bundle. As explained in appendix C, the low energy effective action contains a field $H = H_0 - w_3$, where $H_0 = dB$ is the field strength of the antisymmetric tensor of the NS-sector and w_3 is the Chern-Simons term. The Chern Simons term fulfills the equation

$$dw_3 \sim (tr(R \wedge R) - Tr(F \wedge F)), \tag{1.19}$$

[1]According to recent developments, phenomenological string models might be in accord with large extra dimensions in so-called brane world scenarios, but we do not consider these models here.

where the two terms are just the second Chern-classes of the tangent and the gauge bundle. The second Chern-class of the tangent bundle as well as the background value of dw_3 are generically non-zero in the compactification manifold. Both equations (1.18) and (1.19) impose non-trivial restrictions on the gauge bundle of the internal manifold. Any class of solutions to these equations leads to a consistent compactification. The most obvious one is to set $dw_3 = 0$, this implies that the connection of the tangent and the gauge bundle have to be identified such that $tr(R \wedge R) = Tr(F \wedge F)$. This is called embedding the spin connection into the gauge connection. Thus if the internal manifold has $SU(3)$ holonomy, the gauge bundle must be an $SU(3)$ bundle. The gauge group that is preserved in the four-dimensional theory is given by the centralizer of the group of the gauge bundle of the Calabi-Yau manifold [2]. This leads to a gauge group $E_6 \times E_8$ or $SO(26)$ in four dimensions. In chapter 3, we consider compactifications of the heterotic string with several different gauge bundles. They all satisfy eqns. (1.18) and (1.19).

From the point of view of the two-dimensional conformal field theory, the compactification implies that the target space of the map $X : \Sigma \to \mathbb{R}^{(1,3)} \times M_6$ is not flat anymore. The decomposition of the coordinate $X^M = (X^\mu, u^a)$ leads to two conformal field theories with maps $X : \Sigma \to \mathbb{R}^{(1,3)}$ and $u : \Sigma \to M_6$. The internal metric g_{ab} of eqn. (1.15) is not flat but depends on the coordinates of the Calabi-Yau manifold. For the conformal field theory (1.8), this implies that the action contains terms of the form

$$\partial^\alpha X^M \partial_\alpha X_M \to \partial^\alpha X^\mu \partial_\alpha X_\mu + g_{ab}(u) \partial^\alpha u^a \partial_\alpha u^b. \tag{1.20}$$

The internal part describes an interacting theory, a non-linear sigma model. For a review of strings on Calabi-Yau manifolds, see for example [4]. The compactification of the $N = (1,1)$ supersymmetric conformal field theory of the type II strings (1.8) on a Calabi-Yau manifold to four dimensions is explained in detail in appendix B.1. To ensure that the target space M_6 is a Calabi-Yau threefold, one needs to introduce an additional global $N = (2,2)$ supersymmetry on the worldsheet. In the case of the heterotic string, the additional worldsheet supersymmetry is $N = (0,2)$ and for the type I string it is $N = 2$. Considering the compactification from the conformal field theory point of view can be very instructive. In appendix B.1 we explain in detail that the analysis of the $N = (2,2)$ sigma model leads to the conclusion that type IIA theory compactified on a Calabi-Yau threefold Y is identical to type IIB theory compactified on the so-called mirror manifold Y^*. This is called mirror symmetry and was first considered in [5].

1.3 Dualities

String theory as described in the last two sections is a purely perturbative theory. The construction in terms of a superconformal field theory is sufficient for the derivation of scattering amplitudes, but it does not allow the construction of non-perturbative objects which are in general non-polynomial in the coupling constant. A description of non-perturbative aspects became possible only after the discovery of dualities relating the strong coupling limit of the five string theories to weakly coupled theories that are well under control. These dualities are called non-perturbative dualities. In addition, the strongly coupled theories imply the existence of an underlying eleven-dimensional theory,

[2] This is also explained in appendix A.4.2.

called M-theory. There are also dualities relating superstring theories in the perturbative regime. The mirror symmetry between the type IIA and the type IIB string described at the end of the last section is an example of a perturbative duality. It can be established at a purely perturbative level as a symmetry of the conformal field theories. In the following we summarize the dualities relating the five superstring theories and the low energy limit of the M-theory. More details about dualities are given for example in [3, 6].

A very important tool for dualities is the existence of higher dimensional objects, called D-branes, in addition to the fundamental strings. D-branes can be constructed in terms of open string theory. Instead of moving freely, the open strings obey Dirichlet boundary conditions, which means that their ends are fixed at points in the ten-dimensional spacetime. It is possible to have open strings with mixed Dirichlet and free (Neumann) boundary conditions. This means that the open strings are fixed on subplanes in the ten-dimensional space-time. Open strings with Neumann boundary conditions in (p+1) directions and Dirichlet boundary conditions in (9-p) directions have ends which are fixed on (p+1)-dimensional planes. These planes are called Dp-branes. Similar as strings, the D-branes have some tension. The particle spectrum of a D-brane is identified with the spectrum of the open strings that begin and end on the D-brane. Dp-branes are considered as dynamical objects in their own right. For a more detailed explanation of D-branes, see [7].

In addition to the mirror symmetry described at the end of the last section, there is a perturbative duality relating type IIA and type IIB theories called T-duality. Consider type IIA theory compactified on a circle of radius R_{IIA} in the direction X^9. It was discovered that this theory is equivalent to the type IIB string compactified on a circle with the inverse radius $R_{IIB} = 1/R_{IIA}$ with Dirichlet boundary conditions in X^9. This is called T-duality. The D9-brane of the type IIB theory obtained in this way has a tension that is proportional to the inverse of the type IIB string coupling constant. It is possible to T-dualize more than one direction. If one T-dualizes (d-p), with (d-p) odd, directions in type IIA theory, one obtains the dual type IIB theory including Dp-branes (with p odd). T-dualizing (d-p) directions in type IIA theory with (d-p) even leads back to type IIA theory including Dp-branes, with p even. The antisymmetric tensors of rank (p+1) of the R-R sector of the type II theories couple magnetically to the Dp-branes. It can be shown that T-duality can be identified with mirror symmetry in the case of a torus compactification of type II theories. Reviews of T-duality are given in [7, 8].

Another perturbative T-duality was found between the $E_8 \times E_8$ and the $SO(32)$ heterotic string. One can compactify both theories on a circle with so called "Wilson lines", which are constant background values for the gauge fields on the circle. Choosing the background fields such that both the $E_8 \times E_8$ and the $SO(32)$ gauge groups are broken to $SO(16) \times SO(16)$, one finds that the spectra of the theories coincide if the radius of the $E_8 \times E_8 \to SO(16) \times SO(16)$ compactification is the inverse of the radius the $SO(32) \to SO(16) \times SO(16)$ compactification. T-duality in the context of toroidal compactifications of the heterotic string was first considered in [9, 10].

A non-perturbative duality is the relation between the $SO(32)$ heterotic string and the type I string [11]. Type I string theory contains an antisymmetric tensor of rank two of the R-R sector which couples to a D1-brane. One can show that the massless spectrum originating from open strings that begin and end on the D1 brane plus the massless spectrum of strings that have one end on the D1 brane is identical to that of the weakly coupled $SO(32)$ heterotic string. The tension of the D1 brane is proportional to the inverse of the string coupling constant. Increasing the coupling constant means that the tension of the D1-brane decreases from infinity to finite values and finally becomes smaller than the tension of the fundamental string. Finally, the D1-brane plays the role of the fundamental object of the theory. Because the spectrum of this theory is identical to that of the weakly coupled $SO(32)$ heterotic string, one identifies the two theories. Thus, the strong coupling limit of type I theory is related to the weak coupling limit of the $SO(32)$ heterotic string. This non-perturbative duality is called S-duality.

So far, we have considered dualities relating the perturbative superstring theories respectively their strong coupling limits to other superstring theories. Duality however implies also the existence of a sixth theory which is connected to the string theories on the level of the low energy effective actions [12, 13, 14]. Consider type IIA string theory including a D0-brane. One can show that the structure of the spectrum of a D0 brane corresponds to the spectrum one typically obtains by compactifications on one periodic dimension, where the radius of the compact dimension corresponds to the inverse of the D0-brane tension. The D0-brane tension is inverse proportional to the IIA coupling constant. Thus for a large coupling constant, the radius of the periodic dimension becomes large and the theory looks like an additional eleventh dimension appears. The eleven-dimensional theory cleary cannot be a superstring theory, because these are consistent in ten dimension only. On the level of low energy effective field theories, one can indeed find a theory which, compactified on a circle, reproduces type IIA supergravity in ten dimensions. This theory is eleven-dimensional supergravity with 32 supercharges. Being a classical field theory, it is assumed that eleven-dimensional supergravity is the low energy limit of an underlying eleven-dimensional theory. This underlying theory is called M-theory. This picture implies that the superstring theories are not only related by dualtities, but in addition the different limits of a single underlying fundamental M-theory.

The $E_8 \times E_8$ heterotic string is related to M-theory in a similar way [15]. The eleventh dimension is not a circle as for the IIA string, but an interval. The length of the interval is proportional to the heterotic string coupling. Again, one can verify this on the level of low energy effective field theories. Eleven-dimensional supergravity compactified on an interval reproduces the heterotic low energy effective action in the limit where the interval approaches zero length. Anomaly cancellation requires an $E_8 \times E_8$ gauge group in the compactification from eleven to ten dimensions. Also, the compactification on an interval breaks half of the 32 supersymmetries, leaving ten-dimensional N=1 supersymmetry. This model is the subject of chapter 3, where we give a more detailed explanation.

All in all, the picture is the following. It is assumed that there is an underlying eleven-dimensional theory, called M-theory, whose fundamental formulation has not been found yet. The six known limits of M-theory are the five consistent superstring theories and eleven-dimensional supergravity.

As the last point of the introdution, we explain F-theory [16]. F-theory describes a non-perturbative compactification of type IIB string theory, for a review see for example [6]. In IIB string theory one can form the complex scalar

$$\lambda = a + ie^{\phi/2} \tag{1.21}$$

containing the R-R scalar a and the dilaton ϕ. In perturbative compactifications one takes λ to be constant. F-theory are compactifications of type IIB theory where λ varies over the compact manifold. Consider some complex manifold B and the manifold M obtained by erecting a copy of a torus at every point in B. Such a manifold is called an elliptically fibered manifold, see appendix A.2.1 for an introduction to elliptically fibred manifolds. The complex structure of the elliptic fibre is denoted by $\tau(s)$, where s are the coordinates on the base B. F-theory on M is defined as IIB theory compactified on B with the identification

$$\lambda(s) = \tau(s). \tag{1.22}$$

As the coupling constant of the type IIB theory varies over the compactification manifold B, F-theory cannot be described by some perturbation theory but is by definition a non-perturbative theory. In contrast to M-theory, F-theory is usually not considered as an underlying fundamental theory but rather as a tool to formulate non-perturbative aspects of string theory that originate in M-theory. Because there is no fundamental formulation of M-theory, F-theory has proven very uselful in the past to gain information abut the non-perturbative behaviour of string theory.

For the discovery of the above dualities it was essential to consider string theory compactified to other dimensions than four. Although these models cannot describe realistic scenarios, they are important for gaining insight into the structure of string repectively M-theory. Following this spirit, we consider type IIA and heterotic compactifications on Calabi-Yau manifolds to different dimensions than four in the context of dualities in this thesis. In appendix A, Calabi-Yau manifolds of complex dimensions one to four are explained.

1.4 Topics and Organization of the Thesis

It is essential for the progress in finding a fundamental formulation of M-theory to explore the known limits of M-theory, the five superstring theories and eleven-dimensional supergravity, in the context of the dualities relating the different limits. The thesis is divided in two parts, both of which deal with different aspects of these dualities.

As mentioned above, of greatest interest are compactifications of string theory to four dimensions. This implies that the superstring theories have to be compactified on complex three-dimensional Calabi-Yau manifolds. To construct a formulation that includes non-perturbative aspects, one can consider the dual formulation in terms of F-theory compactified on complex four-dimensional Calabi-Yau manifolds, see for example [17]. Thus Calabi-Yau fourfolds are important in that context. Calabi-Yau fourfolds have a more complicated structure than Calabi-Yau threefolds. They do not admit [18, 20] for example the special geometry explained in appendix A.4, which simplifies threefold compactifications significantly. In order to gain more information about F-theory on Calabi-Yau fourfolds, it can be instructive to consider related theories compactified on fourfolds first that are better under control than F-theory. F-theory compactified on a fourfold times a circle for example is dual to M-theory compactified on the same Calabi-Yau fourfold [16]. These theories live in three dimensions. Taking the decompactification limit in which the radius of the F-theory circle becomes large, eleven-dimensional supergravity compactified on a fourfold can be uselful to gain some information about F-theory in four dimensions. Compactifying further on another circle, one obtains F-theory on fourfold times a torus.

This theory is dual to M-theory compactified on a Calabi-Yau fourfold times a circle, which in turn is dual to type IIA theory compactified on the fourfold. Thus, type IIA theory in two dimensions is a candidate for gaining some information about F-theory in four dimensions. Type IIA theory compactified on Calabi-Yau fourfolds is also the topic of the first part of the thesis, section 2. Type IIA theory compactified on Calabi-Yau fourfolds has been also considered in [19, 20, 21, 22, 23, 24, 25, 26].

A generic four-fold compactification of the type IIA string requires switching on non-vanishing background values for certain R-R field strengths in the fourfold [19]. The R-R fields couple to D-branes as mentioned in the last section. Switching on constant background values for R-R field strengths corresponds to wrapping the D-branes on certain cycles in the Calabi-Yau fourfold. Thus consistent type IIA compactifications on Calabi-Yau fourfolds generically include D-branes wrapped in the compactification manifold. This is in contrast to type IIA compactifications on Calabi-Yau threefolds, where it is possible, but not required for consistency to include branes or equivalently background fluxes. The D-branes wrapped on the cycles in the Calabi-Yau fourfold generate a potential in the low energy effective action of the theory. This was considered in [24, 25]. The goal of section 2 is to derive this potential including "stringy corrections", which are non-perturbative in the world-sheet coupling constant α'. These corrections are called worldsheet instantons, and they depend on the geometry of the fourfold.

The central task of the instanton calculation is to consider the special class of fourfolds which are threefold fibrations. For an explanation of that, see appendix A.5. Taking the large base limit one can show that the stringy corrections of the potential can be extracted from the geometry of the threefold fibre alone [18, 20]. Thus the corrections are fixed by the special geometry of the threefold fibre. This simplifies the calculations significantly and we are able to give the full potential including all stringy corrections for the class of fourfolds described above.

A possible application of the calculation can be found in string dualities. First, one can consider the duality of type IIA theory on a Calabi-Yau fourfold and the heterotic string on a Calabi-Yau threefold times a torus. If the duality between these theories is valid, one would expect that the potential including the worldsheet instanton corrections of the IIA theory should somehow also turn up in the dual heterotic theory. A similar analysis, without taking into account worldsheet instanton corrections on the IIA side, was done in four dimensions in [27]. Second, it turns out that, for a specific choice of background fluxes, the potential coincides with a potential proposed in [28] for compactifications of type IIB string theory on Calabi-Yau threefolds.

The first part of the thesis, section 2, is organized as follows. In 2.1 we give an introduction to the compactification of type IIA string theory on Calabi-Yau fourfolds. We explain the form of the potential generated by the background fluxes. The worldsheet instanton corrections to the potential are derived in 2.2. The corrections are derived by using mirror symmetry, which is explained in detail in appendix B. In section 2.3 we consider the superpotential generated by four-form flux only and rederive the result of the last section using the methods of topological field theory. We show that the results are in agreement. Finally, we make some remarks about the potential in the context of heterotic-type IIA duality in 2.4. The results of section 2 are published in [29].

The second part of the thesis deals with the strong coupling limit of the $E_8 \times E_8$

heterotic string, which is given by M-theory compactified on an interval. To be precise, the interval is the one-dimensional orbifold $S^1/\mathbb{Z}_2$, which is obtained by identifying the coordinate $x = -x$ of the circle, where $x \in [-\pi, \pi]$. This model was first considered in [15]. The identification of the strongly coupled heterotic string with M-theory compactified on $S^1/\mathbb{Z}_2$ can be shown expicitly on the level of the low energy effective actions. Anomaly cancellation requires a gauge group $E_8 \times E_8$ on the M-theory side.

It is interesting to consider further compactifications to lower dimensions. The strongly coupled heterotic string compactified on some manifold X should be given by M-theory compactified on $S^1/\mathbb{Z}_2 \times$X. This does not lead to any particular difficulties unless X is an orbifold. Orbifolds are a generalization of the class of compactification spaces from manifolds, which are smooth by definition, to spaces which may contain quotient singularities. One example is the $S^1/\mathbb{Z}_2$ orbifold with the quotient group $\mathbb{Z}_2$, which is singular at the fixed points $x = 0, \pi$.

Recently, two groups [89, 90] have analyzed the case that X is the orbifold limit of a K3 manifold, that is $T^4/\mathbb{Z}_N$. The six-dimensional case is of particular interest, because anomaly cancellation is particularly restrictive in six dimensions, as is explained in appendix C. This is of great importance because a formulation of the underlying theory, M-theory on $S^1/\mathbb{Z}_2 \times T^4/\mathbb{Z}_N$, is not known and one has to extract the information about the theory purely from consistency conditions such as anomaly cancellation. Anomaly cancellation has lead the authors of [89, 90] to an interesting set of rules for the spectrum and the gauge group of the theory. These rules are highly non-intuitive. Some of the rules have been justified recently in [91] by considering the compactification of type IIA theory on $S^1/\mathbb{Z}_2$, called type I' theory. The six-dimensional rules of [90, 89] can be generalized to describe orbifold compactifications to four dimensions [92].

We take another approach towards understanding the rules in six dimensions by considering the dual F-theory formulation of M-Theory on $S^1/\mathbb{Z}_2 \times T^4/\mathbb{Z}_N$. To obtain a six-dimensional theory, F-theory has to be compactified on a Calabi-Yau threefold. The hope is that the geometry of the correct threefold contains some information about the non-perturbative sector of the dual M-theory. To switch off corrections which may complicate the problem, we take the limit of a large K3 orbifold on the M-theory side. For the F-theory threefold, this means that the manifold undergoes the stable degeneration explained in appendix A.3.3. In this limit, we construct the F-theory threefold. We show the threefold gives a good description of the dual M-theory locally around each fixed point. It turns out that the F-theory formulation indeed gives some additional information about the gauge group of the theory. Thus we are able to give a new understanding to some of the rules developped in [89, 90] directly from F-theory.

The second part of the thesis, section 3, is organized as follows. In 3.1, we give an introduction to M-theory compactified on $S^1/\mathbb{Z}_2$. We explain the conditions for anomaly cancellation in some detail. Compactifying the theory consistently further to six dimensions on $T^4/\mathbb{Z}_2$ gives rise to some interesting rules arising from six-dimensional anomaly-cancellation, this is reviewed in 3.2. In 3.3, we concentrate on the compactification with a perturbative gauge group $SO(16) \times [E_7 \times SU(2)]$. For this gauge group, we construct the dual F-theory model in 3.4 explicitly. We show that some of the rules can be understood directly from F-theory. Finally, in 3.5 we extend the results to six-dimensional models with other gauge groups. The results of section 3 will be published in [93] shortly after the submission of the thesis.

2 Type IIA String Theory with Background Fluxes in d=2

We consider compactifications of type IIA string theory on Calabi-Yau fourfolds. Consistency of a generic compactification requires switching on a four-form flux in the Calabi-Yau manifold [19]. Switching on background fluxes generates a superpotential in the low energy effective action. The goal of this chapter is to derive the superpotential generated by switching on all possible background fluxes [24, 25] including all "stringy corrections", i.e. including worldsheet instanton corrections. In 2.1 we review the compactification of the type IIA string on Calabi-Yau fourfolds. The worldsheet instanton corrections to the superpotential of the effective theory are derived in 2.2 using mirror symmetry [18, 20]. We consider the superpotential generated by four-form flux in 2.3 and rederive the result of 2.2 using the topological field theory obtained by twisting the IIA worldsheet theory. We conclude in 2.4 with a discussion of the impact of the results of 2.2 on the duality between the IIA string and the heterotic string theory in two dimensions.

2.1 Type IIA String Theory in d=2

We consider the low energy effective action of type IIA string theory compactified on a Calabi-Yau fourfold. Ten-dimensional type IIA string theory has thirty-two supersymmetries, sixteen of each chirality, and the bosonic spectrum contains the ten-dimensional metric, the dilaton, the anti-symmetric NS-NS two-form and a vector and threeform from the R-R sector of the worldsheet theory. The low-energy effective action is ten-dimensional $N = 2$ type IIA supergravity and is to leading order [3]

$$\begin{aligned} S &= \int d^{10}x\sqrt{-g}\left[e^{-2\phi}\left(\frac{1}{2}R + 2\partial_M\phi\partial^M\phi - \frac{1}{4}|H|^2\right) - \frac{1}{4}\left(|F_2|^2 + |\tilde{F}_4|^2\right)\right] \\ & \quad -\frac{1}{4}\int B\wedge F_4\wedge F_4 \end{aligned} \tag{2.1}$$

where g is the determinant of the ten-dimensional metric, ϕ is the dilaton, $H = dB$ is the field strength of the NS-NS twoform, $F_2 = dC_1$ is the field strength of the RR vector, $F_4 = dC_3$ is the field strength of the RR 3-form and $\tilde{F}_4 = F_4 - C_1 \wedge H_3$.

Compactifying the ten-dimensional theory on a Calabi-Yau fourfold preserves 1/8 of the supersymmetry leading to a two-dimensional theory with $N = (2, 2)$ spacetime supersymmetry. The spectrum depends on the hodge numbers of the compactification manifold. Expanding the ten-dimensional metric in the harmonic forms of the fourfold leads to $h^{(1,1)}$ real scalars g^A, $A = 1, \dots, h^{(1,1)}$ and $h^{(1,3)}$ complex scalars Z^α, $\alpha = 1, \dots, h^{(1,3)}$. Expanding the the NS-NS twoform results in $h^{(1,1)}$ real scalars b^A. For simplicity we set $h^{(1,2)} = 0$, that means the $(1, 2)$-forms from the expansion of the R-R threeform do not contribute to the spectrum. Vectors have no physical degrees of freedom in two dimensions, thus the R-R oneform does not contribute to the spectrum either. All in all the spectrum contains the two-dimensional Dilaton, $h^{(1,1)}$ complex scalars $t^A = (b^A + ig^A)$, these are the the complex Kählermoduli, and $h^{(1,3)}$ complex scalars Z^α, which are the the complex structure moduli of the fourfold. The two-dimensional action is [29]

$$S = \int d^2x \sqrt{-g^{(2)}}\, e^{-2\phi^{(2)}} \left(\frac{1}{2} R^{(2)} + 2\partial_\mu \phi^{(2)} \partial^\mu \phi^{(2)} - \frac{1}{2} G_{A\bar{B}} \partial_\mu t^A \partial^\mu \bar{t}^{\bar{B}} - \frac{1}{2} G_{\bar{\alpha}\beta} \partial_\mu \bar{Z}^{\bar{\alpha}} \partial^\mu Z^\beta \right) \quad (2.2)$$

where ϕ^2 and g_2 are the two-dimensional dilaton and metric. $G_{A\bar{B}}(t, \bar{t})$ and $G_{\bar{\alpha}\beta}(Z, \bar{Z})$ are the metrics on the moduli space of the Kähler- and complex structure moduli

$$\begin{aligned} G_{A\bar{B}}(t, \bar{t}) &= \partial_A \bar{\partial}_{\bar{B}} K, \\ G_{\alpha\bar{\beta}}(Z, \bar{Z}) &= \partial_\alpha \bar{\partial}_{\bar{\beta}} K, \end{aligned} \quad (2.3)$$

with the Kähler potential

$$K = -\ln\left(\int_{Y_4} \Omega \wedge \bar{\Omega} \right) - \ln \mathcal{V}. \quad (2.4)$$

$\mathcal{V}$ is the volume of the fourfold and can be expressed in terms of the Kähler form $J = g^A e_A$, where e_A denotes a basis of the (1,1)-forms, as

$$\mathcal{V} = \frac{1}{4!} \int_{Y_4} J \wedge J \wedge J \wedge J = \frac{1}{4!} d_{ABCD} (t^A - \bar{t}^A)(t^B - \bar{t}^B)(t^C - \bar{t}^C)(t^D - \bar{t}^D), \quad (2.5)$$

where d_{ABCD} are the classical intersection numbers $d_{ABCD} = \int_{Y_4} e_A \wedge e_B \wedge e_C \wedge e_D$.

Of special importance in fourfold compactifications is a ten-dimensional higher order term not considered in (2.1). This term contains the NS-NS two form and the curvature [19] [3]

$$S \sim \int B \wedge X_8(R) \quad (2.6)$$

with $X_8(R) = \frac{1}{4!}\left(\frac{1}{8} tr R^4 - \frac{1}{32} tr (R^2)^2\right)$. Integrating over the eight-dimensional compactification manifold Y_4 leaves a one-point interaction of the two-form B proportional to

$$\int_{Y_4} X_8(R) = -\frac{1}{4!} \chi, \quad (2.7)$$

where χ is the Euler number of the fourfold. Onepoint functions violate Lorenz invariance of the theory and thus to obtain a consistent compactification this interaction has to be to cancelled. Compactifying the last term in (2.1) leads to a onepoint function of the two-form B proportional to

$$\int_{Y_4} F_4 \wedge F_4. \quad (2.8)$$

By giving the four-form field strength F_4 a non-vanishing background value in Y_4 it is possible to cancel the above term. For integer $\frac{\chi}{4!}$ cancellation is also possible by filling the two-dimensional spacetime with N fundamental strings. The cancellation condition is [19]

$$\frac{\chi}{4!} = N + \frac{1}{8\pi^2} \int_{Y_4} F_4 \wedge F_4. \quad (2.9)$$

[3] The eleven-dimensional M-theory version of this term also plays an important role in the discussion of the strongly coupled heterotic string, see chapter 3

For non-integer $\frac{\chi}{4!}$ consistency requires switching on R-R background fourform fluxes. This is in contrast to lower-dimensional compactifications such as $K3$ manifolds or Calabi-Yau threefolds, where background fluxes can be turned on but are not required for consistency.

In addition to the four-form field strength we turn on zero-, two-, six-, and eight form fluxes F_0, F_2, F_6 and F_8 in Y_4. In [24, 25] it was shown that by including the background fluxes a non-trivial potential is switched on in the low energy effective action. If the $(1,2)$-moduli are frozen, the potential can be expressed in terms of two superpotentials W and $\tilde{W}$ [26]

$$V = e^K \left(G^{-1A\bar{B}} D_A \tilde{W} D_{\bar{B}} \bar{\tilde{W}} + G^{-1\alpha\bar{\beta}} D_\alpha W D_{\bar{\beta}} \bar{W} - |W|^2 - |\tilde{W}|^2 \right), \tag{2.10}$$

where W depends on the complex structure moduli Z^α and $\tilde{W}$ on the complex Kähler moduli t^A. The Kähler covariant derivatives are defined as

$$D_A \tilde{W} = \partial_A \tilde{W} + \tilde{W} \partial_A K \ , \qquad D_\alpha W = \partial_\alpha W + W \partial_\alpha K. \tag{2.11}$$

The purpose of this chapter is to derive the superpotentials.

We are going to consider the background fluxes from the D-brane perspective. A Dp-brane is a magnetic source of a R-R p+1-form potential located on the worldvolume of the brane. Thus instead of considering R-R fluxes in the Calabi-Yau fourfold we can consider the corresponding D-brane wrapped on some submanifold of the fourfold. These submanifolds are the supersymmetric cycles in Y_4 such that the brane configurations are BPS states preserving half of the supersymmetry on the worldvolume. These brane configurations generate the potential (2.10). There are two different kinds of supersymmetric cycles. A special Lagrangian cycle S is a cycle in the homology class $H_4(Y_4, \mathbb{Z})$ (more generally in $H_d(Y_d, \mathbb{Z})$ for a complex d-dimensional compactification manifold) and its volume is given by

$$Vol(S) = \int_S \Omega = \int_{Y_4} \Omega \wedge \bar{S}. \tag{2.12}$$

$\bar{S}$ is an element of the cohomology class $H^4(Y_4, \mathbb{Z})$ which is dual to the cycle S in $H_4(Y_4, \mathbb{Z})$. A holomorphic cycle $C^{(p)}$ is a cycle in the homology class $H_p(Y_4, \mathbb{Z})$, $p = 0, 2, 4, 6, 8$ (in $H_p(Y_d, \mathbb{Z})$, $p = 0, \ldots, 2d$ for a complex d-dimensional compactification manifold), and its volume is

$$Vol(C^{(p)}) = \int_{C^{(p)}} t \wedge \ldots \wedge t = \int_{Y_4} t \wedge \ldots \wedge t \wedge \bar{C}^{(8-p)}, \tag{2.13}$$

where the integrand contains $p/2$ komplex Kähler forms $t = t^A e_A$ and $\bar{C}^{(8-p)}$ is an element of $H^{8-p}(Y_4, \mathbb{Z})$ and dual to the cycle $C^{(p)}$. Note that the volume of a special Lagrangian cycle depends on the complex structure moduli while the volume of a holomorphic cycle depends on the Kähler moduli only.

A very useful observation in string theory is that brane configurations correspond to solitons of the low energy effective action. A Dp- brane wrapped on some supersymmetric cycle of real dimension p has one additional direction in the flat spacetime. As our theory is 1+1-dimensional this configuration has codimension one in spacetime. In the low energy effective action such a state is a domain wall. It is known from field theory that the superpotential generated by a BPS domain wall is equal to the mass of the soliton, see for example [64]. The mass of the soliton translated to the D-brane language is the volume of the supersymmetric cycle the brane is wrapped on [24, 25]. This relation is only true for BPS states, that is why we consider BPS branes and wrapped on supersymmetric cycles only. The superpotentials are

$$W = Vol(S), \quad \tilde{W} = \sum_p Vol(C^{(p)}). \tag{2.14}$$

Note that this formula is valid in any dimension for D-brane configurations with codimension one in spacetime.

Wrapping a D4-brane on a four-dimensional special Lagrangian cycle generates the superpotential

$$W = \frac{1}{2\pi} \int_{Y_4} \Omega \wedge F_4, \tag{2.15}$$

where F_4 denotes the RR 4-form flux. There is a subtlety here that should be mentioned. What was really shown in [24] is that the change of the superpotential when crossing the domain wall in the two-dimensional space-time is equal to the volume of the four-cycle $C^{(4)} \in H_4(Y_4, \mathbb{Z})$ wrapped by the D4-brane: $\Delta W = \int_{C^{(4)}} \Omega = \frac{1}{2\pi} \int_{Y_4} \Omega \wedge \Delta F_4$. It is the change of the 4-form flux $\frac{\Delta F_4}{2\pi} \in H^4(Y_4, \mathbb{Z})$ that is Poincaré dual to the four-cycle $C^{(4)}$ rather than the flux itself. Note that in general not $\frac{F_4}{2\pi}$ but $\frac{F_4}{2\pi} - \frac{p_1}{4}$ does take values in $H^4(Y_4, \mathbb{Z})$, where p_1 is the first Pontryagin class [31]. There are no such subtleties for the other fluxes however, but one might keep in mind that we consider only a subspace of possible fourform fluxes if we assume that $\frac{F_4}{2\pi}$ is an element of the integral cohomology.

The second superpotential is generated by wrapping Dp-branes, $p = 0, 2, 4, 6, 8$, on holomorphic cycles $C^{(p)} \in H_p(Y_4, \mathbb{Z})$ with the same real dimension p [24, 25]

$$\tilde{W}_{\rm cl} = \frac{1}{2\pi} \int_{Y_4} (t \wedge t \wedge t \wedge t\ F_0 + t \wedge t \wedge t \wedge F_2 + t \wedge t \wedge F_4 + t \wedge F_6 + F_8)\ , \tag{2.16}$$

where $\frac{F_p}{2\pi} \in H^p(Y_4, \mathbb{Z})$ is the RR p-form flux which is Poincaré dual to the $(8-p)$-cycle $C^{(8-p)} \in H_{8-p}(Y_4, \mathbb{Z})$ (for $p = 4$ see the above remark).

The effective theory has a $(2, 2)$ supersymmetric vacuum in a Minkowskian background if

$$D_A \tilde{W}|_{\rm min} = D_\alpha W|_{\rm min} = \tilde{W}|_{\rm min} = W|_{\rm min} = 0 \tag{2.17}$$

holds. Depending on the background fluxes this puts a severe constraint on the moduli space and for some fluxes no supersymmetric vacuum exists at all. For example, for the superpotential eq. (2.15) the supersymmetry condition (2.17) implies [30]

$$F_4^{(0,4)} = 0 = F_4^{(1,3)}\ , \tag{2.18}$$

where the last equation arises from the fact that $\partial_\alpha \Omega$ takes values in $H^{4,0} \otimes H^{3,1}$. Since the Hodge decomposition of H^4 depends on the complex structure, eq. (2.18) is a strong constraint on the moduli space of the complex structure. It leaves only the subspace of complex structure deformations which respect (2.18) as the physical moduli space.

The superpotential $\tilde{W}$ depends on the Kählermoduli and receives quantum corrections on the worldsheet while W is exact. Mirror symmetry demands that once all quantum corrections are properly taken into account the two superpotentials should obey [69, 23, 25]

$$\tilde{W}(Y_4) = \tilde{W}_{\rm cl}(Y_4) + \text{quantum corrections} = W(Y_4^*) \ , \tag{2.19}$$

where Y_4^* is the mirror fourfold of Y_4. Mirror symmetry and worldsheet corrections are explained in the appendix B. The quantum corrections can be derived by computing W on the mirror manifold and performing the mirror map. This is done in the next section.

2.2 Worldsheet Corrections of the Superpotential

Our goal is to compute $\tilde{W}(Y_4)$ including the worldsheet corrections. To do this we have to find the mirror manifold Y_4^*, evaluate the potential

$$W(Y_4^*) = \frac{1}{2\pi}\int_{Y_4^*} \Omega \wedge F_4^*, \tag{2.20}$$

and perform the mirror map. We denote the fourform flux by F_4^* in order to distinguish it from the fourform flux on Y_4. Before we perform the calculation we make a few simplifications.

The fourfold Y_4 is chosen to be a Clalabi-Yau threefold fibered over a base $\mathbb{P}^1$ in the large volume limit $\text{Vol}(\mathbb{P}^1) \to \infty$. This limit simplifies the quantum corrections of the superpotential significantly. In addition we assume that the background fluxes F_p and F_4^* are elements of the primary subspaces of the cohomologies $H^{(p/2,p/2)}(Y_4)$ and $H^{(4-k,k)}(Y_4)$ introduced in A.5, that is necessary for using mirror symmetry. The vertical primary cohomology is the subspace of the vertical cohomology $\oplus_{k=0}^{d} H^{(k,k)}(Y_d)$ obtained by taking the wedge products of k $(1,1)$-forms. The horizontal primary cohomology is the subspace of the horizontal cohomology $\oplus_{k=0}^{d} H^{(d-k,k)}(Y_d)$ generated by successive derivatives $D^k\Omega$ of the holomorphic $(d,0)$-form Ω [42, 18].

The elements of the horizontal primary subspace are mapped via mirror symmetry to the elements the vertical primary subspace of the mirror manifold and vice versa [18, 20, 98]. The (k,k)-forms in the vertical primary subspace are mapped to the $(d-k,k)$-forms in the horizontal primary subspace.

Let us restrict to the class of fourfolds which are Y_3-fibred over a base $\mathbb{P}^1$ [20]. The vertical primary subspace can be obtained by taking the wedge product of the elements of the vertical primary subspace of Y_3 with the zero- or $(1,1)$-forms on the $\mathbb{P}^1$ base. This leads to the following Hodge numbers of the vertical primary subspace of Y_4

$$\begin{aligned} h^{(0,0)}(Y_4) &= 1 = h^{(4,4)}(Y_4), \\ h^{(1,1)}(Y_4) &= h^{(1,1)}(Y_3) + 1 = h^{(3,3)}(Y_4) \ , \\ h^{(2,2)}_{vp}(Y_4) &= 2h^{(1,1)}(Y_3) \ , \end{aligned} \tag{2.21}$$

where the subscript vp refers to the vertical primary subspace. Except for $h^{(2,2)}_{vp}$ the Hodge numbers of the vertical primary subspace coincide with those of the full vertical cohomology. Thus we omit the subscript vp except for the $(2,2)$-forms. In the following we also use the formulation of the vertical primary subspace in terms of the dual homology. In the homology the vertical primary subspace is obtained by joining the even-dimensional cycles $M^{(n)}$, $n = 0, 2, 4, 6$, of Y_3 with the zero- or two-cycles of the base $\mathbb{P}^1$.

Mirror symmetry implies the following relations for the Hodge numbers of the horizontal primary subspace of Y_4^*

$$\begin{aligned} h^{(0,4)}(Y_4^*) &= h^{(4,4)}(Y_4), \quad h^{(4,0)}(Y_4^*) = h^{(0,0)}(Y_4), \\ h^{(3,1)}(Y_4^*) &= h^{(1,1)}(Y_4), \quad h_{hp}^{(2,2)}(Y_4^*) = h_{vp}^{(2,2)}(Y_4), \end{aligned} \tag{2.22}$$

where the subscript hp refers to the horizontal primary subspace. The members of the vertical respectively the horizontal primary cohomology are observables in the A-respectively B-model. The A- and the B-model are two topological sigma-models with the Calabi-Yau manifold Y_4 as a target space, which are obtained by twisting the worldsheet sigma-model in two different ways [48]. The observables and correlation functions of the A-model on Y_4 are related via mirror symmetry to those of the B-model on Y_4^* and vice versa.

The R-R fluxes $F^{(p)}$ are mapped to elements of the horizontal primary subspace on the mirror manifold Y_4^*. The dimension $h_{hp}^4(Y_4^*)$ of the horizontal primary subspace can be expressed in terms of the number of $(1,1)$-forms of the fibre of Y_4:

$$h_{hp}^4(Y_4^*) = 2h^{(4,0)}(Y_4^*) + 2h^{(3,1)}(Y_4^*) + h_{hp}^{(2,2)}(Y_4^*) = 4(h^{(1,1)}(Y_3) + 1). \tag{2.23}$$

Thus it is possible to introduce a basis for the horizontal primary homology of Y_4^* by $(A^I, \tilde{A}_I, B^I, \tilde{B}_I)$, where $I = 0, \ldots, h^{(1,1)}(Y_3)$. The $(A^I, \tilde{A}_I)$ are those homology cycles that correspond via mirror symmetry to the elements of the vertical primary subspace of Y_4 which are obtained by joining the even-dimensional cycles of the threefold-fibre with the zero-cycle of the base. Analogously, $(B^I, \tilde{B}_I)$ are the cycles which are related to the elements of the vertical primary subspace which are obtained by joining the even-dimensional cycles of the threefold with the two-cycle of the base [20]. As noted in [20] in the large base limit at leading order the cycles $(A^I, \tilde{A}_I)$ all have vanishing intersections with each other and the same is true for the $(B^I, \tilde{B}_I)$ cycles. The only non-vanishing intersections are between A-cycles and B-cycles and the intersection form is given by that of the Y_3-fibre. For a certain choice of cycles and in terms of the Poincaré dual forms $(a^I, \tilde{a}_I, b^I, \tilde{b}_I)$ this amounts to

$$\int_{\tilde{Y}_4} a^I \wedge \tilde{b}_J \;=\; \delta^I_J \;=\; -\int_{\tilde{Y}_4} \tilde{a}_J \wedge b^I \tag{2.24}$$

with all other intersection pairings vanishing.

In order to evaluate the superpotential $W(Y_4^*)$ using eq. (2.15) we expand the 4-form flux on Y_4^* in this basis

$$\frac{F_4^*}{2\pi} = \mu_I a^I - \tilde{\mu}^I \tilde{a}_I + \nu_I b^I - \tilde{\nu}^I \tilde{b}_I, \qquad (\mu_I, \tilde{\mu}^I, \nu_I, \tilde{\nu}^I) \in \mathbb{Z}\,. \tag{2.25}$$

The fluxes are not all independent but have to obey the consistency condition (2.9). This implies

$$\frac{1}{4!}\,\chi(Y_4^*) = N + (\nu_I \tilde{\mu}^I - \mu_I \tilde{\nu}^I)\,. \tag{2.26}$$

Inserting (2.25) into (2.15) we arrive at

$$W = \mu_I \int_{A^I} \Omega \; - \; \tilde{\mu}^I \int_{\tilde{A}_I} \Omega \; + \; \nu_I \int_{B^I} \Omega \; - \; \tilde{\nu}^I \int_{\tilde{B}_I} \Omega \; . \qquad (2.27)$$

In order to evaluate the period integrals $\int \Omega$ on Y_4^* one has to note that they are mapped via mirror symmetry to the periods on Y_4, which give to leading order in the large volume limit and in special coordinates the classical volumes of the corresponding cycles. These leading terms in general get quantum corrections. In the large base limit the corrections from the base are suppressed and the quantum corrections only arise from the threefold fibre. Thus the periods on the fourfold are given by those of the threefold, multiplied by the classical volume t^V of the base for those cycles which contain the base [20]. More specifically one has [4]

$$\begin{aligned} \int_{A_I} \Omega &= (1, t^i) + \mathcal{O}(e^{it^V}) \; , \qquad \int_{B_I} \Omega = t^V (1, t^i) + \mathcal{O}((t^V)^0) + \mathcal{O}(e^{it^V}) \; , \qquad (2.28) \\ \int_{\tilde{A}_I} \Omega &= (\mathcal{F}_i, \mathcal{F}_0) + \mathcal{O}(e^{it^V}) \; , \qquad \int_{\tilde{B}_I} \Omega = t^V (\mathcal{F}_i, \mathcal{F}_0) + \mathcal{O}((t^V)^0) + \mathcal{O}(e^{it^V}) \; , \end{aligned}$$

where $i = 1, \ldots, h^{(1,1)}(Y_3)$. The vector

$$\Pi = (1, t^i, \mathcal{F}_i, \mathcal{F}_0) \qquad (2.29)$$

corresponds to the periods of the threefold with $\mathcal{F}_i = \partial_{t^i} \mathcal{F}$ and $\mathcal{F}_0 = 2\mathcal{F} - t^i \mathcal{F}_i$. The $N = 2$ prepotential is given by [60]

$$\mathcal{F} \;=\; \mathcal{F}_{\text{pol}} - \frac{1}{(2\pi)^3} \sum_{\{d_i\}} n_{\{d_i\}} Li_3(e^{2\pi i \sum t^i d_i}) \; , \qquad (2.30)$$

where

$$\mathcal{F}_{\text{pol}} = \frac{1}{6} d_{ijk} t^i t^j t^k + b_i t^i + \frac{1}{2} c \; , \qquad Li_3(x) \equiv \sum_{j=1}^{\infty} \frac{x^j}{j^3} \; . \qquad (2.31)$$

The polynomial part $\mathcal{F}_{\text{pol}}$ contains the classical intersection numbers of the threefold d_{ijk} and the coefficients b_i, c are given in [60]. The non-polynomial part contains the contribution of the worldsheet instantons. Worldsheet instantons arise if the compactification manifold contains isolated holomorphic curves $\mathcal{C}$ around which fundamental strings are wrapped. This effect is called worldsheet instanton because the corrections are proportional to the inverse exponential of the worldsheet coupling constant. d_i is the instanton number of the i-th $(1,1)$-form e_i, $d_i = \int_{\mathcal{C}} e_i$, and $n_{\{d_i\}}$ is the number of isolated holomorphic curves $\mathcal{C}$ of multi-degree $(d_1, \ldots, d_{h^{(1,1)}})$ in the threefold fibre. The sum over j takes into account multiple coverings of a fundamental string wrapped on a given curve. Worldsheet instanton corrections are explained in detail in the appendix.

Inserting (2.28) into (2.27) and assuming that the μ_I and $\tilde{\mu}^I$ are large so that the $\mathcal{O}((t^V)^0)$ term can be neglected we get

$$W(Y_4^*) = \mu_0 + \mu_i t^i - \tilde{\mu}^i \mathcal{F}_i - \tilde{\mu}^0 \mathcal{F}_0 + \nu_0 t^V + \nu_i t^i t^V - \tilde{\nu}^i \mathcal{F}_i t^V - \tilde{\nu}^0 \mathcal{F}_0 t^V \; . \qquad (2.32)$$

[4] We thank P. Mayr for a clarifying discussion concerning the period integrals.

The period integrals (2.28) have been used in [20] to derive the Kähler potential in the large base limit

$$K = -\ln[\int_{Y_4^*} \Omega \wedge \bar{\Omega}] = -\ln\left[(t^V - \bar{t}^V)(2(\mathcal{F} - \bar{\mathcal{F}}) - (t^i - \bar{t}^i)(\mathcal{F}_i + \bar{\mathcal{F}}_i))\right] . \qquad (2.33)$$

The superpotential can be written in a more suggestive way by expressing it not in terms of special coordinates t^i but rather in terms of the homogeneous coordinates $X^I = (X^0, X^i)$ [60]. These coordinates are commonly used in $N = 2$ supergravity and are holomorphic functions of the special coordinates $X^I(t^i)$. Furthermore, one has a pre-potential $F(X)$ which is a homogeneous function of the X^I of degree two. The special coordinates are just the particular coordinate choice $t^0 = X^0/X^0 = 1, t^i = X^i/X^0, \mathcal{F}(t^i) = (X^0)^{-2}F(X)$. In homogeneous coordinates the periods of the threefold are

$$\Pi = (X^I, F_I) = X^0(1, t^i, \mathcal{F}_i, \mathcal{F}_0) , \qquad (2.34)$$

and the Kähler potential reads

$$K = -\ln\left[(t^V - \bar{t}^V)(\bar{X}^I F_I - X^I \bar{F}_I)\right] . \qquad (2.35)$$

This coincides with the Kähler potential given in (2.33) up to a Kähler transformation that amounts to a different normalization of the (4, 0)-form Ω. Inserting (2.34) into (2.32) finally yields

$$W(\tilde{Y}_4) = \alpha_I X^I - \beta^I F_I , \qquad (2.36)$$

where we abbreviated $\alpha_I = \mu_I + t^V \nu_I$, $\beta^I = \tilde{\mu}^I + t^V \tilde{\nu}^I$ and discarded an X^0-factor by the same Kähler transformation.

Curiously the superpotential (2.36) coincides with the superpotential derived in ref. [28]. It arises in type IIB compactifications on a Calabi-Yau threefold with non-vanishing RR- and NS 3-form fluxes studied in refs. [33, 28, 34]. In this case the type IIB dilaton plays the role of t^V in (2.36). It also is very closely related to the BPS-mass formula studied in refs. [35, 36] and the entropy formula of $N = 2$ black holes [37]. This fortunate coincidence saves us from a detailed analysis of the supersymmetric minima of (2.36) and we can simply refer the reader to refs. [33, 28, 34, 39, 36]. One finds that for generic fluxes no supersymmetric vacuum exists. However, if the α_I, β^I are appropriately chosen supersymmetric ground states can exists. This can happen if the fluxes are aligned with cycles of the threefold which can degenerate at specific points in the moduli space [32, 33, 28, 34, 39]. These points (or subspaces) then correspond to supersymmetric ground states. They also coincide with the supersymmetric attractor points studied in refs. [37]. Note that in ref. [28] the consistency of the compactification required $\nu_I \tilde{\mu}^I - \mu_I \tilde{\nu}^I = 0$ while in our case this is replaced with the generalized condition given in eq. (2.26).

Before we evaluate $\tilde{W}(Y_4)$ let us briefly discuss the symmetry properties of $W(Y_4^*)$ as obtained in eq. (2.36). In homogeneous coordinates the period vector Π transforms as a symplectic vector according to

$$\begin{pmatrix} F_I \\ X^I \end{pmatrix} \rightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F_I \\ X^I \end{pmatrix} , \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2h^{(1,1)} + 2, \mathbb{Z}) . \qquad (2.37)$$

This transformation leaves the symplectic product $(\bar{X}^I F_I - X^I \bar{F}_I)$ invariant and via eq. (2.35) also the Kähler potential (in homogeneous coordinates). The fact that (F_I, X^I) transform as a symplectic vector implies that also $(\tilde{a}_I, a^I)$ and $(\tilde{b}_I, b^I)$ transform according to (2.37). Indeed one verifies that the intersection matrix given in (2.24) is left invariant if $(\tilde{a}_I, a^I)$ as well as $(\tilde{b}_I, b^I)$ transform as symplectic vectors. Since F_4^* has to be symplectically invariant one infers from (2.25) that in turn the fluxes $(\mu_I, \tilde{\mu}^I)$ and $(\nu_I, \tilde{\nu}^I)$ have to transform as symplectic vectors, i.e. according to (2.37). Since t^V is invariant we conclude that (α_I, β^I) form a symplectic vector. Thus W of (2.36) is invariant under symplectic tranformations.

Having derived the expression (2.32) for $W(Y_4^*)$ we can use eq. (2.19) to determine the quantum corrections of $\tilde{W}(Y_4)$. By matching the classical terms of $\tilde{W}(Y_4)$ as given in eq. (2.16) with the classical part of (2.32) using (2.30), (2.31) one is led to the identification

$$-\tilde{\nu}^0 \mathcal{F}_0 t^V = \frac{1}{2\pi} \int_{Y_4} t \wedge t \wedge t \wedge t \; F_0 + \text{quantum corrections} \; , \tag{2.38}$$

$$-\tilde{\mu}^0 \mathcal{F}_0 - \tilde{\nu}^i \mathcal{F}_i t^V = \frac{1}{2\pi} \int_{Y_4} t \wedge t \wedge t \wedge F_2 + \text{q.c.} \; , \tag{2.39}$$

$$\nu_i t^i t^V - \tilde{\mu}^i \mathcal{F}_i = \frac{1}{2\pi} \int_{Y_4} t \wedge t \wedge F_4 + \text{q.c.} \; , \tag{2.40}$$

$$\nu_0 t^V + \mu_i t^i = \frac{1}{2\pi} \int_{Y_4} t \wedge F_6 \; , \tag{2.41}$$

$$\mu_0 = \frac{1}{2\pi} \int_{Y_4} F_8 = \text{const.} \; . \tag{2.42}$$

The right hand sides of (2.41), (2.42) are two- and one-point functions in the topological A-model explained in the appendix and therefore do not receive instanton corrections [18, 62]. From eq. (2.40) using (2.30), (2.31) we learn that the term including $\mathcal{F}_i$ has a polynomial piece and instanton corrections while the second term $\nu_i t^i t^V$ is purely classical. This can also be understood by considering the corresponding correlation functions in the A-model. Let us now go through this computation in more detail.

2.3 The Superpotential Generated by Four-Form Flux

We switch on only fourform flux and rederive the superpotential $\tilde{W}(Y_4)$ including quantum corrections by evaluation correlation functions in the topological sigma model. The topological sigma model is explained in appendix B.2. We denote the observables of the A-model by $\mathcal{O}_M^{(k)} \in H^{(k,k)}(Y_4)$, where the index takes the values $M = 1, \ldots, h_{vp}^{(k,k)}(Y_d)$. The observables of the A-model are exactly the elements of the primary vertical subspace of $H^{(k,k)}(Y_d)$. The two-point functions

$$\eta_{MN}^{(k)} = \langle \mathcal{O}_M^{(k)} \mathcal{O}_N^{(d-k)} \rangle = \int_{Y_d} \mathcal{O}_M^{(k)} \wedge \mathcal{O}_N^{(d-k)} \tag{2.43}$$

receive no instanton corrections and define a flat metric on the vertical primary cohomology [62]. Note that this metric for $k = 1$ is not the metric on the moduli space of

$(1,1)$-forms, the Zamolodchikov metric $G_{A\bar{B}} = 1/V \int_{Y_d} e_A \wedge * e_B$, where $*e_B$ is the Hodge dual of e_B. Although the Zamolodchikov metric contains also observables of the A-model (because $e_A = \mathcal{O}^{(1)}_A$ is an observable of the A-model) and looks similar as the topological metric above, it depends on the Kählerpotential as $G_{A\bar{B}} = \partial_A \partial_{\bar{B}} K$ and does receive quantum corrections as can be seen from eq. (2.33). The relation between the metric of the $(1,1)$-moduli and the metric of the A-model was the subject of [63].

The threepoint functions

$$Y^{(k)}_{KLM} = \langle \mathcal{O}^{(1)}_K \mathcal{O}^{(k)}_L \mathcal{O}^{(d-k-1)}_M \rangle, \tag{2.44}$$

do receive instanton corrections. Because of their factorization properties all other amplitudes can be expressed in terms of the two- and three-point functions $\eta^{(k)}_{MN}, Y^{(k)}_{KLM}$ [18].

Choosing the 4-form flux as

$$\frac{F_4}{2\pi} = \sum_{N=1}^{h^{2,2}_{vp}} \lambda^N \mathcal{O}^{(2)}_N \ , \tag{2.45}$$

ref. [23] proposed the following formula

$$\partial_{t^A} \partial_{t^B} \tilde{W}(Y_4) = \sum_N \lambda^N \langle \mathcal{O}^{(1)}_A \mathcal{O}^{(1)}_B \mathcal{O}^{(2)}_N \rangle \ . \tag{2.46}$$

In the following we evaluate this three-point function for a threefold-fibred fourfold in the large base limit and show that (2.46) is consistent with (2.40).

Let us first consider the part of F_4 which has one component in the base

$$\frac{F_4}{2\pi} \ = \ \sum_{N=1}^{h^{1,1}(Y_3)} \lambda^N \mathcal{O}^{(2)}_N = \sum_{i=1}^{h^{(1,1)}(Y_3)} \lambda^{Vi} e_V \wedge e_i, \tag{2.47}$$

where λ^{Vi} is large. In the large base limit, there are no instanton corrections from the base so that according to the classical intersection numbers e_V occurs at most once in any correlation function. The divisor which is dual to e_V is the threefold-fibre and projects the amplitude of the fourfold to the threefold. In particular, for the three-point function with the 4-form flux as in (2.47), the three-point function on the fourfold is projected to the three-point function on the threefold [20]

$$\partial_{t^i} \partial_{t^j} \tilde{W}(Y_4) = \sum_N \lambda^N \langle \mathcal{O}^{(1)}_i \mathcal{O}^{(1)}_j \mathcal{O}^{(2)}_N \rangle_{Y_4} = \sum_{k=1}^{h^{(1,1)}(Y_3)} \lambda^{Vk} \langle \mathcal{O}^{(1)}_i \mathcal{O}^{(1)}_j \mathcal{O}^{(1)}_k \rangle_{Y_3} = \sum_{k=1}^{h^{(1,1)}(Y_3)} \lambda^{Vk} Y_{ijk}, \tag{2.48}$$

where

$$\mathcal{O}^{(2)}_N = e_V \wedge e_k \ , \qquad \mathcal{O}^{(1)}_k = e_k \ . \tag{2.49}$$

As explained in appendix A.4, in special coordinates the three-point function Y_{ijk} on Y_3 is the third derivative of a holomorphic prepotential [60]

$$Y_{ijk} = \mathcal{F}_{ijk}, \qquad \mathcal{F}_{ijk} = \partial_{t^i} \partial_{t^j} \partial_{t^k} \mathcal{F} \ . \tag{2.50}$$

Inserting (2.50) in eq. (2.48) we learn that $\partial_{t^i}\partial_{t^j}\tilde{W}(Y_4)$ can be integrated and indeed coincides with the instanton corrected part of the expression (2.40).

In order to get the full superpotential, we still have to consider the part with the 4-form flux restricted to the threefold fibre, i.e.

$$\frac{F_4}{2\pi} = \sum_{i=1}^{h^{1,1}(Y_3)} \lambda^i \mathcal{O}_i^{(2)}. \tag{2.51}$$

We have to distinguish two cases. First we consider the three-point function which contains one observable $\mathcal{O}^{(1)}$ corresponding to the base. In this case the three-point function of the fourfold is projected to the two-point function of the threefold fibre:

$$\partial_{t^V}\partial_{t^j}\tilde{W}(Y_4) = \lambda^i \langle \mathcal{O}_V^{(1)}\mathcal{O}_j^{(1)}\mathcal{O}_i^{(2)}\rangle_{Y_4} = \lambda^i \langle \mathcal{O}_j^{(1)}\mathcal{O}_i^{(2)}\rangle_{Y_3} = \lambda^i \eta_{ji}^{(1)}. \tag{2.52}$$

As already mentioned above, two-point functions receive no worldsheet instanton corrections and the classical part of the amplitude is already the exact expression. Integrating twice we obtain the term in (2.40) which does not contain instanton corrections.

Finally, the contribution to the superpotential which has no component in the base is subleading in the limit where λ^{Vi} and t^V are large. To summarize, we confirmed the expression given in eq. (2.40) by considering correlation functions in the topological A-model, that is without using mirror symmetry. Altogether we thus have

$$\tilde{W}(Y_4) = \lambda^k \eta_{jk}^{(1)} t^j t^V + \lambda^{Vk}\mathcal{F}_k, \tag{2.53}$$

with the relations $\lambda^k \eta_{jk}^{(1)} = \nu_j, \lambda^{Vk} = -\tilde{\mu}^k$.

2.4 The Duality of IIA Theory and the Heterotic String Including the Superpotential

In order to mention some applications of the above calculations, we finally make some remarks about the superpotential in connection with type IIA-heterotic duality. This chapter is very brief, for a longer discussion see [29, 65]. Type IIA string theory compactified on a Calabi-Yau fourfold Y_4 is dual to the heterotic string compactified on $Y_3 \times T^2$. For simplicity, we assume that the fourfold Y_4 is a $K3$ fibration over a large base $\mathbb{F}_n$. The heterotic compactification manifold Y_3 is an elliptic fibration fibration over a large base $\mathbb{F}_n$. It is interesting to compare the Kähler potentials of the two theories including the worldsheet corrections on the IIA side.

The Kähler potential of the IIA string on the $K3$ fibred fourfold in the limit of a large $\mathbb{P}^1$ base of $\mathbb{F}_n$, that is $t^V \to \infty$, is given by eq. (2.33). We denote the Kähler modulus of the $\mathbb{P}^1$ fibre of $\mathbb{F}_n$ by t^U. In the limit $t^U \to \infty$ the worldsheet instanton corrections $\sim e^{-it^U}$ are suppressed and the Kählerpotential reads

$$K = -\ln\left[(t^V - \bar{t}^V)(2(\mathcal{F} - \bar{\mathcal{F}}) - (t^i - \bar{t}^i)(\mathcal{F}_i + \bar{\mathcal{F}}_i))\right] \tag{2.54}$$

with

$$\mathcal{F} = t^U \eta_{\hat{i}\hat{j}} t^{\hat{i}} t^{\hat{j}} + \mathcal{F}^{(1)}(t^{\hat{i}}) + \mathcal{O}(e^{it^U}), \tag{2.55}$$

where $t^{\hat{\imath}}$ denote the moduli of the $K3$-fibre and η is the intersection matrix of $K3$

$$\eta_{\hat{\imath}\hat{\jmath}} = \begin{pmatrix} 0 & 1/2 & \\ 1/2 & 0 & \\ & & -\mathbf{I} \end{pmatrix}. \tag{2.56}$$

$\mathcal{F}^{(1)}(t^{\hat{\imath}})$ contains the world-sheet instanton corrections of the $K3$ fibre

$$\mathcal{F}^{(1)}(t^{\hat{\imath}}) = -\frac{1}{(2\pi)^3} \sum_{\{d_{\hat{\imath}}\}} n_{\{d_{\hat{\imath}}\}} Li_3(e^{2\pi i \sum t^{\hat{\imath}} d_{\hat{\imath}}}). \tag{2.57}$$

On the heterotic side we denote the complex Kähler moduli of the $\mathbb{F}_n$ base of the threefold by u and v. The Kähler modulus of the torus T^2 of $Y_3 \times T^2$ is called τ and the Kähler modulus of the elliptic fibre of Y_3 is called ρ. Finally there are complex scalars n^a, $a = 1, \ldots l$ charged under the gauge group $(U(1))^l$. They have their origin in the compactification of the ten-dimensional gauge vectors charged under the $E_8 \times E_8$ or $SO(32)$ gauge group on the torus. We keep the number l of $U(1)$ gauge groups arbitrary. The Kählerpotential in the large base limit $u, v \to \infty$ is

$$\hat{K} = -\ln[(u - \bar{u})(v - \bar{v})((\tau - \bar{\tau})(\rho - \bar{\rho}) - (n^a - \bar{n}^a)^2)]. \tag{2.58}$$

The duality map relates the moduli of the two theories as

$$\begin{aligned} \left\{t^{\hat{\imath}}\right\} &\leftrightarrow \{\tau, \rho, n^a\}, \\ \left\{t^U, t^V\right\} &\leftrightarrow \{u, v\}. \end{aligned} \tag{2.59}$$

Note that the number of $U(1)$ gauge groups l is related to the number of $(1,1)$-forms of the IIA $K3$ fibre. The two Kählerpotentials are mapped to each other using these relations. Switching off the worldsheet instantons in (2.54) this can be verified easily. In addition the duality suggests that the world-sheet instantons of the IIA Kählerpotential are mapped to some corrections of the heterotic Kählerpotential that have not been taken into account above. This is a standard procedure that has been also studied in detail in four-dimensional compactifications [38], where world-sheet instanton corrections on the IIA side are mapped to loop-corrections in the heterotic theory.

We can also make a statement about the superpotential. Inserting (2.55) into (2.32) one obtains the superpotential for $K3$-fibred fourfolds in the large $\mathbb{F}_n$ limit. If one assumes that the duality map given in (2.59) continues to hold in the presence of background fluxes one can use it to derive a heterotic superpotential $W_{\text{het}}(\tau, \rho, n^a, u, v)$. For a discussion of this, see [29].

2.5 Summary and Outlook

In contrast to higher dimensional compactifications, type IIA string theory compactified to two dimensions on a Calabi-Yau fourfold Y_4 generically contains non-vanishing background R-R fourform fluxes in the compactification manifold [19]. R-R background fluxes generate a potential in the low energy effective action of the theory. We consider the potential generated by switching on all possible R-R background fluxes [24, 25]. For a large worldsheet coupling constant α', the potential receives quantum corrections of the worldsheet theory containing worldsheet instantons. The goal of chapter 2 is to derive these worldsheet corrections. Worldsheet instanton corrections generally can be derived using mirror symmetry. Mirror symmetry states that a classical potential derived on the mirror manifold Y_4^* can be mapped to the potential on Y_4 including the worldsheet quantum corrections. We restrict the Calabi-Yau fourfold Y_4 to the special class of threefold fibred fourfolds in the large base limit. This implies that the worldsheet instanton corrections are given completely in terms of the threefold fibre [20]. In particular, it is possible to use the the special geometry of the threefold fibre, which simplifies the mirror symmetry calculations significantly. As a result, we are able to give the general form of the full superpotential including all worldsheet corrections for the class of threefold fibred fourfolds in the large base limit in section 2.2. In section 2.3, we rederive a part of the result of section 2.2 in terms of scattering amplitudes of a topological field theory, called the A-model, obtained by twisting the worldsheet theory. We show that the results of section 2.2 and section 2.3 agree.

The results of the calculations are interesting in the context of string dualities. Firstly, the potential including the quantum corrections coincides with the potential obtained in [28] for compactifications of type IIB string theory on Calabi-Yau threefolds including threeform fluxes from the R-R and the NS-NS sector. An explanation for this phenomenon has not been given yet as far as we are aware. One possible direction for finding an explanation might be to examine in detail the T-duality between the type IIA and the type IIB string [5]. This remains to be done in the future. Also, a dual four-dimensional formulation in terms of type IIB theory could provide a useful link for constructing a non-perturbative formulation in terms of F-theory.

Another aspect is the duality between IIA theory compactified on a fourfold and the heterotic string compactified on a Calabi-Yau threefold times a torus. If the duality is expected to hold including the potential of the IIA theory, one would expect mechanisms in the heterotic theory which generate the corresponding potential. In particular, the worldsheet corrections of the IIA theory should have their counterpart in the heterotic theory. This also remains a task for future investigations.

[5]We thank P. Mayr for a discussion of that issue.

3 F-Theory Duals of M-Theory on $S^1/\mathbb{Z}_2 \times T^4/\mathbb{Z}_N$

We consider the strong coupling limit of the heterotic string [15] compactified on an orbifold to six dimensions [89, 90]. The strongly coupled heterotic string is described by M-theory compactified on an interval, where the length of the interval corresponds to the heterotic dilaton. Compactifying the theory to six dimensions on certain orbifolds, anomaly cancellation dictates some rules for the gauge group and the massless spectrum of the theory that are highly non-intuitive. The goal of this chapter is to find explanations for some of these rules by considering the F-theory formulation of the strongly coupled heterotic string. In 3.1, we review the strong coupling limit of the heterotic string in ten dimensions. In 3.2, we consider the compactification to six dimensions on an orbifold. We review the rules given in [89, 90] for the gauge group and the massless spectrum that follow from anomaly cancellation. As an instructive example, we consider the gauge group $SO(16)\times[E_7\times SU(2)]$ in detail in 3.3. The main part of this chapter is 3.4, where we construct the F-theory formulation of the six-dimensional model. We show that F-theory contains some additional information about the gauge group of the theory. This enables us give an explanation of some of the rules that have been established in [89, 90]. In the last section, we generalize the results to other gauge groups.

3.1 M-Theory on $S^1/\mathbb{Z}_2$

The strong coupling limit of the ten-dimensional $E_8 \times E_8$ heterotic string is M-theory compactified on a one-dimensional orbifold $\mathbb{R}^{10} \times S^1/\mathbb{Z}_2 = \mathbb{R}^{10} \times I$ [15], where I is the unit interval. To be precise, it was shown that the low-energy limit of M-theory eleven-dimensional supergravity compactified on $\mathbb{R}^{10}\times I$ describes the low energy limit of the strongly coupled heterotic $E_8\times E_8$ string. The bosonic part of eleven-dimensional supergravity to lowest order in the coupling constant is [73]

$$S_{11} = \frac{1}{\kappa^2}\int d^{11}x\sqrt{g}\left\{-\frac{1}{2}R-\frac{1}{96}G_{\mu\nu\rho\lambda}G^{\mu\nu\rho\lambda}\right\}+\frac{1}{12\kappa^2}\int_{M^{11}} C\wedge G\wedge G, \quad (3.1)$$

where $\mu = 0,\ldots,10$. The action contains the curvature scalar R, the field strength $G = dC$ (the fourform is defined as $G = 1/4!\partial_\mu C_{\nu\rho\lambda}dx^\mu\wedge dx^\nu\wedge dx^\rho\wedge dx^\lambda$), the determinant of the metric g and the eleven-dimensional gravitational coupling constant κ.

The one-dimensional compactification manifold I is given by the identification $x^{11} = -x^{11}$ of the eleventh coordinate $x^{11}\in[-\pi,\pi]$ parametrizing the circle S^1. The Chern-Simons term $\int C\wedge G\wedge G$ is invariant under the projection if

$$\begin{aligned} C_{abc}(x^{11}) &= -C_{abc}(-x^{11}), \quad a,b,c = 1\ldots 10, \\ C_{ab,11}(x^{11}) &= C_{ab,11}(-x^{11}). \end{aligned} \quad (3.2)$$

Thus C_{abc} is projected out and $C_{ab,11}$ is kept on the ten-planes. For the field strength this means that G_{abcd} is kept and $G_{abc,11}$ is projected out. The orbifold breaks half of the thirty-two supersymmetries of eleven-dimensional supergravity resulting in a ten-dimensional theory with one chiral supersymmetry.

Consistency requires anomaly-freedom of the compactified theory. A detailed knowledge of M-theory is not necessary for deriving the anomalies, because anomalies arise in the infrared sector of the theory. As anomalies occur only in chiral theories there are no anomalies in the eleven-dimensional bulk. But the inflow of the eleven-dimensional fields on the boundaries gives rise to ten-dimensional chiral anomalies [15, 74].

The action of the theory is eleven-dimensional supergravity in the bulk plus a ten-dimensional gauge theory on the ten-dimensional fixed planes. The bulk action is just the supergravity action (3.1), but anomaly cancellation on the ten-dimensional boundaries requires including a higher order term

$$S_{11}^{\text{higher order}} = -\frac{\lambda^2}{(4\pi\kappa)^2}\int_{M^{11}} G\wedge X_7(R), \tag{3.3}$$

with a seven-form X_7 given by $dX_7 = X_8$, $X_8(R) = \frac{1}{((2\pi)^3 4!)}\left(\frac{1}{8}trR^4 - \frac{1}{32}(trR^2)^2\right)$ and the gauge coupling λ. The notation as well as the relation between the gauge coupling λ and the gravitational coupling κ will be explained later. In the early literature such as [15], the term $C\wedge X_8$ was used alternatively for (3.3). A careful analysis in reference [75] however showed that the terms are not equivalent and that only for the higher order term $G\wedge X_7$ the anomalies cancel. The necessity of such a term was known before Horava-Witten theory from string dualities and also from cancellation of the M-theory five-brane anomaly [76, 77, 78].

The ten-dimensional action on the boundaries is obtained by compactifying the eleven-dimensional action (3.1)+(3.3) on a circle S^1 and projecting onto the modes which are invariant under $\mathbb{Z}_2$. Using equation (3.2), the Chern-Simons and the higher order term lead to the ten-dimensional action

$$S \sim \int_{M^{10}} G\wedge X_6 + B\wedge G\wedge G, \tag{3.4}$$

where X_6 is a six-form and related to the gauge variation of X_7 as $dX_6 = \delta X_7$ and $B_{ab} = C_{ab,11}$. This expression will be explained in detail later and it turns out that it plays the role of a Green-Schwarz term in the anomaly cancellation. For details of anomaly cancellation see appendix C.

It was shown in [15] that compactifying (3.1)+(3.3) on I to ten dimensions in a gauge invariant way breaks the ten-dimensional $N=1$ supersymmetry. Restoring the supersymmetry requires a modified Bianchi-identity of the four-form field strength, $dG\neq 0$. This leads to a modification of the field strength $G = dC +$ (additional terms) and destroys the gauge invariance of the ten-dimensional action. A violation of gauge invariance gives rise to anomalies which have to be cancelled in a consistent theory.

To cancel these anomalies it is necessary to include additional ten-dimensional gauge fields on the two fixed planes. The explicit calculation shows that the theory is anomaly-free only if the gauge fields transform in the adjoint representation of the gauge group E_8 on each fixed plane. Additional states located at the fixed points in orbifold compactifications are familiar from string theory as twisted states. The E_8 gauge fields are equivalent to the string theory twisted states. Due to a lack of understanding the underlying M-theory, there has been no direct way to explain the existence of these fields. The ten-dimensional action is

$$S_{10} = \frac{1}{4\lambda^2}\int d^{10}x\sqrt{g_{10}}\left\{TrF^2 - \frac{1}{2}trR^2\right\}, \tag{3.5}$$

with the E_8 field-strength F. Note that we have two of these actions, one on each fixed plane.

Let us review the anomalies of the theory in greater detail. In ten dimensions there are gauge as well as gravitational and mixed anomalies [79, 80]. The one-loop anomalies on the ten-planes, excluding the inflow of $G \wedge X_7(R)$ and $C \wedge G \wedge G$, are almost identical to those of the ten-dimensional $E_8 \times E_8$ heterotic string. As explained in appendix C.1, the heterotic one-loop anomaly polynomial is

$$I_{12} = -\frac{15}{2(2\pi)^5 6!} Y_4 Z_8 \tag{3.6}$$

where Y_4 and Z_8 are four- and eight forms defined in C.1. The factorization of the anomaly polynomial into a four- and an eight-form is necessary for cancellation of the anomaly by the Green-Schwarz mechanism, for the details see the appendix.

The anomalies in M-theory compactified on $R^{10} \times I$ are localized on the two ten-planes at the end of the interval with an E_8 gauge group each. The ten-planes are separated by a finite distance. The anomaly polynomial is the sum of two polynomials, one on each ten-plane as denoted by the indices $1, 2$

$$\hat{I}_{12} = \hat{I}_{12}^1 + \hat{I}_{12}^2, \tag{3.7}$$

and takes the form, as explained in appendix C.2,

$$\begin{aligned} \hat{I}_{12} &= \frac{\pi}{3}\left(\hat{Y}_4^{\,1}(R, F_1)\right)^3 + X_8(R)\hat{Y}_4^{\,1}(R, F_1) \\ &+ \frac{\pi}{3}\left(\hat{Y}_4^{\,2}(R, F_2)\right)^3 + X_8(R)\hat{Y}_4^{\,2}(R, F_2), \end{aligned} \tag{3.8}$$

where X_8 is the polynomial defined below equation (3.3). The two terms $\sim (\hat{Y}_4^{1,2})^3$ are cancelled by projection of the eleven-dimensional Chern-Simons term $\int C \wedge G \wedge G$ to the ten-planes and the terms $\sim X_8(R)\hat{Y}_4^{1,2}$ are cancelled by the projection of $\int G \wedge X_7$. The anomaly calculation is explained in detail in appendix C.2.

Because there are no scalars in the theory, one might wonder how the heterotic dilaton arises in the M-theory picture. The answer is that the length l of the compact eleventh dimension is related to the ten-dimensional heterotic dilaton [15]

$$l = \phi_{\text{het}}^{2/3}. \tag{3.9}$$

The situation is similar to the duality of type IIA string theory and M-theory compactified on S^1, where the type IIA dilaton is identified with the radius of the circle S^1 [88]. As the heterotic string coupling constant is determined by the dilaton, the length of the interval is related to the heterotic string coupling. If the two fixed planes are pushed together by letting the length of the interval approach zero, the heterotic coupling becomes small. This limit becomes ten-dimensional and describes the weakly coupled heterotic string. If the planes are far away from each other, which means l and ϕ_{het} are large, the theory describes the strong coupling limit of the heterotic string.

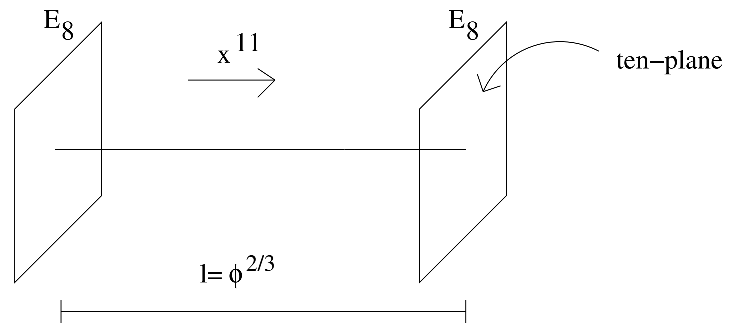


Figure 1: $\mathbb{R}^{10} \times I$ with the length $l = \phi_{\text{het}}^{2/3}$ of the interval and an E_8 gauge group on each ten-plane

3.2 M-Theory on $S^1/\mathbb{Z}_2 \times T^4/\mathbb{Z}_2$

Consider the strong coupling limit of the heterotic string compactified to six dimensions on an orbifold $T^4/\mathbb{Z}_2$ with gauge group $\mathcal{G} = G \times G' \subset E_8 \times E_8$. This has been studied in [89, 90, 94, 95, 96, 97, 100]. The strong coupling limit of the heterotic string compactified on orbifolds to four dimensions has been considered in [92]. For a review on heterotic M-theory compactifications see [101]. The starting point is M-theory compactified on two orbifolds $T^4/\mathbb{Z}_2 \times S^1/\mathbb{Z}_2$ with the gauge group E_8 broken to G on one ten-plane and E_8 broken to G' on the other ten-plane. There are two $\mathbb{Z}_2$ projections in that model, one acts on the coordinate of S^1 and one on the four coordinates of T^4. The combined action of both $\mathbb{Z}_2$'s defines the fixed planes of the theory. The theory has eight supercharges which is $N = 1$ supersymmetry in six dimensions. The orbifold projection acting on S^1 leaves two fixed ten-planes already considered above. The action of the second $\mathbb{Z}_2$ leaves sixteen fixed seven-planes. The combined action of both orbifolds results in thirty-two fixed six-planes, which are the intersections of the fixed ten-planes and the fixed seven-planes. Consistency of the theory requires anomaly cancellation on all the fixed ten-, seven- and six-planes. Anomaly cancellation on the ten-planes was already considered above, so the requirements of anomaly cancellation on the seven- and six-planes remain to be examined. Anomaly cancellation can be useful to gain information about the spectrum and the gauge group of an unknown theory as was done successfully in the ten-dimensional compactification on $\mathbb{R}^{10} \times I$. There are no chiral anomalies in odd dimensions though, so the seven-planes are anomaly-free anyway and at first sight there is no direct way of determining the gauge group on the seven-planes. The anomalies on the six-planes however are not blind to the spectrum on the seven planes which intersect them. It is indeed possible to determine the gauge group on the seven-planes and the twisted spectrum on the six-planes by demanding anomaly cancellation on the six-planes. Demanding anomaly cancellation in six dimensions is a powerful tool as there are gauge as well as gravitational and mixed anomalies.

Consider the seven-dimensional compactification of M-theory on $T^4/\mathbb{Z}_2$. The theory has sixteen supercharges and the only multiplets in seven dimensions are supergravity and vector-multiplets. Compactifying further on $S^1/\mathbb{Z}_2$ breaks half of the supersymmetry and decomposes the seven-dimensional vector-multiplet into a six-dimensional hypermultiplet and a six-dimensional vector multiplet. The six-dimensional multiplets are chiral and contribute to the anomaly on the six-planes. The vector multiplet contains vectors $C_{\mu,yz}$,

$\mu = 1, \ldots, 6$, $y, z = 7, \ldots, 10$ arising from the eleven-dimensional three-form C with one space-time index μ and two internal indices (y, z) of $T^4/\mathbb{Z}_2$. There are no vectors with one index in the eleventh direction. If the internal five-dimensional space is a direct product $T^4/\mathbb{Z}_2 \times S^1/\mathbb{Z}_2$, the vector changes sign under the $x^{11} \to -x^{11}$ projection (see (3.2)) and is projected out. It turns out that this is not consistent with anomaly cancellation [89, 90]. Instead it is assumed that the compactification manifold is not really a direct product but a "twisted" product which allows the vector components to survive the orbifold projection in certain cases. We follow the literature [89, 90] in dealing with this by using the "twisted" product structure of the geometry without specifying how the geometry gets distorted. The twisted product is defined by obtaining a spectrum such that anomaly cancellation works. There has been no direct formulation of this problem in terms of the proper geometry.

The effect of the twisted product structure can be summarized as follows [89, 90]. It should be stressed again that there has been no derivation of the following rules from first principles, it is only possible to show that the anomalies cancel using this receipe.

If the gauge group on the seven-plane does not coincide with the perturbative gauge group on the ten-plane, the six-dimensional vector multiplet survives the projection while the six-dimensional hypermultiplet is projected out [90]. In this case the gauginos in the chiral vector multiplet contribute to the anomaly. But as the fields are localized on the seven-planes and not on the six-planes the anomaly is half of the standard anomaly resulting from six-dimensional gauginos charged under the same gauge group.

If the seven-plane gauge group coincides with a factor of the perturbative gauge group, the six-dimensional hypermultiplet survives the projection and contributes to the anomaly while the six-dimensional vector multiplet is projected out. The contribution of the hyperinos in the hypermultiplet to the anomaly is again half the standard anomaly given by six-dimensional hyperinos charged under the same gauge group.

Another equivalent way of describing the twisting of the internal manifold is to impose different boundary conditions on the six-dimensional $N = 1$ vector and hypermultiplets [89]. Free (Neumann) boundary conditions on one six-plane imply fixed (Dirichlet) boundary conditions on the six-plane which is connected to the first one by a seven-plane for the same multiplet. Neumann boundary conditions on the vector component implies Dirichlet boundary conditions on the hyper components on the same six-plane and vice versa. Imposing Dirichlet boundary conditions on a six-plane means that the corresponding multiplet is invisible on that plane and projected out by the $\mathbb{Z}_2$ symmetry.

If the seven-plane gauge group does not coincide with the perturbative gauge group Neumann boundary conditions are imposed on the six-plane on the vector multiplets and Dirichlet boundary conditions on the hypermultiplets.

If the gauge group on the seven-plane coincides with a factor of the perturbative gauge group Dirichlet boundary conditions are imposed on the vector multiplets and Neumann boundary conditions on the hypermultiplets. Again, there has been no derivation of these boundary conditions from first principles but one has to us the above receipe to obtain anomaly freedom of the theory.

If the gauge group on the seven-plane is broken to some subgroup by the action of the orbifold projection $T^4/\mathbb{Z}_2$, the seven-dimensional fields decompose into six-dimensional fields according to representations determined by the breaking of the gauge group. The case that has been considered explicitly so far is a gauge group $SU(N)$ broken to its Cartan subgroup $(U(1))^{N-1}$. There are $(N-1)$ six-dimensional vector multiplets charged under $(U(1))^{N-1}$ surviving the projection. The projection of the (N^2-1) six-dimensional hypermultiplets charged under $SU(N)$ has to be determined by anomaly cancellation again with the result that $2(N-1)$ six-dimensional hypermultiplets survive the projection.

3.3 The Gauge Group $SO(16) \times [E_7 \times SU(2)]$

As an example consider the gauge group $G \times G' = SO(16) \times [E_7 \times SU(2)]$, which was analyzed in detail both in [89] and [90]. The breaking of the gauge group $E_8 \rightarrow SO(16)$ leads to a decomposition $(\mathbf{248}) \rightarrow (\mathbf{120}) \oplus (\mathbf{128})$. There are 120 vector multiplets transforming in the adjoint representation and 128 hypermultiplets transforming in the spinor representation of $SO(16)$. The breaking of the gauge group $E_8 \rightarrow [E_7 \times SU(2)]$ on the other ten-plane leads to a decomposition $(\mathbf{248}) \rightarrow (\mathbf{133}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{56}, \mathbf{2})$. The first factor are 133 vector multiplets in the adjoint representation of E_7, the second factor are three vector multiplets in the adjoint representation of $SU(2)$ and the third one are 112 hypermultiplets in the bifundamental representation. In addition the compactification leads to four moduli which are hypermultiplets and gauge singlets and there is one universal tensor multiplet which includes the dilaton. This spectrum is just the untwisted massless spectrum of the weakly coupled heterotic theory. For the construction of heterotic spectra see [102, 103]. Thus the untwisted spectrum can be reproduced successfully in the M-theory formulation by taking eleven-dimensional supergravity on $\mathbb{R}^{10} \times S^1/\mathbb{Z}_2$ including the decomposition of the "twisted" E_8 gauge fields and compactifying the flat ten-dimensional space on T^4 to six dimensions projecting onto the modes which are invariant under $\mathbb{Z}_2$ acting on the coordinates of the torus.

The twisted states of the $T^4/\mathbb{Z}_2$ orbifold are less straight forward. The twisted states of the heterotic theory are sixteen half hypermultiplets [6] transforming as $(\mathbf{16}, \mathbf{1}, \mathbf{2})$ of $SO(16) \times [E_7 \times SU(2)]$. The problem is that they transform under the gauge groups on *both* ten-planes. It is not clear at first sight how to reproduce these states in M-theory. The twisted states are located at the fixed points of the theory, either on the sixteen fixed points on one ten-plane or on the sixteen fixed points on the other ten-plane. As the ten-planes are separated by the interval of length l, the twisted states should be charged under only one of the gauge groups of the ten-planes, never under both. This problem has been resolved in [89],[90]. Both references come to the conclusion that it is necessary to include a non-perturbative $SU(2)$ gauge group on each of the sixteen fixed seven-planes such that the gauge groups on the fixed six-planes are $SO(16)^{\text{pert}} \times SU(2)^{\text{non-pert}}$ and $[E_7 \times SU(2)]^{\text{pert}} \times SU(2)^{\text{non-pert}}$. Let us analyze the effect of the additional sixteen non-perturbative $SU(2)$ gauge groups. On the six-plane with a perturbative $[E_7 \times SU(2)]$ gauge group there is a non-perturbative gauge group $SU(2)$ as well as a perturbative $SU(2)$. According to the above prescription there are $SU(2)$ hypermultiplets surviving the $x^{11} \rightarrow -x^{11}$ projection on these six-planes. The anomalies on the six-planes including the contributions of the hypermultiplets indeed cancel. On the six-planes with the perturbative $SO(16)$ gauge group it is the the vector multiplets that survive the projection and contribute to the anomalies. To cancel the anomalies one

[6] A half hypermultiplets contains exactly half the massless spectrum of a hypermultiplet

indeed has to include additional sixteen half hypermultiplets transforming as $(\mathbf{16},\mathbf{2})$ of the gauge group $SO(16)^{\rm pert}\times SU(2)^{\rm non-pert}$.

The anomaly calculation was done explicitly in [89],[90], we summarize the calculation in appendix C.4. There are several contributions to the one-loop anomaly on the six-planes resulting from the chiral projection of fields which live in the seven-dimensional bulk and on the fixed ten-, seven- and six-planes. Adding up all contributions on the six-planes with gauge group $[E_7\times SU(2)]$ leads to the resulting anomaly

$$\begin{aligned} I^{\rm resulting}_{12,\ [E_7\times SU(2)]} &= \frac{1}{(2\pi)^3 4!}\Big(-\frac{1}{32}-\frac{g_1}{8}\Big)trR^4+\Big(\frac{1}{128}+\frac{g_1}{32}+\frac{3g_1}{16}-\frac{3\eta}{4}\Big)(trR^2)^2 \\ &+ \Big(\frac{3}{32}+\frac{3g_1}{4}+\frac{3\eta}{2}\Big)trR^2trF^2_{E_7}+\Big(-\frac{21}{32}+\frac{3g_1}{4}+\frac{3\rho}{4}+\frac{3\eta}{2}\Big)trR^2trF^2_{SU(2)} \\ &+ \Big(-\frac{3}{16}-\frac{3g_1}{4}\Big)(trF^2_{E_7})^2+\Big(\frac{21}{16}-\frac{3g_1}{4}-\frac{3\rho}{2}\Big)(trF^2_{SU(2)})^2 \\ &+ \Big(\frac{9}{8}-\frac{6g_1}{4}-\frac{3\rho}{2}\Big)trF^2_{E_7}trF^2_{SU(2)} \end{aligned} \tag{3.10}$$

on each six-plane and the resulting anomaly on the six-planes with gauge group $SO(16)$ is

$$\begin{aligned} I^{\rm resulting}_{12,\ SO(16)} &= \frac{1}{(2\pi)^3 4!}\Big(\frac{1}{32}-\frac{g_2}{8}\Big)trR^4+\Big(\frac{33}{384}+\frac{g_2}{32}-\frac{3g_2}{16}-\frac{3\eta}{4}\Big)(trR^2)^2 \\ &+ \Big(-\frac{9}{32}+\frac{3g_2}{4}-\frac{3\eta}{2}\Big)trR^2trF^2_{SO(16)}+\Big(-\frac{3}{4}+\frac{3\rho}{4}\Big)trR^2trF^2_{SU(2)} \\ &+ \Big(\frac{3}{16}-\frac{3g_2}{4}\Big)(trF^2_{SO(16)})^2+(-1+1)trF^4_{SO(16)}+(-1+1)(trF^2_{SU(2)})^2 \\ &+ \Big(\frac{3}{2}-\frac{3\rho}{2}\Big)trF^2_{SO(16)}trF^2_{SU(2)}, \end{aligned} \tag{3.11}$$

where g_1 and g_2 are the magnetic charges on the six-planes with $[E_7\times SU(2)]$ and $SO(16)$ gauge group and η and ρ parametrize the coupling of the thereeform C to seven-dimensional fields as explained in appendix C.4. Vanishing of both anomalies fixes

$$g_1=-1/4,\quad g_2=1/4,\quad \eta=1/16\ \text{ and }\ \rho=1. \tag{3.12}$$

The charges g_1, g_2 arise in the anomaly via the inflow of the eleven-dimensional Chern-Simons term $\int C\wedge G\wedge G$ and the higher order term $\int G\wedge X_7$ on each six-plane, see eqn. (C.64) of the appendix. To be precise, the four-form field strength G fulfills a modified Bianchi-identity $dG\neq 0$ with source terms on the fixed planes of the theory. The contributions of the $S^1/\mathbb{Z}_2$ fixed ten-planes are explained in the appendix, see eqn. (C.35). Embedded into each ten-plane are the 16 fixed six-planes of the $T^4/\mathbb{Z}_2$ orbifold, which give rise to a source term

$$dG=\sum_{i=1}^{32} g_i\delta^{(5)}_{(6-plane)_i}, \tag{3.13}$$

where $\delta^{(5)}_{(6-plane)_i}$ is a five-form with support on the i-th fixed six-plane. The charge g_i includes the gauge field F and the curvature term R with indices on $T^4/\mathbb{Z}_2$ only,

$$g_i=\int_{T^4/\mathbb{Z}_2}\Big((F\wedge F)_i-\frac{1}{2}(R\wedge R)_i\Big) \tag{3.14}$$

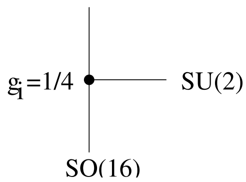
on the i-th fixed six-plane. The sum of the sixteen charges embedded into the same ten-plane is the second Chern class of the gauge bundle $c_2 = \sum_{j=1}^{16} \int_{T^4/\mathbb{Z}_2} (F \wedge F)_j$ and the curvature $c_2 = \sum_{j=1}^{16} \int_{T^4/\mathbb{Z}_2} (R \wedge R)_j$, where the index j counts the fixed six-planes in one ten-plane (the index i counts all fixed six-planes).

The curvature term can be determined by using the fact that the Euler number of a $K3$ manifold is[7] $\chi = \int_{K3} R \wedge R = 24$. The $K3$-orbifold $T^4/\mathbb{Z}_2$ is smooth everywhere except at the orbifold points. Thus the curvature term is concentrated at the sixteen fixed-points of $T^4/\mathbb{Z}_2$ and we get $(R \wedge R)_i = \frac{\chi}{16} = 3/2$ and

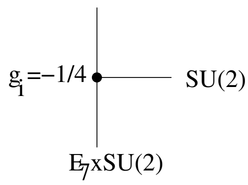
$$g_i = \int_{T^4/\mathbb{Z}_2} (F \wedge F)_i - \frac{3}{4}. \qquad (3.15)$$

In the case of a gauge group $SO(16) \times [E_7 \times SU(2)]$ the charge was determined in the last section in eqn. (3.12) as $g_i = \pm 1/4$. From this one can easily determine the instanton number $c_i = \int (F \wedge F)_i$,

$$SO(16)_{(10-plane)} \times SU(2)_{(7-plane)} : \qquad c_i = 1, \qquad (3.16)$$



$$E_7 \times SU(2)_{(10-plane)} \times SU(2)_{(7-plane)} : \qquad c_i = \frac{1}{2}, \qquad (3.17)$$



Pointlike instantons are explained in appendix A.4.2. The point-like instantons located at the $\mathbb{Z}_2$ fixed points of the theory and thus have non-trivial holonomy. This will turn out to be crucial for the construction of the dual F-theory compactification in the next chapter.Note that the sum of the magnetic charges of one $SO(16)$ and one E_7 vertex vanishes. This means in particular that in the limit of a weak coupling constant, or small eleventh direction, the magnetic source vanishes on each fixed point of $K3$ as expected from string theory. After glueing together sixteen pairs (3.16) and (3.17) we have an overall vanishing magnetic charge, which is a necessary condition for anomaly cancellation.

The main result of the analysis of [90, 89] reviewed above is the existence of additional $SU(2)$ gauge groups on the seven-planes. It is not clear however how to extract the single heterotic $SU(2)^{\text{het}}$ gauge group from the seventeen $SU(2)$ gauge groups, one perturbative $SU(2)$ on the ten-plane and sixteen non-perturbative $SU(2)$'s on the seven-planes, of M-theory [8]. Reference [89] suggests that the heterotic gauge group is the diagonal subgroup

$$SU(2)^{\text{het}} = diag[SU(2)^{\text{pert}} \times (SU(2)^{\text{non-pert}})^{16}], \qquad (3.18)$$

[7]The second Chern class of a two-fold is equal to the Euler number of the manifold, see appendix A.2.2

[8]The gauge groups on the seven-planes are called non-perturbative because the seven-planes, and thus the additional gauge groups do not exist in the perturbative weakly coupled heterotic string. The gauge groups on the ten-planes do exist in the weakly coupled theory, thus they are called perturbative.

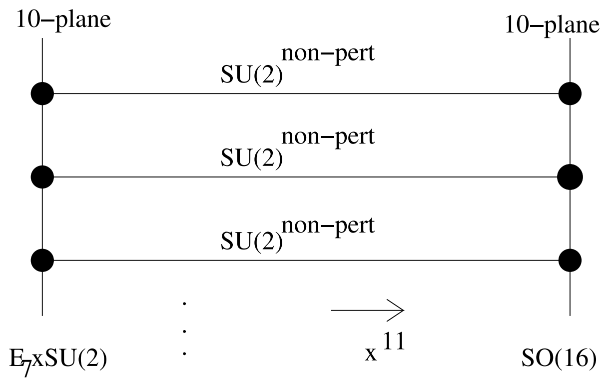


Figure 2: First possibility: The diagonal subgroup of sixteen non-perturbative $SU(2)^{\text{non-pert}}$ gauge groups and the perturbative $SU(2)$ is identified with the heterotic $SU(2)^{\text{het}}$

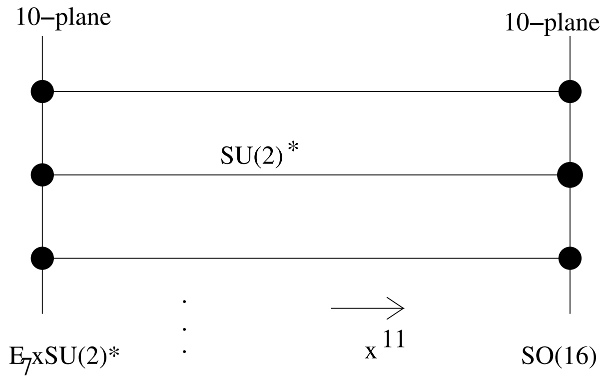


Figure 3: Second possibility: The gauge group $SU(2)^* = SU(2)^{\text{het}}$ with non-perturbative and perturbative contributions

and each of the sixteen twisted states transforms as $(\mathbf{16}, \mathbf{2})$ of $SO(16) \times SU(2)^{\text{non-pert}}$. This is sketched in figure 2.

Reference [90] makes another suggestion, which is to identify the perturbative $SU(2)$ and the sixteen non-perturbative $SU(2)^{\text{non-pert}}$'s to form a single gauge group

$$SU(2)^{\text{het}} = SU(2)^* \tag{3.19}$$

as is sketched in figure 3.

The question of the correct identification of the heterotic $SU(2)$ has not been considered so far and this is the central point of this chapter. To address this problem we construct the dual F-theory formulation in the following.

3.4 The Dual F-Theory Formulation

As explained in the introduction, F-theory has been a useful tool in the past for gaining insight into the non-perturbative behaviour of string theories. Following this idea, we consider the F-theory formulation of the strongly coupled heterotic string and find out that F-theory indeed resolves the puzzle about the heterotic $SU(2)$ gauge group.

F-theory compactified on an elliptically fibred manifold M with base B is defined as IIB theory compactified on B, where the complex structure of the elliptic fibre of M is identified with the complex dilaton of type IIB theory. From the duality (IIA theory $\leftrightarrow$ M-theory on S^1), one obtains via compactification on a circle and T-duality in one direction the duality (IIB on $S^1 \leftrightarrow$ M-theory on T^2) of type IIB theory on a circle S^1 and M-theory on a torus. The non-perturbative formulation of this duality in terms of F-theory is

$$\mathrm{F-theory\ on}\ N \leftrightarrow \mathrm{M-theory\ on}\ T^2, \tag{3.20}$$

where N is an elliptic fibration over S^1. This duality can be used to construct the F-theory formulation of the strongly coupled heterotic string. In the case that is relevant in this section, M-theory is compactified on $K3 \times I$ in the the orbifold limit of the $K3$ manifold. Note that the orbifold limit is a singular limit of an elliptically fibered $K3$ manifold. Thus it is justified to consider an elliptically fibred $K3$. We have to take the fibration of (3.20) over a $\mathbb{P}^1$ and in addition include the extra dimension of the interval I of the M-theory side. This duality was studied in [104]. It was found that the dual F-theory is compactified on an elliptically fibred threefold Y_3 which is also a $K3$ fibration over $\mathbb{P}^1$. This manifold is an elliptic fibration over a Hirzebruch surface, which is a $\mathbb{P}^1$ fibred over $\mathbb{P}^1$. It was shown in [104] the manifold obtained by squeezing the fibre $\mathbb{P}^1$ of the threefold to an interval I can be identified with the M-theory compactification manifold. Thus one has the duality

$$\mathrm{F-theory\ on}\ Y_3 \leftrightarrow \mathrm{M-theory\ on}\ K3 \times I. \tag{3.21}$$

We use this duality in the orbifold limit $T^4/\mathbb{Z}_2$ of the K3. The duality between F-theory and M-theory in the context of the heterotic string has also been studied for example in [97, 99, 100].

We consider the F-theory formulation of the strongly coupled heterotic string with gauge group $SO(16) \times [E_7 \times SU(2)]$. The goal is to find an answer to the question of the correct identification of the heterotic $SU(2)$ gauge group discussed at the end of section 3.3.

To obtain the six-dimensional theories, we assume that F-theory is compactified on an elliptically fibered Calabi-Yau threefold with a section (see appendix A.2.2) and the heterotic string is compactified on an elliptically fibred $K3$. We will explain at the end of this section that this assumption is not quite correct. However, it simplifies the problem significantly and gives a good description of the theory locally around each fixed point. It turns out that this is enough to understand the $SU(2)$ gauge groups. We keep the problem as simple as possible and restrict ourselves to the classical geometry of the heterotic $K3$ manifold, which means that both the base $\mathbb{P}^1$ and the elliptic fibre are large.

To understand the effect of the large volume limit of the base $\mathbb{P}^1$ in the dual F-theory, we consider the dual pair IIA string theory compactified on a threefold and the heterotic string on K3$\times T^2$. The Kähler modulus of the base of the heterotic $K3$ is mapped to the IIA dilaton. Thus in the large area limit of the base of the heterotic $K3$ the spacetime instanton corrections in the dual IIA theory are suppressed. Decompactifying the heterotic T^2 leads to the duality between F-theory on a Calabi-Yau three-fold and the heterotic string on K3.

To understand the effect large volume limit of the elliptic fibre, we consider eight-dimensional F-theory on a K3 manifold and the heterotic string on T^2 in the large volume limit of the heterotic torus. Below, we are going to fibre both manifolds over a $\mathbb{P}^1$ to regain the six-dimensional theories. The elliptically fibred F-Theory K3 manifold can be written in Weierstrass form (see appendix A.2.1)

$$y^2 = x^3 + a(s)x + b(s), \tag{3.22}$$

where s is a coordinate on the projective space $\mathbb{P}^2$ describing the elliptic fibre and $a(s)$ and $b(s)$ are polynomials of degree 8 and 12 in s. As explained in appendix A.3.2 , the elliptic fibre is singular if $\delta = 24(b(s))^2 + 4(a(s))^3 = 0$. This is a polynomial of degree 24 and thus $\delta = 0$ has generically 24 solutions. As explained in appendix A.3.2, several singular fibres located at the same point s lead to non-trivial gauge groups. The singular fibres can be classified due to the order $O(a), O(b)$ and $O(\delta)$ of which the polynomials a, b and δ vanish at the point s of the singular fibre, see table A.3.2. From table A.3.2 we see that in order to obtain a gauge group $SO(16) \times [E_7 \times SU(2)]$, we have to put one I_4^* fibre at $s = 0$ and one II^* plus one I_2 fibre at $s = \infty$. The remaining three singular fibres are I_1 fibres and located at finite points s.

Taking the large volume limit of the heterotic torus means pushing two I_1 singularities to $s \to 0$ and the other I_1 singularity to $s \to \infty$. These are the same points at which the singular fibres generating the gauge groups are located. This results in two singularities of degree 12 at $s = 0$ and $s = \infty$. As explained in A.3.2, singularities of degree 12 are not possible for Calabi-Yau manifolds. Thus the points $s = 0$ and $s = \infty$ require a blow-up. This blow up is explained in detail in appendix A.3.3 and leads to the stable degeneration of the F-theory threefold, as was first shown in [106] and explained in detail in [107]. The base $\mathbb{P}^1$ degenerates into two $\mathbb{P}^1$'s intersecting at a point C_*. One $\mathbb{P}^1$ has one I_4^* fibre and two I_1 fibres and the other $\mathbb{P}^1$ has one III^*, one I_2 and one I_1 fibre. The important point for the heterotic/F-theory duality is now that the fibre at the intersection point C_* has to be identified with the heterotic torus T^2.

The next step is to compactify further to six dimensions by taking the fibration of both the heterotic torus and the degenerated F-theory K3 over a base $\mathbb{P}^1$. The F-theory threefold is an elliptic fibration over a Hirzebruch surface $\mathbb{F}_n$, which has degenerated to two intersecting $\mathbb{F}_n$. The intersection locus is now a curve instead of a point. This curve together with the fibres along that curve form the K3 manifold which is to be identified with the heterotic K3 in the large volume limit of the fibre.

In the language of divisors, $\mathbb{F}_n$ (before the stable degeneration) is described by the exceptional section C_0 with self intersection number $C_0 \cdot C_0 = -n$ and the class of divisors f of the $\mathbb{P}^1$ fibre of $\mathbb{F}_n$ with self-intersection number $f \cdot f = 0$. The section C_0 intersects each fibre once, $f \cdot C_0 = 1$. We denote the divisor associated to the conormal bundle of the elliptic fibre by L, as explained in appendix A.2.2. L is related to the canonical class $K_{\mathbb{F}_n}$ of the base $\mathbb{F}_n$ by $L = -K_{\mathbb{F}_n}$. This follows from the fact that the canonical class of the Calabi-Yau threefold X is zero and from the adjunction formula (see appendix A.2.2)

$$K_X = \pi^*(K_{\mathbb{F}_n} + L), \tag{3.23}$$

where $\pi : X \to \mathbb{F}_n$ is the elliptic fibration of X. As explained in appendix A.2.2, the divisor L is given by

$$L = 2C_0 + (n + 2)f. \tag{3.24}$$

We denote by A, B, and Δ the divisors associated to the polynomials $a = 0$, $b = 0$ and $\delta = 0$ of the Weierstrass form. These divisors are fixed by the above line bundle,

$$A = 4L, \quad B = 6L, \quad \Delta = 12L. \tag{3.25}$$

The $SO(16)$ and E_7 singularities are put along the sections C_0 and $C_\infty = C_0 + nf$. They do not intersect each other. Before we go into the details of our model we take the stable degeneration limit. Instead of one we have two intersecting $\mathbb{F}_{n,1}$ and $\mathbb{F}_{n,2}$, which have a section C_0 and $C_* = C_0 + nf$ each. $\mathbb{F}_{n,1}$ and $\mathbb{F}_{n,2}$ intersect along C_*. The line bundle L of each of the intersecting threefolds is given by (3.24) minus the intersection curve $-C_*$ (see appendix A.3.3),

$$L = C_0 + 2f. \tag{3.26}$$

This implies that the divisors defined by the polynomials $a = 0$, $b = 0$ and $\delta = 0$ are

$$\begin{aligned} A &= 4C_0 + 8f, \\ B &= 6C_0 + 12f, \\ \Delta &= 12C_0 + 24f \end{aligned} \tag{3.27}$$

in the stable degeneration limit. Note that we consider only one of the two intersecting manifolds. This tells us that in the stable degeneration limit each of the intersecting threefolds can be described in Weierstrass form

$$y^2 = x^3 + a(s,t)x + b(s,t), \tag{3.28}$$

where s, t parametrize the base $\mathbb{P}^1$ and the fibre $\mathbb{P}^1$, the curve $a(s,t) = 0$ is the above divisor $A = 4C_0 + 8f$ and the curve $b(s,t) = 0$ is the divisor $B = 6C_0 + 12f$. Thus we know for example from (3.27) that a is a polynomial of degree 4 in s and b is a polynomial of degree 6 in s.

The next step is to find the explicit form of the polynomials a and b of (3.28) which describe that model. Reproducing the correct gauge group restricts the possible form of the polynomials. In addition we know from the last section that the dual M-theory has non-vanishing instanton numbers on each orbifold fixed point. Being located at the fixed points means that the instantons have a discrete $\mathbb{Z}_2$ holonomy. Thus we are looking for an F-theory compactification manifold which also has instantons with a discrete $\mathbb{Z}_2$ holonomy. This class of models, i.e. F-theory compactifications on orbifolds including instantons with discrete holonomy, were considered in [110]. For the sake of clarity and compactness, we do not give an explanation of the rather involved calculations done in [110] but use the results only. The stable degeneration of the F-theory threefold including instantons with $\mathbb{Z}_2$ holonomy has the Weierstrass form

$$\begin{aligned} a(s,t) &= a_4(s,t) - \frac{1}{3}a_2(s,t)^2, \\ b(s,t) &= \frac{1}{27}a_2(s,t)(2a_2(s,t)^2 - 9a_4(s,t)), \\ \delta(s,t) &= a_4(s,t)^2(4a_4(s,t) - a_2(s,t)^2), \end{aligned} \tag{3.29}$$

where s, t parametrize the base $\mathbb{F}_n$ and the subscripts denote the degree of the polynomials a, b under rescaling $(x, y) \rightarrow (\lambda^2 x, \lambda^3 y)$ of the Weierstrass equation (3.22). As expected for the manifold after the stable degeneration, $a(s,t)$ is of degree 4 and $b(s,t)$ is of degree 6.

The curves of the $SO(16)$ and E_7 singularities after the stable degeneration are located along C_0 of $\mathbb{F}_{n,1}$ and $\mathbb{F}_{n,2}$. Let us build the model which includes the fractional instantons with $\mathbb{Z}_2$ holonomy considered in (3.29). $SO(16)$ and $E_7 \times SU(2)$ are indeed the only possible gauge groups for models with a discrete $\mathbb{Z}_2$ holonomy, because they commute with the holonomy group of the instantons. This is explained in appendix A.4.2 in greater detail. Placing the perturbative gauge groups $SO(16)$ and E_7 along the curves $s = 0$ and $s = \infty$ fixes the shape of the remaining $SU(2)$ curve as we show in the following. We denote by c_0 the polynomial associated to C_0. In other words C_0 is defined by the curve $c_0 = 0$.

Let us first consider the half-plane $\mathbb{F}_{n,1}$ with gauge group $SO(16)$. The polynomials a, b and δ vanish to orders $2, 3$ and 10 on C_0. To reproduce this in the Weierstrass model, the polynomials a_2 and a_4 have to be of the form

$$\begin{aligned} a_2(s,t) &= c_0(s,t) g(s,t) \\ a_4(s,t) &= c_0(s,t)^4 h(s,t), \end{aligned} \tag{3.30}$$

where $g = 0$ is a divisor in the class $C_0 + 4f$ and $h = 0$ is a curve in the class $8f$. As the curves in the class $8f$ generically split into eight distinct fibres the polynomial is of the form $h = f_1 \dots f_8$. Thus the Weierstrass model (3.29) is

$$\begin{aligned} a(s,t) &= c_0(s,t)^2 \left(c_0(s,t)^2 f_1(s,t) \dots f_8(s,t) - \frac{1}{3} g(s,t)^2 \right) \\ b(s,t) &= \frac{1}{27} c_0(s,t)^3 g(s,t) \left(2g(s,t)^2 - 9c_0(s,t)^2 f_1(s,t) \dots f_8(s,t) \right), \\ \delta(s,t) &= c_0(s,t)^{10} f_1(s,t)^2 \dots f_8(s,t)^2 \left(4c_0(s,t)^2 f_1(s,t) \dots f_8(s,t) - g(s,t)^2 \right) \end{aligned} \tag{3.31}$$

In addition to the perturbative gauge group $SO(16)$, there are eight curves $f_i = 0$ that vanish of degree $O(a) = 0$, $O(b) = 0$ and $O(\delta) = 2$. Comparison with table A.3.2 tells us that we have eight I_2 fibres which lead to eight additional $SU(2)$ gauge groups. The remaining discriminant

$$\delta' = \delta(s,t) / \left(c_0(s,t)^{10} f_1(s,t)^2 \dots f_8(s,t)^2 \right) = 4c_0(s,t)^2 f_1(s,t) \dots f_8(s,t) - g(s,t)^2 \tag{3.32}$$

describes a curve of I_1 fibres and does not lead to any further gauge symmetry. The discriminant curve $\delta' = 0$ is a divisor in the class $2C_0 + 8f$. Intersection theory tells us that the discriminant curve intersects C_0 exactly $(2C_0 + 8f) \cdot C_0 = (8 - 2n)$ times. From (3.32) we see that at $c_0 = 0$ (on C_0), $f_1^2 \dots f_8^2$ =constant the discriminant δ' vanishes quadratically. Thus the discriminant curve has really $(4 - n)$ intersections with C_0 of degree 2. Each fibre intersects the discriminant curve $(2C_0 + 8f) \cdot f = 2$ times, but from (3.32) we see that there is really one intersection of degree 2 at each curve $f_i = 0$, c_0 finite. Finally the discriminant curve has $(2C_0 + 8f) \cdot (2C_0 + nf) = 8$ transversal intersections with C_*. This is illustrated in figure 4.

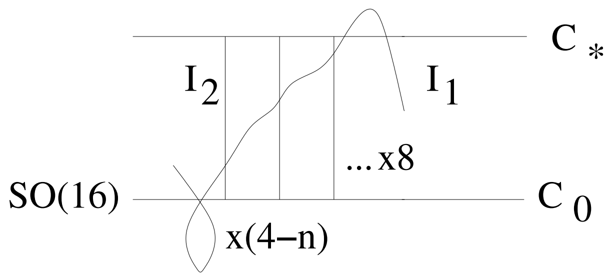


Figure 4: $\mathbb{F}_{n,1}$ with a perturbative gauge group $SO(16)$ and instantons with $\mathbb{Z}_2$ holonomy

The half-plane $\mathbb{F}_{n,2}$ with gauge group E_7 has vanishing polynomials $O(a) = 3, O(b) = 5$ and $O(\delta) = 9$. The polynomials take the form

$$\begin{aligned} a_2(s,t) &= c_0(s,t)^2 g(s,t) \\ a_4(s,t) &= c_0(s,t)^3 h(s,t), \end{aligned} \tag{3.33}$$

where $g = 0$ is a curve in the class $4f$ and $h = 0$ is a curve in the class $(C_0 + 8f)$. In the Weierstrass formulation we have

$$\begin{aligned} a(s,t) &= c_0(s,t)^3 \left(h(s,t) - \frac{1}{3} c_0(s,t) g(s,t)^2 \right) \\ b(s,t) &= \frac{1}{27} c_0(s,t)^5 g(s,t) \left(2c_0(s,t) g(s,t)^2 - 9h(s,t) \right), \\ \delta(s,t) &= c_0(s,t)^9 h(s,t)^2 \left(4h(s,t) - c_0(s,t) g(s,t)^2 \right). \end{aligned} \tag{3.34}$$

In addition to the E_7 gauge symmetry on the curve $c_0 = 0$, there are I_2 fibres and thus an $SU(2)$ gauge group along the curve $h(s,t) = 0$. The curve $h = 0$ is in the class $(C_0 + 8f)$. The remaining discriminant is

$$\delta' = \delta(s,t)/c_0(s,t)^9 h(s,t)^2 = 4h(s,t) - c_0(s,t) g(s,t)^2 \tag{3.35}$$

and $\delta' = 0$ describes a curve of I_1 fibres in the class $(C_0 + 8f)$, see figure 5. The curve $\delta' = 0$ has $(C_0 + 8f) \cdot C_0 = (8 - n)$ transversal intersections with C_0. The number of intersections of C_0 with the curve $h = 0$ with I_2 fibres is $(C_0 + 8f) \cdot C_0 = (8 - n)$. The curve $\delta' = 0$ with I_1 fibres intersects the curve $h = 0$ with I_2 at $(C_0 + 8f) \cdot (C_0 + 8f) = 8$ points. Each intersection point is of degree 2 as for $h = 0$, $c_0 =$constant the discriminant δ' vanishes quadratically. Thus we have really four intersection of degree 4. Both curves with the I_1 and the I_2 fibres intersect C_* eight times transversally.

To regain the complete degenerated threefold, we have to glue the two half planes together along C_* such that each intersection of the $SU(2)$ curve on the E_7 half-plane with C_* is also an intersection point of an $SU(2)$ curve on the $SO(16)$ half-plane with C_*. The same is true for the I_1 curves. C_* has 24 singular fibres as is necessary for the $\mathbb{P}^1$ base of a $K3$ manifold.

Let us first analyze the $SU(2)$ gauge groups. We see that the E_7 half-plane has just one $SU(2)$ gauge group, whereas the $SO(16)$ half-plane has eight separate non-perturbative $SU(2)$'s. After glueing together the half-planes as described above the $SU(2)$ curves are connected and the manifold has only one $SU(2)$ curve with components both in

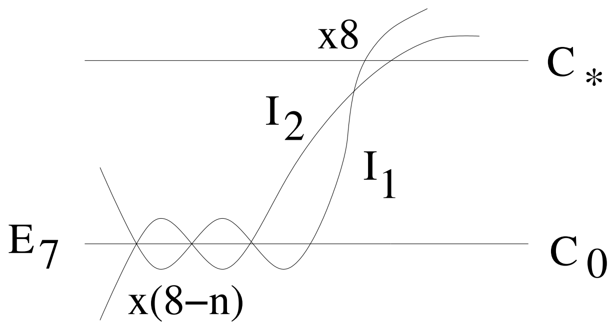


Figure 5: $\mathbb{F}_{n,2}$ with perturbative gauge group E_7 and instantons with $\mathbb{Z}_2$ holonomy

the fibre and the base direction of $(\mathbb{F}_{n,1} \vee \mathbb{F}_{n,2})$. The same should be true for the dual M-theory formulation: To take the M-theory limit of the F-theory model we have to shrink all two-cycles in the fibre $\mathbb{P}^1$ to one-cycles to create a one-dimensional interval. Such a compactification manifold is often called squeezed manifold in the literature [104, 105]. This leaves us with a five-dimensional compactification manifold and the interval is identified with the M-theory interval $S^1/\mathbb{Z}_2$. Note that this shrinking process includes the stable degeneration limit. In the stable degeneration limit a one-cycle in the fibre $\mathbb{P}^1$ is shrunk to zero size and we have two intersecting $\mathbb{P}^1$'s. If we shrink all one-cycles in the same class to zero size instead of just one, we squeeze the $\mathbb{P}^1$ to the one-dimensional interval which is identified with the M-theory $S^1/\mathbb{Z}_2$.

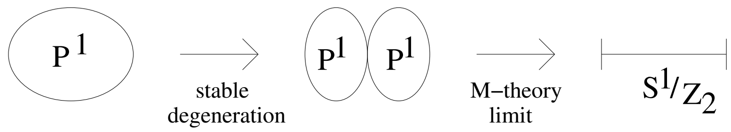


Figure 6: The stable degeneration and M-theory limit of the $\mathbb{P}^1$ fibre of $\mathbb{F}_n$

The result we get from considering the dual F-theory is that we really have only one $SU(2)$ gauge group in M-theory. This seems to be in agreement with the picture suggested by [90] and the heterotic gauge group is really $SU(2)^{\text{het}} = SU(2)^*$.

Let us analyze the above F-theory model in greater detail. We can count the instanton numbers at the fixed points in the following way: The half planes have a total instanton number of $c = 12 - n$ each, where n characterizes the base $\mathbb{F}_n$. On the $SO(16)$ half-plane the I_1 curve intersects C_0 $(4 - n)$ times, these points are $(4 - n)$ pointlike instantons of charge 1. The remaining fractional instantons with discrete holonomy are located at the fixed points of the dual heterotic theory [110]. Every intersection of the I_2 curve with C_* is such a $\mathbb{Z}_2$ fixed point. There are eight fixed points in our case. Thus to get a resulting instanton number of $c = 12 - n$ we need to have eight fractional instantons with charge $c = 1$ each. This is locally consistent with the M-theory model close to a fixed point as anomaly cancellation requires an instanton number $F \wedge F = c = 1$ at each fixed point as sketched in (3.16). The counting works equivalently on the other half plane with gauge group E_7. There are $(8 - n)$ pointlike instantons of charge one and eight fractional instantons at the fixed points with charge $c = 1/2$ each. This is consistent again with the M-theory formulation in (3.17).

Although the instanton numbers are locally correct, they do not describe the model we want globally. The total fractional instanton number on the $SO(16)$ plane in the

M-theory formulation is $c = 16$, whereas the F-theory model has a total $c = 8$. The total fractional instanton number on the E_7 plane is $c = 8$ in M-theory but $c = 4$ in our F-theory model.

To put it another way the heterotic theory dual to our F-theory model has only eight fixed points. As $T^4/\mathbb{Z}_2$ has sixteen fixed points, our model is clearly not dual to the heterotic string compactified on $T^4/\mathbb{Z}_2$ or to its strong coupling limit M-theory on $T^4/\mathbb{Z}_2 \times I$, but describes a different model. To resolve that problem we take the limit in which two intersection points of curves with I_2 fibres with C_* and two intersection points of curves with I_1 fibres with C_* are pushed to the same point. We take the same limit C_0, two curves with I_2 fibres and two curves with I_1 fibres intersect C_0 at the same point. This is possible only if we fix $n = 4$ on the E_7 half-plane and $n = -4$ on the $SO(16)$ plane. Before the stable degeneration, this corresponds to choosing the base of the threefold to be $\mathbb{F}_n$ with $n = 4$. The four intersection points along C_* now have singularities of type D_4 and give an $SO(8)$ gauge group each. Every fixed-point contributes an instanton number $c = 4$ on the $SO(16)$ half-plane and $c = 2$ on the E_7 half-plane. Each singularity along C_* can now be blown up resulting in four $SU(2)$ singularities. There is no Weierstrass formulation of that model, but we can see that we end up with four $\mathbb{Z}_2$ singularities at four points in C_*. Thus the heterotic $K3$ manifold we construct from C_* after the blow-up and the fibres along that curve has sixteen $\mathbb{Z}_2$ singularities, this is the $T^4/\mathbb{Z}_2$ orbifold we wanted. The instanton numbers at the fixed points are correct both locally and globally, we have $c = 1$ for each of the sixteen fixed points on the $SO(16)$ half plane and $c = 1/2$ for each of the sixteen fixed points on the E_7 half plane. We have found the F-theory formulation of the strong coupling limit of the heterotic string compactified on $T^4/\mathbb{Z}_2$ with gauge group $SO(16) \times [E_7 \times SU(2)]$.

3.5 Other Gauge Groups

In this section we apply the above procedure to models with other gauge groups. M-theory compactified on $T^4/\mathbb{Z}_2 \times S^1/\mathbb{Z}_N$ with many different gauge groups is considered in [89],[90].

3.5.1 M-Theory on $S^1/\mathbb{Z}_2 \times T^4/\mathbb{Z}_N$ with other Gauge Groups

To construct a consistent M-Theory compactification on $S^1/\mathbb{Z}_2 \times T^4/\mathbb{Z}_2$, it is necessary to fulfill the local anomaly cancellation conditions on each six-plane, as was explained for gauge group $SO(16)\times[E_7\times SU(2)]$. The anomaly cancellation conditions fix the magnetic charge g_i and thus the instanton number on each six-plane and one can construct vertices of the form (3.16) and (3.17). Starting with the perturbative gauge group one can derive the seven-plane gauge group from six-dimensional anomaly cancellation. To obtain a model which is also globally consistent, that is with a vanishing overall magnetic charge, one has to glue together the vertices such that the total magnetic charge vanishes. This is not always possible in perturbative compactifications and in some cases five-branes have to be included. Apart from the two consistent vertices (3.16) and (3.17) there are additional ones with gauge groups $E_8^{\mathrm{pert}} \times U(1)^{\mathrm{non-pert}}$ and magnetic charge $g_i = -3/4$ and $[E_7 \times SU(2)]^{\mathrm{pert}} \times U(1)^{\mathrm{non-pert}}$ with $g_i = 3/4$ on each six-plane, where the $U(1)$

gauge group in both cases can be either an $SU(2)$ broken to $U(1)$ or a $U(1)$ from the beginning[9].

Consider the compactification with a perturbative gauge group $E_8 \times E_8$ [90, 89]. The untwisted spectrum includes the vector multiplets transforming as $(\mathbf{248}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{248})$ of the unbroken gauge group, four neutral supergravity moduli and the universal tensor multiplet. Anomaly cancellation determines the gauge group on the seven-planes to be $(U(1))^{16}$ which can be either sixteen $SU(2)$ gauge groups broken to $U(1)$ or sixteen $U(1)$ gauge groups from the start. Both theories are consistent and equivalent from the point of view of anomaly cancellation. The spectrum consists of sixteen neutral vector multiplets surviving the projection on the seven-plane and of thirty-two neutral twisted hypermultiplets on the six-planes. The fixed six-planes have zero instanton number and thus a magnetic charge of $g_i = -3/2$ each. To cancel this charge it is necessary to include additional 24 five-branes with magnetic charge $g = 1$ each. The five-branes are generically located away from the fixed-planes and lead to additional 24 tensor multiplets in the spectrum. This model is an example of a non-perturbative compactification due to the presence of the five-branes.

Another perturbative example is the strong coupling limit of the standard embedding of the heterotic string with a perturbative gauge group $E_8 \times [E_7 \times SU(2)]$ [90, 89]. The untwisted spectrum consists of vector multiplets transforming as $(\mathbf{248}, \mathbf{1}, \mathbf{1})$ from the unbroken E_8 and of vector and hypermultiplets transforming as $(\mathbf{1}, \mathbf{133}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{3}) \oplus (\mathbf{1}, \mathbf{56}, \mathbf{2})$ from the other ten-plane with gauge group $E_8 \to [E_7 \times SU(2)]$. The non-perturbative gauge group on the seven-planes can be derived again locally on each fixed six-plane first. For the $[E_7 \times SU(2)]$ vertices we get either the picture (3.17) with a non-perturbative $SU(2)$ gauge group on the seven-plane or a $U(1)$ which can be either an $SU(2)$ gauge group broken to $U(1)$ or a $U(1)$ from the start. The E_8 vertex has either a $U(1)$ gauge group on the seven-plane or an $SU(2)$ broken to $U(1)$ as mentioned in the last example with unbroken $E_8 \times E_8$ gauge group. If one E_8 and one $[E_7 \times SU(2)]$ vertex are glued together, there has to be the same gauge group on the seven-plane which connects the two vertices. This rules out the gauge group $U(1)$ on each seven-plane, which can possibly be an $SU(2)$ broken to $U(1)$. If one restricts to the case of a $U(1)$ gauge group on the seven-planes from the start, the spectrum contains in addition sixteen neutral vector multiplets, sixteen twisted neutral hypermultiplets on the E_8 six-plane and sixteen hypermultiplets transforming as $[\frac{1}{2}(\mathbf{1}, \mathbf{56}, \mathbf{1}) \oplus \mathbf{2}(\mathbf{1}, \mathbf{1}, \mathbf{2})]$ on the $[E_7 \times SU(2)]$ six-planes. The case where the seven-plane gauge group is $SU(2) \to U(1)$ has not been considered explicitly but has the same spectrum as the case with the $U(1)$ gauge group on the seven-planes and does lead to the same results.

None of the last two examples has twisted states which are charged under both ten-plane gauge groups. These states only exist in the first example with gauge group $SO(16) \times [E_7 \times SU(2)]$. To be more general, this kind of twisted states only exist if the non-perturbative gauge group coincides with one factor in the perturbative gauge groups on the ten-planes, in this case there is a non-perturbative as well as a perturbative $SU(2)$. This can be verified also in compactifications on higher-order orbifolds. One example is the compactification of M-theory on the orbifold $\mathbb{R}^{10} \times T^4/\mathbb{Z}_3 \times S^1/\mathbb{Z}_2$

[9]For each vertex mentioned so far there is another one with the same gauge group but a magnetic charge $g_i + 1$, these are also considered in [90]. These vertices have an additional point-like instanton with charge one pushed into the six-plane.

[89, 90, 110] with gauge group $SU(9) \times [E_6 \times SU(3)]$. The orbifold $T^4/\mathbb{Z}_3 \times S^1/\mathbb{Z}_2$ has eighteen fixed six-planes which are the intersections of the nine fixed seven-planes of the $\mathbb{Z}_3$ orbifold and the two fixed ten-planes of the $\mathbb{Z}_2$ orbifold. In addition to the perturbative gauge group there are nine non-perturbative $SU(3)$ gauge groups on the seven-planes. The untwisted spectrum on the ten-plane with gauge group $E_8 \to SU(9)$ has vector multiplets transforming as $(\mathbf{80})$ and hypermultiplets transforming as $(\mathbf{84})$. On the other ten-plane the vector multiplets transform as $(\mathbf{78},\mathbf{1}) \oplus (\mathbf{1},\mathbf{8})$ and the hypermultiplets transform as $(\mathbf{27},\mathbf{3})$ of $[E_7 \times SU(2)]$. The untwisted spectrum also includes the universal tensor multiplet and two neutral moduli which are hypermultiplets. The twisted states are localized on the six-planes with the perturbative $SU(9)$ gauge group and are charged under $SU(9)^{\mathrm{pert}} \times SU(3)^{\mathrm{non-pert}}$. The situation is similar to that of the $\mathbb{Z}_2$ orbifold with gauge group $SO(16) \times [E_6 \times SU(3)]$. It is not obvious again to determine whether there is a single gauge group $SU(3)$ on the five-dimensional compactification manifold or whether there are seventeen $SU(3)$'s whose diagonal subgroup is identified with the heterotic $SU(3)$.

Finally we briefly mention a model which was considered in [94, 95, 96] and in [97] in the context of the duality between M-theory on $\mathbb{R}^{10} \times T^4/\mathbb{Z}_2 \times S^1/\mathbb{Z}_2$ and F-theory compactified on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. The gauge group is $(SO(8))^8$, four of which are perturbative and four are non-perturbative and put along the seven-planes. We come back to this model at the end end of the next section and show how this model is connected to the compactifications with sixteen non-perturbative $SU(2)$ gauge groups considered above.

In the next section we consider F-theory compactifications on Calabi-Yau threefolds dual M-theory models described above.

3.5.2 The Dual F-Theory Formulation with other Gauge Groups

Let us first consider the the strong coupling limit of the heterotic string on $T^4/\mathbb{Z}_3$ with a gauge group $[E_6 \times SU(3)] \times SU(9)$ mentioned above. In the M-theory picture, anomaly cancellation requires a non-perturbative $SU(3)$ gauge group on each fixed seven-plane. Thus the non-perturbative gauge group coincides with a factor of the perturbative gauge group. The twisted states of the heterotic theory transform as $(\mathbf{1},\mathbf{3},\mathbf{9})$ and are located at the nine six-planes which are the intersection of the $SU(9)$ ten-plane with the seven-planes. The i-th fixed six-plane carries a magnetic charge

$$g_i = (F \wedge F - \frac{1}{2} R \wedge R)_i = (F \wedge F)_i - 4/3 \tag{3.36}$$

due to the nine fixed points of $T^4/\mathbb{Z}_3$.

Again one can ask the question whether there is one single $SU(3)$ or ten $SU(3)$'s whose diagonal subgroup is identified with the heterotic $SU(3)$. The methods used in 3.4 easily extend to this case. We consider F-theory compactified on a threefold including fractional instantons, this time with a discrete $\mathbb{Z}_3$ holonomy. As mentioned in [97], due to the holonomy group, the only possible perturbative gauge groups are $E_6 \times SU(3)$ and $SU(9)$. The polynomials a, b and δ are of the form

$$a \;=\; a_1(s,t) \left(\frac{1}{2} a_3(s,t) - \frac{1}{48} a_1(s,t)^3 \right)$$

$$
\begin{aligned}
b &= \frac{1}{4}a_3(s,t)^2 + \frac{1}{864}a_1(s,t)^6 - \frac{1}{24}a_1(s,t)^3 a_3(s,t) \\
\delta &= \frac{1}{16}a_3(s,t)^3(27a_3(s,t) - a_1(s,t)^3)
\end{aligned} \tag{3.37}
$$

Again the variables s, t parametrize the base $\mathbb{F}_n$ of the threefold and the subscripts denote the degree of the coefficients under rescaling $(x, y) \to (\lambda^2 x, \lambda^3 y)$ of the Weierstrass equation.

On the half-plane with a perturbative gauge group E_6 along C_0 the polynomials a, b and δ vanish to order $O(a) = 3$, $O(b) = 4$ and $O(\delta) = 8$. The polynomials $a_1(s,t)$, $a_3(s,t)$ are of the form

$$
\begin{aligned}
a_1(s,t) &= c_0(s,t)g(s,t) \\
a_3(s,t) &= c_0(s,t)^2 h(s,t),
\end{aligned} \tag{3.38}
$$

where $g = 0$ is a curve in the class $2f$ and $h = 0$ is a curve in the class $(C_0 + 6f)$. The Weierstrass equation (3.37) is [10]

$$
\begin{aligned}
a &= c_0(s,t)^3 g(s,t)\left(\frac{1}{2}h(s,t) - \frac{1}{48}c_0(s,t)g(s,t)^3\right) \\
b &= c_0(s,t)^4\left(\frac{1}{4}h(s,t)^2 + \frac{1}{864}c_0(s,t)^2 g(s,t)^6 - \frac{1}{24}c_0(s,t)g(s,t)^3 h(s,t)\right) \\
\delta &= \frac{1}{16}c_0(s,t)^8 h(s,t)^3\left(27h(s,t) - c_0(s,t)g(s,t)^3\right)
\end{aligned} \tag{3.39}
$$

We have one curve with I_3 fibres at $h = 0$ in the class $(C_0 + 6f)$ which leads to an additional $SU(3)$ gauge group. From the discriminant

$$
\delta' = \delta / \frac{1}{16}c_0(s,t)^8 h(s,t)^3 = 27h(s,t) - c_0(s,t)g(s,t)^3 \tag{3.40}
$$

we extract that the curves with I_1 fibres are in the class $(C_0 + 6f)$ and intersect C_0 $(C_0 + 6f) \cdot C_0 = (6 - n)$ times. Also a curve with I_3 fibres intersects C_0 $(C_0 + 6f) \cdot C_0 = (6 - n)$ times. The I_1 intersects the I_3 curve twice of degree 3 as $(C_0 + 6f) \cdot (C_0 + 6f) = 6$ and the discriminant δ' vanishes to order 3 at $h = 0$, c_0 constant. Both the I_1 and I_3 curves intersect C_* six times. We have $(6 - n)$ pointlike instantons and in order to get total instanton number $c = 12 - n$ the six fractional instantons need to have charge one each.

Considering the other plane with gauge group $SU(9)$ we have

$$
\begin{aligned}
a_1(s,t) &= g(s,t) \\
a_3(s,t) &= c_0(s,t)^3 h(s,t),
\end{aligned} \tag{3.41}
$$

where $g = 0$ is a curve in the class $(C_0 + 2f)$ and $h = 0$ is a curve in the class $6f$ and thus splits as $h(s,t) = f_1(s,t) \dots f_6(s,t)$, where each $f_i = 0$ is a curve in the class f. The

[10]The polynomial g really splits into two factors $g = f_1 f_2$ but as this is of no importance in this case we use the function g

Weierstrass equation is

$$
\begin{aligned}
a &= g(s,t)\left(\frac{1}{2}c_0(s,t)^3 f_1(s,t)\dots f_6(s,t) - \frac{1}{48}g(s,t)^3\right)\\
b &= \frac{1}{4}c_0(s,t)^6 f_1(s,t)^2\dots f_6(s,t)^2 + \frac{1}{864}g(s,t)^6 - \frac{1}{24}c_0(s,t)^3 g(s,t)^3 f_1(s,t)\dots f_6(s,t)\\
\delta &= \frac{1}{16}c_0(s,t)^9 f_1(s,t)^3\dots f_6(s,t)^3\left(27c_0(s,t)^3 f_1(s,t)\dots f_6(s,t) - g(s,t)^3\right) \qquad (3.42)
\end{aligned}
$$

There are six curves with I_3 fibres along $f_i = 0$ and one I_1 curve along along $\delta' = 0$ with

$$\delta' = \delta / \frac{1}{16}c_0(s,t)^9 f_1(s,t)^3 \dots f_6(s,t)^3 = 27c_0(s,t)^3 f_1(s,t)\dots f_6(s,t) - g(s,t)^3. \qquad (3.43)$$

The I_1 curves are in the class $(3C_0 + 6f)$. The curve with the I_3 fibres intersect curves in C_* six times and the I_1 curves have $(2-n)$ intersections with C_0. Thus we have six fractional instantons with charge $c = 5/3$.

Glueing together the two half-planes along C_* leads to a vanishing magnetic charge $g = \sum_{i=1}^{18} g_i$ as required by anomaly cancellation. Again we have to take the blow-up of the correct limit of the base-manifold to obtain the model which is dual to the heterotic theory we started with. The limit is to push two intersections of the I_3 curve and two intersections of the I_1 curve with C_* to the same point. We push the pointlike instantons to intersection points of I_3 with C_0 and fix $n = 3$ on the E_6 half-plane and $n = -3$ on the SU(9) half-plane. There are three points with E_6 singularities along C_* with instanton number $c = 3$ on the E_6 plane and $c = 5$ on the $SU(9)$ plane. The points in C_* with E_6 singularities are blown up to three exceptional divisors with an I_3 singularity each. This leads to the correct manifold $T^4/\mathbb{Z}_3$ with nine fixed points in the dual heterotic theory and instanton number $c = 1$ and $c = 5/3$ on the fixed points.

Let us finally apply the same methods to the model with gauge group $(SO(8))^8$ considered in [97]. Four $SO(8)$'s are perturbative gauge groups and four are non-perturbative. The compactification manifold of the F-theory model is the threefold $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. The base of the manifold is $T^2/\mathbb{Z}_2 \times T^2/\mathbb{Z}_2$ which is a singular limit of $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and the base should be invariant under exchanging the two $\mathbb{P}^1$'s. This means that the theory should be invariant under exchanging the perturbative and the non-perturbative gauge groups, which is indeed the case. This model is dual to M-theory compactified on $T^4/\mathbb{Z}_2 \times S^1/\mathbb{Z}_2$ and to the heterotic string compactified on $T^4/\mathbb{Z}_2$ with the same gauge group. Again there are fractional instantons at the fixed points of the theory. Comparison with [110] tells us indeed that for F-theory compactified on a manifold including instanton with a discrete $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ symmetry the only possible perturbative gauge groups are $SO(8) \times SO(8)$ and $SO(12) \times SO(4)$. The polynomials a, b and δ are of the form

$$
\begin{aligned}
a &= \frac{1}{3}(b_2(s,t)c_2(s,t) - b_2(s,t)^2 - c_2(s,t)^2)\\
b &= -\frac{1}{27}(b_2(s,t) + c_2(s,t))(b_2(s,t) - 2c_2(s,t))(2b_2(s,t) - c_2(s,t))\\
\delta &= -b_2(s,t)^2 c_2(s,t)^2 (b_2(s,t) - c_2(s,t))^2. \qquad (3.44)
\end{aligned}
$$

The instanton-number of each half-plane after the stable degeneration is $c = 12$ as the base of the threefold is $T^2/\mathbb{Z}_2 \times T^2/\mathbb{Z}_2$ which is a singular limit of $\mathbb{F}_n$ with $n = 0$.

We consider a perturbative gauge group $SO(8)\times SO(8)$ on each half-plane, which means that a, b and δ vanish as $O(a)=2\cdot 2$, $O(b)=2\cdot 3$ and $O(\delta)=2\cdot 6$. This determines

$$\begin{aligned} b_2(s,t) &= c_0(s,t)^2 g(s,t) \\ c_2(s,t) &= c_0(s,t)^2 h(s,t), \end{aligned} \tag{3.45}$$

where $g(s,t)=0$ is in the class $4f$ and thus factorizes as $g(s,t)=f_1(s,t)\dots f_4(s,t)$. The same is true for $h(s,t)$ and we get $h(s,t)=\tilde{f}_1\dots\tilde{f}_4$. The Weierstrass equation is

$$\begin{aligned} a &= \frac{1}{3}(c_0(s,t)^2)^2\left(f_1(s,t)\dots f_4(s,t)\tilde{f}_1\dots\tilde{f}_4 - f_1(s,t)^2\dots f_4(s,t)^2 - \tilde{f}_1^2\dots\tilde{f}_4^2\right) \\ b &= -\frac{1}{27}(c_0(s,t)^2)^3\left(f_1(s,t)\dots f_4(s,t)+\tilde{f}_1\dots\tilde{f}_4\right)\left(f_1(s,t)\dots f_4(s,t)-2\tilde{f}_1\dots\tilde{f}_4\right) \\ &\quad \left(2f_1(s,t)\dots f_4(s,t)-\tilde{f}_1\dots\tilde{f}_4\right) \\ \delta &= -(c_0(s,t)^2)^6\left(f_1(s,t)^2\dots f_4(s,t)^2\right)\left(\tilde{f}_1^2\dots\tilde{f}_4^2\right)\left(f_1(s,t)\dots f_4(s,t)-\tilde{f}_1\dots\tilde{f}_4\right)^2. \end{aligned} \tag{3.46}$$

In addition to the perturbative $SO(8)\times SO(8)$ gauge group, there are 12 curves with I_2 fibres. Taking the limit $f_i = j\tilde{f}_i$ with $j=$ constant leads to

$$\begin{aligned} a &= \frac{1}{3}(c_0(s,t)^2)^2 f_1(s,t)^2\dots f_4(s,t)^2(j-1-j^2) \\ b &= -\frac{1}{27}(c_0(s,t)^2)^3(1+j)(1-j)(2-j) \\ \delta &= -(c_0(s,t)^2)^6 f_1(s,t)^6\dots f_4(s,t)^6 j^2(1-j) \end{aligned} \tag{3.47}$$

and gives the desired additional four curves $f_i=0$ with D_4 singularities leading to four $SO(8)$ gauge groups. Glueing together two half-planes leads to the total gauge group $(SO(8))^8$. As the discriminant after dividing by the perturbative gauge group is $\delta' = f_1(s,t)^6\dots f_4(s,t)^6$ there are no pointlike instantons but four fractional instantons with charge $c=3$ each. Again the four intersections of the D_4 singularities with C_* have to be blown up to get the correct heterotic compactification manifold with sixteen $\mathbb{Z}_2$ fixed points. Each fixed point has instanton number $c=3/4$ and the magnetic charge vanishes. Note that [97] does not quite describe the dual F-, M- and heterotic theories but their Weierstrass formulation which includes the four non-perturbative $SO(8)$ gauge groups. After the blow-up, the dual heterotic theory has sixteen $\mathbb{Z}_2$ singularities as required by the orbifold compactification. At this point, the theory has a perturbative gauge group $(SO(8))^4$ and a non-perturbative gauge group $(SU(2))^{16}$. If we want to keep invariance under exchanging the perturbative and the non-perturbative gauge groups the perturbative $SO(8)$ gauge groups have to be broken to $SU(2)$ as well. The model has a gauge group $(SU(2))^{32}$ in this case, sixteen of which are perturbative and sixteen are non-perturbative.

As a last remark note that the methods of this section do not apply to models which include $U(1)$ gauge groups, for example M-theory on $T^4/\mathbb{Z}_2\times S^1/\mathbb{Z}_2$ with gauge group $E_8\times E_8\times(U(1)^{16})$. There is no way in the dual F-theory picture to include geometric singularities into the compactification manifold which produce $U(1)$ gauge groups as can be seen in table A.3.2. Thus we are not able to give a geometric explanation from the F-theory picture for the sixteen non-perturbative $U(1)$ gauge groups on the fixed seven-planes of the M-theory compactification. The problem of $U(1)$ gauge groups was considered in [118] and it was shown that the $U(1)$ gauge groups can be explained in terms of string junctions.

3.6 Summary and Outlook

The strong coupling limit of the $E_8 \times E_8$ heterotic string is given by M-theory compactified on the one-dimensional orbifold $S^1/\mathbb{Z}_2$ [15]. Due to the lack of an underlying formulation of M-theory, one has to rely on physical consistency conditions in constructing the theory. In particular, anomaly cancellation has proven essential for fixing the spectrum and the gauge group of the theory. The spectrum and the gauge group are completely in accord with ten-dimensional heterotic string theory in the limit of a small compact eleventh dimension. This limit just provides the weak coupling limit. However, compactifying M-theory on $S^1/\mathbb{Z}_2$ further on a four-dimensional orbifold $T^4/\mathbb{Z}_N$ to six dimensions, one obtains highly non-intuitive rules for the spectrum and the gauge group of the theory from anomaly cancellation [89, 90]. A complete understanding of these rules from physical principles has not been achieved yet. The target of chapter 3 is to investigate the rules closely in order to contribute to the progress in finding an explanation for them. We address the problem by considering the dual formulation in terms of F-theory. To build an explicit model, we consider the example of an orbifold $T^4/\mathbb{Z}_2$ and a perturbative gauge group $E_7 \times [SO(16) \times SU(2)]$. In section 3.4, we explicitly construct the F-theory compactification manifold which, locally around the fixed points of the orbifold, gives the correct description of the model. We show that the F-theory formulation indeed contains new information about the non-perturbative behaviour. This allows to understand some of the rules developed by anomaly cancellation directly from F-theory. In section 3.5, we use the methods developed in section 3.4 to generalize the results to models on different orbifolds and with other gauge groups.

As mentioned above, the F-theory compactification manifold describes the dual M-theory model correctly locally around each fixed point. Technically spoken, this is the case because we construct the F-theory manifold as a Weierstrass model, which means a restriction on the class of possible manifolds. The compactification manifold which produces the correct picture also globally is more complicated and does not belong to this class of manifolds. However, one might expect that the precise dual F-theory formulation contains more information than the locally correct one. Thus a task for future investigations is to extend the dual local F-theory description to a picture which is also globally valid.

Another aspect for future investigations is to extend the methods developed in chapter 3 to four-dimensional models. We expect that these theories are more complicated than the six-dimensional ones. Having a four-dimensional space-time however, they are of great relevance due to their closer link to realistic scenarios. Four-dimensional models of the strongly coupled heterotic string compactified on orbifolds have been considered in [92]. It would be interesting to construct the dual non-perturbative F-theory formulation of these models.

Appendix

A Calabi-Yau Manifolds

In this appendix we explain some aspects of Calabi-Yau manifolds. Due to the restricted length of the thesis, the general introduction A.1 is kept very brief and rather a summary of facts than a self-contained introduction. We mainly concentrate on the aspects that are relevant for the calculations in the chapters 2 and 3 and explain some important concepts in greater detail. For a general, non-specialized overview of Calabi-Yau manifolds see for example [1, 3, 4] and also [65].

A.1 General Introduction

A.1.1 (Co)Homology

The exterior derivative d takes a p-form to a $(p+1)$-form

$$dA_p = \frac{1}{p!}\partial_\mu A_{\mu_1\ldots\mu_p} dx^\mu \wedge dx^{\mu_1} \wedge \ldots dx^{\mu_p}. \tag{A.1}$$

The exterior derivative is nilpotent, that is $d^2 = 0$. A p-form A_p is closed if the exterior derivative vanishes, $dA_p = 0$, and exact if it is the exterior derivative of some $(p-1)$-form $A_p = dB_{(p-1)}$. From $d^2 = 0$ it follows that an exact form is always closed, that means it fulfills $dA_p = 0$ trivially. The p-forms that provide non-trivial solutions to the equation $dA_p = 0$ are obtained by taking the quotient of the closed p-forms and the exact p-forms forms. This defines the p^{th} de Rham cohomology on some manifold M,

$$H^p(M) = \frac{\{A_p | dA_p = 0\}}{\{B_p | B_p = dC_{(p-1)}\}}. \tag{A.2}$$

The dimension of the de Rham Cohomology is called the Betti number,

$$b^p(M) = \text{dim} H^p(M). \tag{A.3}$$

The Euler-number of the manifold M is

$$\chi(M) = \sum_{p=0}^{d} (-1)^p b_p(M), \tag{A.4}$$

where d is the dimension of M.

The Poincare dual of a p-form is defined by

$$*(A_p) = \frac{1}{p!(d-p)!}\epsilon^{\mu_1\ldots\mu_p}{}_{\nu_{(p+1)}\ldots\nu_d} A_{\mu_1\ldots\mu_p} dx^{\nu_{(p+1)}} \wedge \ldots dx^{\nu_d}, \tag{A.5}$$

which is a $(d-p)$-form. The operator $\Delta = dd^\dagger + d^\dagger d$ is the Laplace operator in M (to be precise it reduces to the Laplace operator in flat space), with $d^\dagger = (-1)^{dp+d+1} * d *$.

A p-form is called harmonic if $\Delta A_p = 0$. It can be shown that the harmonic forms of M are in one-to-one correspondence with the de Rham cohomology. Every harmonic form corresponds to one equivalence class of the cohomology.

The Hodge decomposition theorem states that every p-form can be uniquely written as

$$A_p = dB_{(p-1)} + d^\dagger C_{(p+1)} + A'_p. \tag{A.6}$$

If A_p is closed, $C_{(p+1)}$ vanishes and the decomposition reads $A_p = dB_{(p-1)} + A'_p$.

It can be shown that the Poincare dual $*(A_p)$ is harmonic if and only if A_p is harmonic. This implies

$$b^p(M) = b^{(d-p)}(M). \tag{A.7}$$

The integrals over p-dimensional submanifolds N_p of M, $\int_{N_p}$, form a vector space. The linear combinations of these integrals

$$n_p = \sum_l c_l \int_{N_p^l}, \tag{A.8}$$

where c_l are constants, are called chains. The boundaries of the submanifolds δN_p are defined by a nilpotent boundary operator δ. Just as for the exterior derivative one can define closed and and exact chains with respect to δ. The closed chains are called cycles. From $\delta^2 = 0$ it is clear that exact chains are closed trivially, in other words the boundary δN_p of some submanifold N_p has no boundaries. The p-dimensional submanifolds of M which are closed but not the boundary of some other submanifold form the p^{th} simplicial homology

$$H_p(M) = \frac{\{n_p = \sum_l c_l \int_{N_p^l} | \delta N_p^l = 0\}}{\{o_p = \sum d_l \int_{O_p^l} | O_p^l = \delta P_{(p+1)}^l\}}. \tag{A.9}$$

The de Rham cohomology $H^p(M)$ and the simplicial homology $H_p(M)$ are isomorphic, that is for any p-form A_p there is a $(d-p)$-cycle $N_{(d-p)}$ such that

$$\int_M A_p \wedge B_{(d-p)} = \int_{N_{(d-p)}} B_{(d-p)} \tag{A.10}$$

for any closed $(d-p)$-form $B_{(d-p)}$. Usually, the integrals in (A.9) are omitted and one refers to the cycles as elements of the homology.

Depending on whether the coefficients c_l in (A.8) are real, complex or integer, the homology is called the real, complex or integer homology $H_p(M, \mathbb{R})$, $H_p(M, \mathbb{C})$ or $H_p(M, \mathbb{Z})$.

For two cycles $N \in H_p(M)$, $M \in H_q(M)$, with $p + q = d$, one may define an inner product given by

$$N \cdot M = \#(N \cap M), \tag{A.11}$$

where $\#(N \cap M)$ counts the number of intersections of the cycles N and M in M. Transverse intersections with positive orientation contribute one and intersections with negative orientation contribute with minus one to the total intersection number. If the dimensions p, q of the cycles N, M do not add up to the complex dimension of the manifold, the intersections define not points but $d - p - q$-dimensional submanifolds of M.

A.1.2 Complex Manifolds

A complex manifold is a manifold of even dimension $2d$ with local complex coordinates z^i on patches in M and local complex coordinates z'^i on different patches in M such that the transition functions $z'^i(z^j)$ are holomorphic. A complex manifold admits a complex structure J. A complex structure is an integrable almost complex structure, that means it is an integrable tensor field J on M with one covariant and one contravariant index that obeys $J^2 = -1$. It can be shown that an almost complex structure is integrable if and only if the Nijenhuis tensor

$$N^k_{ij} = J^l_j(\partial_l J^k{}_j - \partial_j J^k{}_l) - J^l{}_j(\partial_l J^k{}_i - \partial_i J^k{}_l) \tag{A.12}$$

vanishes. The existence of a complex structure on some manifold M can be used indeed to define a complex manifold. The complex structure of a complex manifold does not necessarily have to be unique. That means two manifolds which are topologically equivalent can have inequivalent complex structures and thus be inequivalent as complex manifolds.

On a complex manifold it is always possible to choose local holomorphic coordinates $z^i, \bar{z}^{\bar{\imath}}$, $i, \bar{\jmath} = 1, \ldots d$, such that the complex structure takes the form

$$J_j{}^i = i\delta_j{}^i, \quad J_{\bar{\jmath}}{}^{\bar{\imath}} = -i\delta_{\bar{\jmath}}{}^{\bar{\imath}}, \tag{A.13}$$

all other components zero. In these coordinates one can define (p, q)-forms as $p+q$-forms with p holomorphic and q anti-holomorphic indices,

$$A_{(p,q)} = \frac{1}{p!q!} A_{i_1 \ldots i_p \bar{\imath}_1 \ldots \bar{\imath}_q} dz^{i_1} \wedge \ldots dz^{i_p} d\bar{z}^{\bar{\imath}_1} \wedge d\bar{z}^{\bar{\imath}_q}. \tag{A.14}$$

The exterior derivative in terms of the complex coordinates is $d = \partial + \bar{\partial}$ with $\partial^2 = \bar{\partial}^2 = 0$, such that $\partial = dz^i \partial_i$ takes a (p, q)-form to a $(p + 1, q)$-form and $\bar{\partial} = d\bar{z}^{\bar{\imath}} \partial_{\bar{\imath}}$ takes a (p, q)-form to a $(p, q + 1)$-form. The de Rham Cohomology (A.2) splits into the cohomology $H^{(p,q)}_{\partial}(M)$ of the ∂-closed (p, q)-forms on M and the cohomology $H^{(p,q)}_{\bar{\partial}}(M)$ of the $\bar{\partial}$-closed (p, q)-forms on M, they are called the Dolbeaut Cohomology groups. The dimensions of the Dolbeaut Cohomology groups are the Hodge numbers $h^{(p,q)}$. The $\Delta_\partial = \partial\partial^\dagger + \partial^\dagger\partial$ resp. $\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}$ harmonic (p, q)-forms of M are in one-to one correspondence with $H^{(p,q)}_{\partial}(M)$ resp. $H^{(p,q)}_{\bar{\partial}}(M)$. Note that one can also define the Laplace operator Δ_d with $d = \partial + \bar{\partial}$, thus there are three different Laplace operators on complex manifolds.

Every complex manifold admits a Hermitian metric, that is a metric which satisfies

$$g_{ij} = g_{\bar{\imath}\bar{\jmath}} = 0, \tag{A.15}$$

only the components $g_{i\bar{\jmath}}$ are non-vanishing. A Hermitian metric satisfies the equation $g_{i\bar{\jmath}} = J_i{}^k J_{\bar{\jmath}}{}^{\bar{l}} g_{k\bar{l}}$ and multiplying with $J^{\bar{\jmath}}_{\bar{m}}$ leads to $J_{i\bar{m}} = -J_{\bar{m}i}$, where $J_{i\bar{m}} = J_i{}^k g_{k\bar{m}} = ig_{i\bar{m}}$ and $J_{\bar{m}i} = J^{\bar{\jmath}}_{\bar{m}} g_{i\bar{\jmath}} = ig_{\bar{m}i}$. Thus one naturally gets a $(1, 1)$-form

$$J = J_{i\bar{\jmath}} dz^i \wedge dz^{\bar{\jmath}} = ig_{i\bar{\jmath}} dz^i \wedge dz^{\bar{\jmath}} \tag{A.16}$$

on a complex manifold.

A.1.3 Kähler Manifolds

A Kähler manifold is a complex manifold with a Hermitian metric $g_{i\bar{\jmath}}$ such that the natural two-form (A.16) is closed,

$$dJ = 0. \tag{A.17}$$

An equivalent definition is that the Hermitian metric is locally of the form

$$g_{i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}} K(z, \bar{z}), \tag{A.18}$$

where $K(z, \bar{z})$ is the Kählerpotential. The metric is invariant under transformations

$$K'(z, \bar{z}) = K(z, \bar{z}) + f(z) + f(\bar{z}). \tag{A.19}$$

For Kähler manifolds the three Laplace operators are related as

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}. \tag{A.20}$$

This in turn relates the Dolbeaut cohomologies

$$H_\partial^{(p,q)}(M) = H_{\bar{\partial}}^{(p,q)}(M) \tag{A.21}$$

and the Betti numbers with the Hodge numbers

$$b^r = \sum_{(p+q)=r} h^{(p,q)}. \tag{A.22}$$

Furthermore, complex conjugation gives $h^{(p,q)} = h^{(q,p)}$ and Hodge * gives $h^{(d-p,d-q)} = h^{(p,q)}$.

A.1.4 Holonomy

Upon parallel transport around a closed curve in M, tangent vectors are transformed by an orthogonal matrix. These matrices form a group, the Holonomy group. For a Riemannian manifold the Holonomy group is $SO(d)$. For manifolds of even dimension, one can show that

$U(d)$ Holonomy $\leftrightarrow$ M is a Kähler manifold,

this can indeed be used next to (A.17) and (A.18) as a third alternative definition for Kähler manifolds. Another important statement is that

$SU(d)$ Holonomy $\leftrightarrow$ M is a Kähler manifold and Ricci-flat,

where Ricci-flatness means that the Ricci-tensor $R_{i\bar{\jmath}} = R^k{}_{ki\bar{\jmath}}$ obtained from the Riemann tensor $R^i{}_{jk\bar{l}} = -\partial_{\bar{l}}(g^{\bar{m}i}\partial_k g_{j\bar{m}})$ vanishes. The Ricci-form $R = R_{i\bar{\jmath}} dz^i \wedge d\bar{z}^{\bar{\jmath}}$ is closed, $dR = 0$, but in general not exact because it can be written as the derivative of a term locally but not necessarily globally. Therefore the Ricci-form defines an equivalence class in $H^{(1,1)}(M)$. This equivalence class is the first Chern class, $c_1(M) = \left[\frac{R}{2\pi}\right]$. A manifold that admits an exact Ricci-form has vanishing first Chern class.

A.1.5 Calabi-Yau Manifolds

Calabi-Yau manifolds are Kähler manifolds with $c_1 = 0$. It was conjectured by Calabi and proven by Yau that that for a Kähler manifold with vanishing first Chern class there exists a unique Ricci-flat metric for any given Kähler form and complex structure. Using the statements of the last section, it is clear that Calabi-Yau manifolds of complex dimension d have a holonomy group $SU(d)$.

It can be shown that a Kähler manifold has $c_1 = 0$ if and only if there exists a unique nowhere vanishing holomorphic $(d, 0)$-form Ω. The holomorphic $(d, 0)$-form is covariant constant with respect to the Ricci-flat metric. For the Hodge numbers the existence of a unique $(d, 0)$-form implies the relation

$$h^{(p,0)}(M) = h^{(p,d-p)}(M). \tag{A.23}$$

For a complex d-dimensional Calabi-Yau manifold with Holonomy $SU(d)$ (not a subgroup of $SU(d)$) it can be shown that

$$h^{(0,p)} = 0, \quad p \neq 0, d. \tag{A.24}$$

A.1.6 Algebraic Calabi-Yau Manifolds

Of great importance are algebraic Calabi-Yau manifolds, i.e. Calabi-Yau manifolds that can be expressed as complex hypersurfaces embedded in the complex projective space $\mathbb{P}^N$ for some N. The complex projective space $\mathbb{P}^N$ is parametrized by complex variables x_k, $k = 1, \ldots, x_{N+1}$, such that for any complex number λ the family $\{x_k\}$ is identified with the family $\{\lambda x_k\}$ [11]. These coordinates are called homogeneous coordinates. The projective space is a complex, compact manifold. To describe the embedded hypersurface consider a polynomial $f(x_1, \ldots x_{N+1})$ which is homogeneous of degree n, that means $f(\lambda x_1, \ldots \lambda x_{N+1}) = \lambda^n f(x_1, \ldots, x_{N+1})$. The equation $f = 0$ defines a compact surface of complex dimension $d = N - 1$ and degree n. In order to construct a Calabi-Yau manifold, the hypersurface must have vanishing first Chern class. It can be shown that $c_1 = 0$ if and only if

$$n = N + 1. \tag{A.25}$$

An algebraic Calabi-Yau manifold is called singular at some point $(y_1, \ldots, y_{N+1}) \in \mathbb{P}^{N+1}$ if

$$f(x_1, \ldots, x_{N+1})|_{(y_1,\ldots,y_{N+1})} = df(x_1, \ldots, x_{N+1})|_{(y_1,\ldots,y_{N+1})} = 0. \tag{A.26}$$

More generally, algebraic Calabi-Yau manifolds can be expressed as hypersurfaces in weighted complex projective spaces. Weighted projective spaces $\mathbb{P}^N_{[w_1,\ldots,w_{N+1}]}$ are projective spaces with $\lambda(x_1, \ldots, x_{N+1}) = (\lambda^{w_1} x_1, \ldots, \lambda^{w_{N+1}} x_{N+1})$. A hypersurface in the weighted projective space is given by the zero locus $f = 0$ of a quasi-homogeneous polynomial f of some degree n, i.e. a polynomial that satisfies $f(\lambda^{w_1} x_1, \ldots, \lambda^{w_{N+1}} x_{N+1}) = \lambda^n f(x_1, \ldots, x_{N+1})$. The hypersurface has vanishing first Chern-class if and only if

$$n = \sum_{l=1}^{N+1} w^l. \tag{A.27}$$

[11] It is assumed that the coordinates $\{x_k\}$ are not all zero.

For $(w_1, \ldots, w_{N+1}) = (1, \ldots, 1)$ one recovers ordinary projective space and $n = N + 1$. Omitting the weights always implies $(w_1, \ldots, w_{N+1}) = (1, \ldots, 1)$.

To construct additional examples one can consider m polynomials $f_1, \ldots, f_m$, $m \leq N$, of degrees $k_1, \ldots, k_m$ in $\mathbb{P}^N$ (or more generally in weighted projective spaces). The equations $f_1 = \ldots = f_m = 0$ define a $N - m$-dimensional complex submanifold of $\mathbb{P}^N$, these are called complete intersection hypersurfaces. The existence of a nowhere vanishing holomorphic $(N - m, 0)$-form, or $c_1 = 0$, is guaranteed if and only if

$$\sum_{a=1}^{m} k_a = N + 1. \tag{A.28}$$

A.2 The Torus

The torus is the simplest, and lowest-dimensional, Calabi-Yau manifold. The Hodge numbers of the torus are

$$h^{(0,0)} = h^{(1,1)} = h^{(1,0)} = h^{(0,1)} = 1. \tag{A.29}$$

It follows that the Euler number of a torus is zero, $\chi(T^2) = 0$. For a complex d-dimensional manifold the integral of the d-th Chern class is equal to the Euler number, thus for a torus one has $\chi(T^2) = \int c_1(\mathcal{T}_{T^2})$, where $\mathcal{T}_{T^2}$ is the tangent bundle of the torus. This leads to $c_1 = 0$, as required for Calabi-Yau manifolds.

A torus as an algebraic manifold can be described in terms of the zero locus of a third order polynomial $f = 0$ in the projective space $\mathbb{P}^2$

$$f = x^3 - y^2 z + axz^2 + bz^3 = 0, \tag{A.30}$$

where x, y, z are the coordinates on $\mathbb{P}^2$ and a, b are constant parameters. The above equation is called Weierstrass equation. Note that for $z \neq 0$ the coordinates on the projective space can be rescaled as $x \to x/z$, $y \to y/z$, $z \to 1$ leading to the Weierstrass equation

$$y^2 = x^3 + ax + b. \tag{A.31}$$

The new coordinates are called affine coordinates. It can be shown that the complex structure of the torus depends only on the modular parameter of the torus, usually called τ. In terms of the constants a and b of the Weierstrass equation the modular parameter τ is given by

$$j(\tau) = \frac{4(24a)^3}{27b^2 + 4a^3}, \tag{A.32}$$

where $j(\tau)$ is the unique modular invariant function with a pole at $\tau = i\infty$ and zeros at $\tau = e^{i\pi/3}$. The normalization is chosen such that $j(i) = (24)^3$. The denominator $\delta = 27b^2 + 4a^3$ is called the discriminant of (A.31). For vanishing discriminant $\delta = 0$, the torus becomes singular. This can be verified easily solving the equations (A.26), i.e. by solving $f = x^3 - y^2 + ax + b = 0$, $df = (3x^2 + a)dx - 2ydy = 0$.

The complex structure of the torus is invariant under transformations of the modular parameter τ of the torus

$$\tau' = \frac{A\tau - iB}{iC\tau + D} \quad \text{for} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{A.33}$$

A.2.1 Elliptic Fibrations

Starting from the torus one can construct higher-dimensional elliptically fibred algebraic Calabi-Yau manifolds. Consider some complex manifold B. Elliptically fibered Calabi-Yau manifolds Y are Calabi-Yau manifolds obtained by erecting a torus at every point in the base B. Elliptic fibrations can be described by a map $\pi : Y \to B$. We only consider elliptic fibrations that have in addition a global section $\sigma : B \to Y$ such that $\pi \circ \sigma = \mathrm{id}_B$. If a section σ exists at every point in B, the manifold is a direct product of B and the torus. Elliptically fibered Calabi-Yau manifolds with a section can be described by the Weierstrass equation (A.31), with the parameters of the torus depending on the coordinates $\{s\}$ of the base B, $a = a(\{s\})$ and $b = b(\{s\})$. This leads to the equation

$$y^2 = x^3 + a(\{s\})x + b(\{s\}). \tag{A.34}$$

Such a Calabi-Yau manifold is a hypersurface in some projective space of complex dimension $N = (\dim(B)+2)$ and is parametrized by the coordinates $(x, y, \{s\})$. In holomorphic coordinates (x, y, z), one can see that $(0, 1, 0)$ always lies in the hypersurface. This gives rise to the global section mentioned above. There are of course elliptic fibrations that do not admit this global section and thus cannot be written in Weierstrass form.

A.2.2 Some Algebraic Geometry

In this section, we review some useful concepts of algebraic geometry in the context of elliptically fibred Calabi-Yau manifolds. A mathematical introduction to algebraic geometry is given in [121], ref. [120] emphasizes the applications in string theory.

Line Bundles
Line bundles are bundles of rank one, their fibre is isomorphic to $\mathbb{C}$. The tensor products of line bundles of a Calabi-Yau manifold Y form the Picard group $\mathrm{Pic}(X)$. The Picard group of a complex manifold is isomorphic to the space of divisors on Y, $\mathrm{Pic}(Y) \leftrightarrow \mathrm{Div}(Y)$. More precisely, a divisor D is the zero-set $\rho = 0$ of a section ρ of a holomorphic line bundle $\mathcal{L}$. Taking the tensor product of line bundles $\mathcal{L}_1 \otimes \mathcal{L}_2$ corresponds to adding divisors, $D_1 + D_2$. The fibres of a holomorphic vector bundle $\mathcal{M}$ of rank k are isomorphic to $\mathbb{C}^k$. Given a holomorphic vector bundle $\mathcal{M}$ over some complex manifold Y with a holomorphic sub-bundle $\mathcal{N}$ one can always define a quotient bundle $\mathcal{M}/\mathcal{N}$ which is spanned by the elements of $\mathcal{M}$ that differ by an element of $\mathcal{N}$. Instead of dealing with vector bundles it is often more convenient to consider the line bundle defined by the determinant of the vector bundle, $\det\mathcal{M}$, with the the relation $\det\mathcal{M} = \det\mathcal{N} \otimes \det(\mathcal{M}/\mathcal{N})$ for the holomorphic vector bundle $\mathcal{M}$ and a holomorphic sub-bundle $\mathcal{N}$. Considering the determinant of a vector bundle instead of the vector bundle itself is very useful because of the correspondence between line bundles and divisors mentioned above.

The (Co)Tangent Bundle
The tangent bundle of a complex manifold Y is the union of the tangent spaces at all points $p \in Y$, $\mathcal{T}_Y = \cup_{p\in Y} T_p(Y)$, where the tangent space $T_p(Y)$ is the vector space spanned by the tangents to the curves through p. The cotangent bundle $\mathcal{T}^*_Y$ is the holomorphic complex dual of the tangent bundle. Any complex n-dimensional manifold Y has a canonical line bundle $\mathcal{K}_Y$ which is the determinant of the cotangent bundle of

Y,

$$\mathcal{K}_Y = \det \mathcal{T}_Y^*. \tag{A.35}$$

The cotangent bundle is identified with the bundle of holomorphic one-forms on Y and the canonical line bundle is the bundle of holomorphic n-forms, the volume forms of the manifold. Associated to the canonical bundle is the canonical divisor K_Y. The canonical divisor is given by the first Chern-class of the manifold, $K_Y = -c_1(\mathcal{T}_Y)$. This relation is often used as the definition for the canonical class, as for example in [108].

The (Co)Normal Bundle

Consider a Calabi-Yau manifold Y with an elliptic fibration $\pi : Y \to B$ with at least one global section $\sigma : B \to Y$ which maps the base into Y such that $\pi \circ \sigma =$id. The normal bundle of σ is the quotient bundle of the holomorphic tangent bundles over Y restricted to B and the tangent bundle over B,

$$\mathcal{N}_\sigma = \{\mathcal{T}_Y|_B / \mathcal{T}_B\}. \tag{A.36}$$

The divisor associated with the determinant of the conormal bundle of σ, $\det\mathcal{N}_\sigma^*$, is denoted by L.

The Conormal Bundle of Elliptically Fibred Manifolds

Let $\mathcal{E}$ be a holomorphic bundle over Y and ϕ a holomorphic section of $\mathcal{E}$ such that B is the zero locus of the section, $\phi^{-1}(0) = B$. A tangent vector $Z \in \mathcal{T}_Y$ at some point in Y can be represented by $Z^i \nabla_i$ with the covariant derivative $\nabla_i = \partial_i + \Gamma_i$ containing some connection Γ_i. The covariant gradient of the section is $\nabla\phi = Z^i(\partial_i\phi + \Gamma_i\phi)$. If the points in Y at which the tangent vectors Z are taken are restricted to B, that means if we restrict to $\phi : \mathcal{T}_Y|_B \to \mathcal{E}|_B$, then $\phi = 0$ and $\Gamma_i\phi = 0$. The gradient does not depend on the connection in this case. Moreover as ϕ vanishes on B so does the gradient along the tangent directions of B, in other words $\ker(\nabla\phi) = \mathcal{T}_B$. This together with the condition that the gradient $\nabla\phi : \mathcal{T}_Y|_B \to \mathcal{E}|_B$ covers all of $\mathcal{E}|_B$ defines the short exact sequence of vector bundles

$$0 \to \mathcal{T}_B \xrightarrow{i} \mathcal{T}_Y|_B \xrightarrow{\nabla\phi} \mathcal{E}|_B \to 0. \tag{A.37}$$

Using the above definition of the normal bundle of σ, this allows to identify

$$\mathcal{E}|_B = \mathcal{N}_\sigma, \tag{A.38}$$

this is the Adjunction Formula I in [121]. For the line bundles this implies

$$\det\mathcal{T}_Y|_B = \det\mathcal{T}_B \otimes \det\mathcal{N}_\sigma, \tag{A.39}$$

this is the Adjunction Formula II of [121]. In terms of the divisors associated to the line bundles this reads

$$K_Y = \pi^*(K_B + L). \tag{A.40}$$

For Calabi-Yau manifolds the canonical class (or equivalently the first Chern class) vanishes, $K_Y = 0$. Thus for Calabi-Yau threefolds which are elliptically fibered over the base B the determinant of conormal bundle of the section σ is given by the canonical class of the base of the fibration,

$$K_B = -L. \tag{A.41}$$

This result is needed in 3.4 as well as in the following sections.

A.3 K3 Manifolds

A K3 manifold is a complex two-dimensional Calabi-Yau manifold. It can be shown that all K3 manifolds are diffeomorphic, that means they all have the same topological invariants. K3 manifolds have been considered in detail in [108]. The Hodge numbers of a K3 manifold are

$$\begin{matrix} & & h^{(0,0)} & & \\ & h^{(1,0)} & & h^{(0,1)} & \\ h^{(2,0)} & & h^{(1,1)} & & h^{(0,2)} \\ & h^{(2,1)} & & h^{(1,2)} & \\ & & h^{(2,2)} & & \end{matrix} \quad = \quad \begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix} \;. \tag{A.42}$$

The Euler number of K3 manifolds is

$$\chi(K3) = \sum_{p=0}^{2} (-1)^p b_p = 24. \tag{A.43}$$

Using $\chi(K3) = \int c_2(\mathcal{T}_{K3})$, where $\mathcal{T}_{K3}$ is the tangent bundle of the manifold, one obtains for the integral of the second Chern-class $\int c_2(\mathcal{T}_{K3}) = 24$.

A.3.1 Elliptically Fibred K3 Manifolds

An elliptically fibred K3 manifold can be constructed by taking $B = \mathbb{P}^1$ as the base of the elliptic fibration described at the end of A.2. For algebraic manifolds, the K3 can be analyzed in terms of algebraic curves within the manifold, that is curves which are holomorphically embedded into the K3 manifold. These curves are elements of $H_2(K3, \mathbb{Z})$, which implies the existence of a dual two-form in $H^2(K3, \mathbb{Z})$. Because the curve is holomorphically embedded, the dual two-form must also be an element of $H^{(1,1)}(K3)$. This defines the Picard group of the manifold,

$$Pic(K3) = H^2(K3, \mathbb{Z}) \cap H^{(1,1)}(K3), \tag{A.44}$$

and the Picard number is defined by the dimension of the Picard group. The Picard number of an elliptically fibred K3 manifold is at least two, one from the generic fibre and one from the global section. The homology cycles that are dual to the elements of the Picard group are called divisors.

An elliptically fibred K3 manifold with a global section can be expressed in Weierstrass form as

$$y^2 = x^3 + a(s)x + b(s), \tag{A.45}$$

where s parametrizes the base $\mathbb{P}^1$. One can show that in order to obtain vanishing first Chern-class, the polynomials $a(s)$ and $b(s)$ have to be of degree 8 and 12.

A.3.2 Singular Fibres

For an elliptically fibred K3 with base $\mathbb{P}^1$, the zero locus of the discriminant of eqn. (A.34), $\delta = 27b^2(s) + 4a^3(s) = 0$, describes points in the base. Because δ is a polynomial of degree

24, there are generically 24 of these points. As already mentioned in A.2, the vanishing locus of the discriminant gives rise to singularities of the torus. These degenerated fibres are are usually called singular or "bad" fibres. We explain the degeneration of elliptic fibres in greater detail.

There are only a few different kinds of singular fibres which are compatible with a vanishing first Chern-class. A classification of singular fibres was first given by Kodaira, therefore singular fibres are often called Kodaira fibres. For a Calabi-Yau manifold that can be written in Weierstrass form, the classification was given for example in [122]. We explain some examples here.

Given some explicit Weierstrass form of a K3, let $s = 0$ be the location of a singular fibre, i.e. at $s = 0$ one has $\delta = 0$ and in general also $a = 0$ and $b = 0$. Denote by $O(a)$, $O(b)$ and $O(\delta)$ the leading order of the polynomials at $s = 0$. The singular fibre can be classified in terms of the integers $O(a)$, $O(b)$ and $O(\delta)$. In general, the higher the orders are the more singular the fibre gets. If for example $O(a) = 0$, $O(b) = 0$ and $O(\delta) = 1$, the polynomials are of the form

$$\begin{aligned} a(s) &= a_0 + s a_1(s), \\ b(s) &= b_0 + s b_1(s), \\ \delta(s) &= 27b^2(s) + 4a^3(s) \stackrel{!}{=} \delta_0 s + s^2 \delta_1(s), \end{aligned} \tag{A.46}$$

with a_0, b_0, δ_0 constant and some polynomials $a_1, (s), b_1(s), \delta_1(s)$. The Weierstrass equation at $s = 0$ to leading order is simply

$$y^2 = x^3 + a_0 x + b_0. \tag{A.47}$$

The above form of $\delta(s)$ requires that $27b_0^2 + a_0^3 = 0$, and thus

$$b_0 = \mp \frac{4}{3\sqrt{3}} \sqrt{-a_0}^{\,3}. \tag{A.48}$$

The location of the singularity on the torus is given by setting the derivative of the Weierstrass form to zero, $-ydy + (3x^2 + a_0)dx = 0$, leading to $y = 0$ and $x^2 = -a_0/3$. Let us take small coordinates ϵ_y, ϵ_x in the neighborhood of this point, i.e. $y = 0 + \epsilon_y$ and $x = \pm\sqrt{-a_0/3} + \epsilon_x$. Inserting these coordinates and (A.48) into the Weierstrass equation (A.47), one gets to leading order

$$\begin{aligned} \epsilon_y &= \left(\pm\sqrt{\frac{-a_0}{3}}^{\,3} \pm 3\sqrt{\frac{-a_0}{3}}^{\,3} \epsilon_x \right) \pm a_0 \left(\sqrt{\frac{-a_0}{3}} + \epsilon_x \right) \mp 4\sqrt{\frac{-a_0}{3}}^{\,3} \\ &= \pm \epsilon_x \left(3\sqrt{\frac{-a_0}{3}}^{\,3} + a_0 \right). \end{aligned} \tag{A.49}$$

Thus the manifold locally looks like $y^2 = x^2 \cdot$constant close to the singularity. Such a fibre is called an I_1 fibre.

Let us consider some more complicated, higher order singularity. If for example $O(a) = 4$, $O(b) = 5$ and $O(\delta) = 10$, the polynomials are of the form

$$a(s) = s^4 a_0(s),$$

$O(a)$	$O(b)$	$O(\delta)$	Kodaira fibre	singularity	gauge algebra
≥ 0	≥ 0	0	I_0	$-$	$-$
0	0	1	I_1	$-$	$-$
0	0	$2n \geq 2$	I_{2n}	A_{2n-1}	$su(2n)$ or $sp(2n)$
0	0	$2n+1 \geq 3$	I_{2n+1}	A_{2n}	$su(2n+1)$ or $so(2n+1)$
≥ 1	1	2	II	$-$	$-$
1	≥ 2	3	III	A_1	$su(2)$
≥ 2	2	4	IV	A_2	$su(3)$ or $su(2)$
≥ 2	≥ 3	6	I_0^*	D_4	$so(8)$ or $so(7)$ or g_2
2	3	$n+6 \geq 7$	I_n^*	D_{n+4}	$so(2n+8)$ or $so(2n+7)$
≥ 3	1	8	IV^*	E_6	e_6 or f_4
3	≥ 5	9	III^*	E_7	e_7
≥ 4	5	10	II^*	E_8	e_8
≥ 4	≥ 6	≥ 12	$\mathrm{non-minimal}$		

Table A.3.2: Orders of vanishing, fibres, singularities and gauge algebra

$$b(s) = s^5 b_0(s),$$
$$\delta(s) = 27s^{10}b_0^2(s) + 4s^{12}a_0^3(s) = s^{10}(27b_0^2(s) + 4s^2a_0^3(s)) = s^{10}\delta_0(s), \qquad \text{(A.50)}$$

where $a_0(s)$, $b_0(s)$ and $\delta_0(s)$ are nonzero at $s = 0$. Thus the Weierstrass form reads

$$y^2 = x^3 + s^4 x a_0(s) + s^5 b_0(s). \qquad \text{(A.51)}$$

Such a singular fibre is called II^* fibre. The degenerated torus is of the form of a connected union of ten $\mathbb{P}^1$'s intersecting at several points. The intersection points where two or more $\mathbb{P}^1$'s meet are the singular points on the torus. Drawing a diagram of the torus, symbolizing the the $\mathbb{P}^1$'s as lines that intersect at the singular points, one obtains a diagram which has exactly the form the root lattice of the gauge group E_8 [108]. It can be shown that a II^* fibre indeed induces an E_8 gauge group in type IIA compactifications on elliptically fibered K3's. For a more detailed explanation of this, see [108]. One can generate several other gauge groups in this way. The generation of gauge groups in type IIA compactifications by singularites in the compactification manifolds is obviously of great importance in the construction of dual heterotic-type II string models. It is possible to construct IIA compactifications with gauge groups $E_8 \times E_8$, $SO(32)$, etc. The complete classification of singular fibres is given in table A.3.2. It should be stressed that this classification is valid only if the base of the elliptic fibration is not singular at the locus of the degenerated fibre. These "exotic" singularities also occur in Calabi-Yau manifolds, but they cannot be classified as in table A.3.2.

The classification includes singular fibres up to the orders $O(a) = 4$, $O(b) = 5$ and $O(\delta) = 10$ only. It can be shown that all higher order singularities are not compatible with vanishing first Chern-class (or canonical class) of the manifold, leading to a so-called non-minimal manifold which is not Calabi-Yau. However, if such higher degree fibres occur in a manifold, one can preserve the Calabi-Yau conditions by "blowing up" the singularities. In the next section we explain this and we give an example.

A.3.3 Blowing Up Singularities

We explain blowing up singularities by considering an example that gives the reader an idea of the procedure of blowing up singularities as well as explaining a situation that is important in section 3.4. This example explains the "stable degeneration" of K3 manifolds, which was first considered in [106] and explained in detail in [107].

Consider an elliptically fibred K3 manifold with an $E_8 \times E_8$ gauge group generated by two II^* fibres. In addition to the II^* fibres, there have to be four more singular fibres as we explain in the following. The Euler number of a K3 manifold is $\chi(K3) = 24$. The base of the elliptic fibration is a $\mathbb{P}^1$, which has Euler number $\chi(\mathbb{P}^1) = 2$. If the fibres are all smooth, the Euler number of each fibre is $\chi(T^2) = 0$, leading to a total Euler number given by the product of the Euler numbers of base and fibre, $\chi = 0$. This is clearly not a K3 manifold. Thus there have to be some singular fibres in the K3 manifold. The difference between the actual Euler number $\chi(K3) = 24$ and the Euler number of the smooth version of a torus fibred over $\mathbb{P}^1$ counts the number of singular fibres one has to include [120]. Thus for a K3 we need 24 singular fibres in total. Because $O(\delta) = 10$ for II^* fibres, the $E_8 \times E_8$ gauge group counts 20 singular fibres. The remaining four singular fibres should have $O(\delta) = 1$ each and not lead to any further gauge group, thus one has to include four I_1 fibres.

Putting the II^* fibres along $s = 0$ and $s = \infty$ leads to the Weierstrass form

$$y^2 = x^3 + a_4 s^4 x + (b_5 s^5 + b_6 s^6 + b_7 s^7), \tag{A.52}$$

with a_4, b_5, b_6, b_7 constant. The discriminant is

$$\delta(s) \;\; = \;\; s^{10}(4a_4^3 s^2 + 27(b_5 + b_6 s + b_7 s^2)^2). \tag{A.53}$$

For generic coefficients the four I_1 fibres are at distinct finite points in the base $\mathbb{P}^1$. If we push two I_1 fibres to $s = 0$ and two to $s = \infty$, the manifold contains two fibres of degree 12 at these points. To obtain a consistent Calabi-Yau manifold, the singularities have to be blown-up. This was done explicitly in [107], in the context of the F-theory formulation of the heterotic string. This is also what we consider in chapter 3.4. For simplicity we consider the singularity at $s = 0$ only. Close to $s = 0$, the lowest orders in s are the leading terms in the Weiertrass form, leading to

$$y^2 = x^3 + a_4 s^4 x + b_5 s^5, \tag{A.54}$$

and $\delta = s^{10}(4a_4^3 s^2 + 27b_5^2)$. The two I_2 fibres are located at $4a_4^3 s^2 + 27b_5^2 = 0$. To describe the what happens when two I_1 fibres are pushed to $s = 0$, it is useful to introduce a new parameter, t, which parametrizes the distance of the I_1 fibres to $s = 0$. For simplicity we set $a_4 = b_5 = 1$. In a neighborhood of $s = 0$ the model looks like

$$y^2 = x^3 + s^4 x + s^5 t, \tag{A.55}$$

and the discriminant locus is $\delta = s^{10}(4s^2 + 27t^2)$. The I_1 fibres are located at

$$t = \pm \frac{2}{3\sqrt{3}} s. \tag{A.56}$$

Thus for $t = 0$ the I_1 fibres are pushed into the II^* fibre and the point $s = 0$ requires a blow-up. Away from $t = 0$, the distance of the I_1 fibres to $s = 0$ grows linearly and the manifold is consistent. Formally the manifold parametrized by x, y, s, t can be considered as an elliptically fibred threefold, which is a K3 fibration over a complex disk parametrized by t. The point $s = t = 0$ can be blown up by substituting

$$s \to st, \quad t \to t. \tag{A.57}$$

The Weierstrass form (A.55) becomes

$$y^2 = x^3 + s^4t^4x + s^5t^6, \tag{A.58}$$

leading to $a = s^4t^4$, $b = s^5t^6$ and $\delta = s^{10}t^{12}(s^2+1)$. At $t = 0$ there is a new singular fibre with $O(a) = 4$, $O(b) = 6$ and $O(\delta) = 12$. The curve $t = 0$ is called exceptional divisor E. Changing the coordinates as $x \to t^2x$ and $y \to t^3y$, one obtains

$$y^2 = x^3 + s^4x + s^5, \tag{A.59}$$

and $a = s^4$, $b = s^5$, $\delta = s^{10}(s^2 + 1)$. This singularity gives rise to a minimal model. The manifold has undergone an important change though. At the point $t = 0$, which was the point of interest from the start, the base of the elliptically fibred manifold consists of *two* $\mathbb{P}^1$'s intersecting at a point. The new $\mathbb{P}^1$ is just the exceptional divisor E produced by the blow up. Note that there are no singular fibres located on E.

The blow up of the manifold does not lead to a modification of the canonical class of the manifold. The blow up adds the exceptional divisor E to the canonical class of the base. The change of coordinates however modifies L as $L - E$, and using $K_Y = \pi^*(K_{\mathbb{P}^1} + E + L - E)$ it becomes clear that the canonical class of the blown up manifold Y does not change.

Remember that we have considered only one half of the whole manifold so far. Performing the same blow-up at the point $s = \infty$ leads to an elliptically fibred manifold with a base that is a chain of three $\mathbb{P}^1$'s. The middle $\mathbb{P}^1$ is produced by the blow ups and does not have any singular fibres. This implies that $\delta =$const. and nonzero and the modular invariant function $j(\tau)$ of the elliptic fibre is constant. Thus one is free to shrink the middle $\mathbb{P}^1$ of the base manifold to zero size. The base manifold is becomes a product of two intersecting $\mathbb{P}^1$'s, with one II^* and two I_1 fibres each. The new manifold is called the stable degeneration of the former K3 manifold.

To summarize, pushing the 24 singular fibres of a K3 manifold to two points with 12 singular fibres each leads to a non-minimal model and requires a blow-up of these two points. After the blow up the K3 manifold degenerates such that the base of the manifold consists of two intersecting $\mathbb{P}^1$'s. This process is called stable degeneration.

A.4 Calabi-Yau Threefolds

Complex three dimensional Calabi-Yau manifolds have two independent non-trivial hodge numbers, $h^{(1,1)}$ and $h^{(1,2)}$. The Hodge numbers are

$$\begin{array}{ccccccc} & & & h^{(0,0)} & & & \\ & & h^{(1,0)} & & h^{(0,1)} & & \\ & h^{(2,0)} & & h^{(1,1)} & & h^{(0,2)} & \\ h^{(3,0)} & & h^{(2,1)} & & h^{(1,2)} & & h^{(0,3)} \\ & h^{(3,1)} & & h^{(2,2)} & & h^{(1,3)} & \\ & & h^{(3,2)} & & h^{(2,3)} & & \\ & & & h^{(3,3)} & & & \end{array} \quad = \quad \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{(1,1)} & & 0 & \\ 1 & & h^{(2,1)} & & h^{(2,1)} & & 1 \\ & 0 & & h^{(1,1)} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} .$$

The Euler number of a Calabi-Yau threefold Y is

$$\chi(Y) = 2(h^{(1,1)} - h^{(1,2)}), \tag{A.60}$$

where negative values are allowed.

Calabi-Yau threefolds have the very useful property of special geometry, see for example [40, 41, 42, 43, 44], which means that the integrals of the holomorphic threeform can be expressed in terms of a single holomorphic function F. We give a summary of special geometry in the following.

The threefold has a unique holomorphic $(3,0)$-form Ω that depends on the complex structure of the threefold. The integrals of the holomorphic threeform over the closed threecycles $S^a \in H_3(Y,\mathbb{Z})$, $a = 1,\ldots,2(h^{(2,1)}+1)$, of the threefold are called period integrals,

$$\Pi_a = \int_{S^a} \Omega. \tag{A.61}$$

One can expand the periods in terms of a basis A^I and B_I, $I = 0,\ldots,h^{(2,1)}$, of $H_3(Y,\mathbb{Z})$ with intersection numbers $A^I \cdot A^J = B_I \cdot B_J = 0$, $A^I \cdot B_J = \delta^I_J$. The dual basis of the cohomology is a_I, b^I with $\int_{A^I} a_J = \int_Y a_J \wedge b^I = -\int_{B_J} b^I = \delta^I_J$. The holomorphic threeform can be expanded as

$$\Omega \;=\; z^I a_I - F_I b^I \tag{A.62}$$

with $z^I = \int_{A^I} \Omega$ and $F_I = \int_{B_I} \Omega$. The complex coordinates z^I are local coordinates on the moduli space of complex structures [72, 42], they are called local projective coordinates. The full set of projective coordinates is given by z^I and the complex conjugates $\bar{z}^I$, and the $(3,0)$-form Ω depends only on z^I and the $(0,3)$-form $\bar{\Omega}$ depends only on $\bar{z}^I$. This implies that $F_I = F_I(z)$ depends on the coordinates z^I only. The periods are

$$\Pi(z) = (z^I, F_I). \tag{A.63}$$

The derivative of the holomorphic threeform, $\frac{\partial}{\partial z^I}\Omega$, has components in $H^{(3,0)} \oplus H^{(2,1)}$. It follows that $\int \Omega \wedge \frac{\partial}{\partial z^I}\Omega = 0$. Inserting (A.62) it follows that $F_I = \frac{\partial}{\partial z^I}F$ with $F = \frac{1}{2}z^I F_I$ and $F(az^I) = a^2 F(z^I)$. As z^I are projective coordinates we are free to change the coordinates as $\lambda^I = \frac{1}{z^0}z^I$, $F(\lambda) = \frac{1}{(z^0)^2}F(z)$, in other words we make a transformation

$\Omega \to \frac{1}{z^0}\Omega$). Expressed in the new coordinates $\lambda^I = (1, \lambda^\alpha)$, $\alpha = 1, \ldots, h^{(1,2)}(Y, \mathbb{Z})$, the periods are

$$\Pi(\lambda) = (1, \lambda^\alpha, F_\alpha, 2F - \lambda^\alpha F_\alpha)\,, \tag{A.64}$$

with $F_\alpha = \frac{\partial}{\partial \lambda^\alpha} F$. The function F is usually called prepotential. As mentioned above, because the integrals depend on the complex structure of the threefold, the prepotential describes the moduli space of complex structures. For Calabi-Yau threefolds, the moduli space of the complex structures is a special Kähler manifold, that means the Kähler potentials can be given in terms of the prepotential,

$$K = -\ln[2(F - \bar{F}) - (\lambda^\alpha - \bar{\lambda}^\alpha)(F_\alpha + \bar{F}_\alpha)]. \tag{A.65}$$

Special geometry also implies that the moduli space of $(1,1)$-forms of a Calabi-Yau threefold, called the Kähler moduli space, can be expressed in terms of a holomorphic prepotential $\mathcal{F}$. The prepotential $\mathcal{F}$ depends on the Kähler moduli t^i, $i = 1, \ldots, h^{(1,1)}(Y, \mathbb{Z})$.The Kähler moduli space is a special Kähler manifold,

$$K = -\ln[2(\mathcal{F} - \bar{\mathcal{F}}) - (t^i - \bar{t}^i)(\mathcal{F}_i + \bar{\mathcal{F}}_i)]. \tag{A.66}$$

The moduli space of Calabi-Yau threefolds is indeed a direct product of the moduli spaces of complex structures and Kähler moduli. This follows from $h^{(2,0)} = 0$.

A.4.1 Elliptically Fibred Calabi-Yau Threefolds

An elliptically fibred Calabi-Yau threefold can be constructed by fibering an elliptically fibered K3 over another $\mathbb{P}^1$. This manifold is an elliptic fibration over a Hirzebruch surface $\mathbb{F}_n$, where $\mathbb{F}_n$ are the possible fibrations of $\mathbb{P}^1$ over $\mathbb{P}^1$. Considering such a manifold, it is clear that all statements of the last section about singularities etc. in elliptically fibred K3 manifolds are still valid in this section. The base $\mathbb{P}^1$ of the K3 manifold of the last section is the fibre of the base $\mathbb{F}_n$ of the threefold . Thus whenever we considered points in the $\mathbb{P}^1$ base of the fibration in the last section, we now have to consider curves in the $\mathbb{P}^1$ base $\mathbb{F}_n$.

This manifold has at least Picard number three, one from the elliptic fibre, one from the $\mathbb{P}^1$ fibre and one from the $\mathbb{P}^1$ base. The divisors associated with the base and the fibre of $\mathbb{F}_n$ are usually called C and f. The self-intersection within $\mathbb{F}_n$ of the the divisor associated to the base $\mathbb{P}^1$ depends on the nature of the fibration, $C \cdot C = -n$. For $n = 0$, the Hirzebruch surface is just the product of two $\mathbb{P}^1$'s. The other intersection numbers are $f \cdot f = 0$ and $C \cdot f = 1$.

For Calabi-Yau threefolds that are elliptically fibered over $\mathbb{F}_n$ and that admit a global section, the line bundle given by the determinant of conormal bundle of the section is equal to the canonical class of the base of the fibration, see (A.41),

$$K_{\mathbb{F}_n} = -L. \tag{A.67}$$

Because the base $\mathbb{F}_n$ itself is a fibration with a section, the canonical class of $\mathbb{F}_n$ can be derived from the adjunction formula II. One has to replace X in (A.39) by $\mathbb{F}_n$ and B by the divisors $G = \{C_0, f\}$ of $\mathbb{F}_n$,

$$\det\mathcal{T}_{\mathbb{F}_n}|_G = \det\mathcal{T}_G \otimes \det\mathcal{N}_\sigma, \tag{A.68}$$

where $\sigma : \mathbb{P}^1 \to \mathbb{F}_n$ is now the section that maps $\mathbb{P}^1$ to the Hirzebruch surface. The divisor associated with the conormal bundle is just G in this case. The canonical class $K = \det\mathcal{T}^*$ of a complex manifold is equal to the first Chern class $K = -c_1(\mathcal{T})$. The Euler number of a complex n-dimensional manifold E is related to the n-th Chern class as $\chi(E) = \int_E c_n(\mathcal{T}_E)$, thus for the complex one-dimensional submanifold G we have $\chi(G) = \int_G c_1(\mathcal{T}_G)$ [12]. Integrating the canonical class of G leads to $\int K_G = -\chi(G)$ and to the adjunction formula [108]

$$\chi(G) = -G \cdot (G + K_{\mathbb{F}_n}). \qquad \text{(A.69)}$$

The curves $G = \{C_0, f\}$ are topologically spheres, thus they both have Euler-number $\chi = 2$. Using the self-intersection numbers $C_0 \cdot C_0 = -n$ and $f \cdot f = 0$ we get

$$\begin{aligned} 2 &= n - C_0 \cdot K_{\mathbb{F}_n}, \quad \text{for } G = C_0 \\ 2 &= -f \cdot K_{\mathbb{F}_n}, \quad \text{for } G = f. \end{aligned} \qquad \text{(A.70)}$$

Inserting this into the ansatz $K_{\mathbb{F}_n} = mC_0 + nf$, with m and n constant, leads to $m = -2$ and $n = -(n+2)$ and thus to the canonical class $K_{\mathbb{F}_n} = -2C_0 - (n+2)f$. Using (A.41) this gives the divisor associated to the conormal bundle of the section σ of an elliptically fibred threefold

$$L = 2C_0 + (n+2)f. \qquad \text{(A.71)}$$

Using the above formulas we are able to give the divisors associated to the zero loci of the polynomials $a(s,t) = 0$ and $b(s,t) = 0$ of the Weierstrass from (A.34). The affine coordinates y and x are sections of the line bundle $(\det\mathcal{N}^*_\sigma)^3$ and $(\det\mathcal{N}^*_\sigma)^2$. It follows that a and b are sections of $(\det\mathcal{N}^*_\sigma)^4$ and $(\det\mathcal{N}^*_\sigma)^6$. Using eqn. (A.71), the divisors A and B associated to the zero loci of $a(s,t) = 0$ and $b(s,t) = 0$ are ,

$$\begin{aligned} A &= 4L = 8C_0 + 4(n+2)f, \\ B &= 6L = 12C_0 + 6(n+2)f. \end{aligned} \qquad \text{(A.72)}$$

From $\delta = 27b^2(\{s\}) + 4a^3(\{s\})$ it follows that

$$\Delta = 12L = 24C_0 + 12(n+2)f, \qquad \text{(A.73)}$$

where Δ is the divisor associated to the zero locus $\delta = 0$.

A.4.2 Point-like Instantons

In this chapter we consider a special aspect of heterotic string compactifiactions, the point-like instantons [109]. They play an important role for constructing dual heterotic/type IIA pairs with non-trivial gauge groups [99, 100]. For a review, see [108]. In heterotic string compactifications, one needs to specify the compactification manifold Y and also a gauge bundle F. This gauge bundle is in general not flat but has some

[12] Note that we use the same notation $c_1(\mathcal{T}_G)$ for the divisor as well as for the dual $(1,1)$-form. This notation common in the physics literature.

nonzero second Chern-class $c_2(F)$, called instanton number. A point-like instanton is the limit where all the curvature of the gauge bundle is concentrated at points in the mainfold Y. These points are the locations of the point-like instantons. If several instantons coalesce at the same point, the instanton numbers add. Each point-like instanton contributes $c_2 = 1$ if the holonomy around the point-like instanton is trivial, that means if Y is smooth. Aspects of point-like instantons in ehterotic string theory are also discussed in [104, 111, 112, 113]. Point-like istantons at orbifold singularities have non-trivial holonomy. They have in general fractional instanton numbers. Point-like instantons with non-trivial holonomy in the context of type II/heterotic duality have been disussed in [107, 110] [13].

Consider the $E_8 \times E_8$ string compactified on a K3 manifold, i.e. with $c_2(F) = 24$ [108]. The gauge group of the compactified theory depends on the holonomy of the gauge bundle F. The subgroup of $E_8 \times E_8$ that commutes with the holonomy group H is the observed gauge group G. If all 24 instantons are point-like instantons, the curvature of the gauge bundle F is zero everywhere else. Thus by shrinking all instantons to zero size one preserves the full $E_8 \times E_8$ gauge group. If the instantons are large however, F has some curvature which gives rise to a non-trivial holonomy group H. A holonomy group $H = E_8 \times E_8$ for example breaks the gauge group G of the heterotic string completely.

Consider type IIA theory compactified on an elliptically fibred threefold Y. The base of the threefold is a Hirzebruch surface $\mathbb{F}_n$. The dual heterotic theory is compactified on K3$\times T^2$, where the K3 is elliptically fibred. If the theories are dual, the type IIA theory should contain 24 instantons, just as the heterotic theory. Consider the elliptically fibred threefold of the IIA theory. Using again the letters A, B and Δ to denote the divisors associated to the zero loci of the polynomials in the Weierstrass form $a = 0$, $b = 0$ and $\delta = 0$, we have

$$\begin{aligned} A &= 4L = 8C_0 + 4(n+2)f, \\ B &= 6L = 12C_0 + 6(n+2)f, \\ \Delta &= 12L = 24C_0 + 12(n+2)f. \end{aligned} \tag{A.74}$$

The E_8 fibres are located on curves in the base $\mathbb{F}_n$, the curves should be disjoint. Thus one E_8 is put along C_0 and the other along $C_\infty = C_0 + nf$. Substracting the part from A, B and Δ which is due to the II^* fibres one gets

$$\begin{aligned} A' &= 8C_0 + 4(n+2)f - 4C_0 - 4(C_0 + nf) = 8f, \\ B' &= 12C_0 + 6(n+2)f - 5C_0 - 5(C_0 + nf) = 2C_0 + (12+n)f, \\ \Delta' &= 24C_0 + 12(n+2)f - 10C_0 - 10(C_0 + nf) = 4C_0 + (24+2n)f. \end{aligned} \tag{A.75}$$

The remaining discriminant curve Δ' collides with C_0 exactly $\Delta' \cdot C_0 = 2(12-n)$ times and with C_∞ there are $\Delta' \cdot C_\infty = 2(12+n)$ collisions. Because of $\Delta' = 2B'$, each time that B' collides with C_0 or C_∞ the discriminant Δ' hits C_0 or C_∞ twice at the same point. Thus Δ' has $(12-n)$ intersection of degree two with C_0 and $(12+n)$ intersections of degree 2 with C_∞.

[13] A discussion of instantons in terms of D-branes on singular spaces can be found in [114, 115, 116, 117].

These intersection points of the II^* and the two I_1 curves produce singular fibres of degree $O(\delta) = 12$ and thus should be blown-up to obtain a smooth model. Note that locally at each intersection point, the singularity looks like the one described in A.3.3. The blow up introduces a new exceptional divisor E for each intersection point. The blow-up is explained in A.3.3. Blowing up all intersection points leads to 24 exceptional divisors in the base. As in the example of the last section, the blow up does not change the canonical class of the threefold Y. The canonical class of the Hirzebruch surface gets modified as $K_{\mathbb{F}_n} - E$ and the line bundle as $L + E$, which leads to an unaltered canonical class $K_X = 0$. Thus the blown up manifold is still Calabi-Yau.

After the blow-up one has a smooth Calabi-Yau manifold with singularities that produce a gauge group $E_8 \times E_8$. Not blowing up the singularities would imply, in order to keep the manifold smooth, that the gauge group E_8 has to be broken to some subgroup. This would produce lower order singularities at the intersection points. This reminds of the situation that occurs in the dual heterotic theory if the point-like instantons become large, as mentioned at the beginning of the chapter. The above compactification of the type IIA string theory indeed corresponds to the heterotic string on K3$\times T^2$ with gauge group $E_8 \times E_8$, and thus with 24 pointlike instantons. The point-like instantons correspond to the blown up singularities at the intersection of the II^* and the I^1 curve in the dual IIA theory. The gauge bundle of the heterotic string F is a sum of two E_8 bundles, $F = F_1 \oplus F_2$ such that $c_1(F_1) = 12 - n$ and $c_1(F_2) = 12 + n$. The dual IIA theory the $12 + n$ and $12 - n$ blow ups along C_O and C_∞ at the intersections of the II^* with the I^1 curve. Making some of the instantons of the heterotic theory large corresponds to the disappearance of some of the blow ups in the $\mathbb{F}_n$ of the dual IIA theory.

To obtain a total instanton number 24 there is still the freedom of choosing how to distribute the instantons between the two E_8 bundles. Fixing the integer n on the heterotic side corresponds to fixing the Hirzebruch surface $\mathbb{F}_n$ on the IIA side. Choosing a symmetric embedding with $c_1(F_1) = c_2(F_2) = 12$ for example implies that the dual IIA theory is compactified on a threefold with a base $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$.

A.5 Calabi-Yau Fourfolds

Complex four-dimensional Calabi-Yau manifolds [18, 20] have three independent non-trivial Hodge numbers. The Hodge numbers are

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & h^{(1,1)} & & 0 & & \\ & 0 & & h^{(2,1)} & & h^{(1,2)} & & 0 & \\ 1 & & h^{(3,1)} & & h^{(2,2)} & & h^{(1,3)} & & 1 \\ & 0 & & h^{(3,2)} & & h^{(2,3)} & & 0 & \\ & & 0 & & h^{(3,3)} & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array} , \qquad \text{(A.76)}$$

where $h^{(1,1)} = h^{(3,3)}$ and $h^{(1,2)} = h^{(2,1)} = h^{(3,2)} = h^{(2,3)}$. Counting the number of independent Hodge numbers one arrives at $h^{(1,1)}$, $h^{(1,2)}$, $h^{(1,3)}$ and $h^{(2,2)}$ and thus four independent numbers. There is an additional relation between the Hodge numbers of a fourfold [98],

$$h^{(2,2)} = 2(22 + 2h^{(1,1)} + 2h^{(1,3)} - h^{(1,2)}). \tag{A.77}$$

This leaves only three independent Hodge numbers.

The moduli space of a fourfold is a direct product of complex structure moduli and Kähler moduli only if $h^{(1,2)} = 0$. Also the moduli spaces do not have the special geometry structure of Calabi-Yau threefolds and are therefore much more difficult to handle.

The cohomology $H^{(2,2)}$ plays a special role in Calabi-Yau fourfolds. One can make a decomposition

$$H^{(2,2)} = H^{(2,2)}_{vp} \oplus H^{(2,2)}_{hp}, \tag{A.78}$$

where the subscripts denote the vertical and horizontal primary subspaces of the $(2,2)$ cohomology. The vertical primary cohomology of a Calabi-Yau d-fold is the subspace of the vertical cohomology $\oplus_{k=0}^{d} H^{(k,k)}(Y_d)$ obtained by taking the wedge products of k $(1,1)$-forms. The horizontal primary cohomology is the subspace of the horizontal homology $\oplus_{k=0}^{d} H^{(d-k,k)}(Y_d)$ generated by successive derivatives $D^k\Omega$ of the holomorphic $(d,0)$-form Ω [42, 18]. The connection in $D = \partial + w$ is chosen such that the covariant derivative of a $(d-k,k)$-form is a $(d-k-1,k+1)$-form. The ordinary derivative ∂ of a $(d-k,k)$-form is an element of $H^{(d-k,k)}(Y) \oplus H^{(d-k-1,k+1)}(Y_d)$ and the connection is chosen such that it cancels the part in $H^{(d-h,k)}(Y_d)$.

Note that for Calabi-Yau threefolds, each cohomology is either purely vertical (these are $H^{(0,0)}, H^{(1,1)}, H^{(2,2)}, H^{(3,3)}$) or horizontal ($H^{(0,3)}, H^{(1,2)}, H^{(2,1)}, H^{(3,0)}$). The primary subspaces are equal to the full cohomologies. One can get cohomologies with both horizontal and vertical elements only for Calabi-Yau manifolds of even complex dimensions d, because otherwise $d - k = k$ is not possible. For K3 manifolds, it is the $(1,1)$-forms that can be elements of either the vertical or the horizontal cohomology.

For Calabi-Yau fourfolds, there are two independent non-trivial members of the primary vertical cohomology, $H^{(1,1)}$ and $H^{(2,2)}_{vp}$. We omit the subscript vp for the $(1,1)$-forms, because the vertical primary subspace is equal to the full vertical cohomology, $H^{(1,1)}_{vp} = H^{(1,1)}$. The horizontal primary cohomology also has two independent non-trivial elements.

Finally some remarks about elliptically fibred fourfolds. One can construct an elliptically fibred fourfold by fibering an elliptically fibred threefold over another $\mathbb{P}^1$. This fourfold is at the same time an elliptic fibration over a base consisting of three $\mathbb{P}^1$'s, a K3 fibration over a Hirzebruch surface $\mathbb{F}_n$ and a threefold fibration over a base $\mathbb{P}^1$. The Picard number of such a fourfold is at least four.

B Mirror Symmetry

B.1 Conformal Field Theory

Type II string theory is described in terms of a two-dimensional $N = (1,1)$ supersymmetric conformal field theory on the worldsheet Σ of the strings. The scalars X^M, $M = 0, \ldots, 9$, of the two-dimensional field theory are the coordinates of the target space $R^{(1,9)}$ and map the string worldsheet to the ten-dimensional target space, $X : \Sigma \to R^{(1,9)}$ [1, 2]. In superconformal gauge the action reads

$$S = -\frac{1}{2\pi}\int_\Sigma d^2\sigma \left(\partial^\alpha X^M \partial_\alpha X_M - i\bar\psi^M \gamma^\alpha \partial_\alpha \psi_M\right). \tag{B.1}$$

The conformal field theory contains only kinetic terms.

Compactifying the theory on a complex d-dimensional Calabi-Yau manifold Y_d means that the target space is not flat but a product $R^{(1,9-2d)} \times Y_d$. It is in principle possible to consider more general compactifications such that the target space is not a direct product of the internal and the flat space, but we consider only the simple case with the direct product structure. The worldsheet action splits into two parts, one with the $(10-2d)$-dimensional flat space as a target space and another one with the Calabi-Yau manifold as a target space. The first part has the same action as (B.1), just with a lower range of the coordinates $X^M \to X^\mu$, $\mu = 0, \ldots, (9-2d)$. The worldsheet action has local $N = (1,1)$ supersymmetry. The action with the compact target space is more interesting. The target space is chosen such that the worldsheet theory has some global worldsheet supersymmetry in addition to the local supersymmetry. In type II compactifications to four dimensions this is an $N = (2,2)$ supersymmetry. This is necessary in order to have $N = 2$ space-time supersymmetry in four dimensions [45] [14]. The internal conformal field theory is an interacting theory with in general unknown interactions. Strings in general backgrounds are described in [46, 47]. The assumption of a global $N = (2,2)$ supersymmetry makes it possible to handle the conformal field theory to some extent and make general statements about the interaction terms. The action is a non-linear sigma model in left- and right-moving worldsheet coordinates $z, \bar z$ is (see for example [48])

$$\begin{aligned} S \;=\; T\int_\Sigma dz d\bar z \,\Big(& g_{m\bar n}(\partial u^m \bar\partial u^{\bar n} + \bar\partial u^m \partial u^{\bar n}) - i b_{m\bar n}(\partial u^m \bar\partial u^{\bar n} - \bar\partial u^m \partial u^{\bar n}) \\ & - i g_{m\bar n}\rho^m D\chi^{\bar n} - i g_{m\bar n}\rho^{\bar m}\bar D\chi^n - \frac{1}{2}R_{m\bar m n\bar n}\chi^m\chi^{\bar m}\rho^n\rho^{\bar n}\Big), \end{aligned} \tag{B.2}$$

where u^m and $u^{\bar m}$, $m, \bar m = 1, \ldots, 3$, are complex coordinates on Y_3, $\chi^m, \rho^m, \chi^{\bar m}, \rho^{\bar m}$ are the worldsheet fermions of the $N = (2,2)$ supersymmetry and the covariant derivative is $\bar D\chi^m = \bar\partial\chi^m + \Gamma^m_{no}\bar\partial u^n \chi^o$. To provide a consistent string background the sigma model has to be conformally invariant [46, 47].

As the Calabi-Yau manifold is not flat, the metric $g_{mn}(u)$ depends on the coordinates of the target space. The coupling of the worldsheet bosons u^m to the metric contains interaction terms of generically all orders. The metric can be treated as a background

[14] To be precise, it was shown in [45] that four-dimensional heterotic string compactifications with N=1 space-time supersymmetry require an additional global $N = (2,0)$ worldsheet supersymmetry.

field of the worldsheet theory. In addition to the metric the above action includes the antisymmetric tensor $b_{m\bar{n}}$ of the NS-NS sector as a background field. The terms describing the coupling of the worldsheet fields to the background fields are identical to the vertex operators of the corresponding background field at vanishing momentum [15]

$$\begin{aligned} V_g &\sim g_{m\bar{n}}(\partial u^m \bar{\partial} u^{\bar{n}} + \bar{\partial} u^m \partial u^{\bar{n}}) e^{ik\cdot u} + O(k), \\ V_b &\sim b_{m\bar{n}}(\partial u^m \bar{\partial} u^{\bar{n}} - \bar{\partial} u^m \partial u^{\bar{n}}) e^{ik\cdot u} + O(k). \end{aligned} \tag{B.3}$$

Including background fields from the R-R sector in this way is not possible because the vertex operators have branch cuts [16]. Background fields from the R-R sector are usually not considered in non-linear sigma models.

The superconformal algebra of the sigma-model has four generators: the energy-momentum tensor T, two fermionic super-currents $T_F^{\pm}$ and a $U(1)$ current J. We consider only the left-moving part of the worldsheet algebra, the right-moving part is obtained by $z \to \bar{z}$. Expanding the generators as

$$\begin{aligned} T(z) &= \sum L_n z^{-n-2}, \quad L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z) \\ T_F^{\pm}(z) &= \sum G_r^{\pm} z^{-r-3/2}, \quad G_r^{\pm} = \oint \frac{dz}{2\pi i} z^{r+\frac{1}{2}} T_F^{\pm}(z) \\ J(z) &= \sum J_n z^{-n-1}, \quad J_n = \oint \frac{dz}{2\pi i} z^n J(z), \end{aligned} \tag{B.4}$$

with $r \in \mathbb{Z}$ in the Ramond sector and $r \in \mathbb{Z} + 1/2$ in the Neveu-Schwarz sector, the algebra is

$$\begin{aligned} [L_m, L_n] &= (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \\ \{G_r^{\pm}, G_s^{\mp}\} &= 2L_{r+s} \pm (r-s)J_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}, \quad \{G_r^{\pm}, G_s^{\pm}\} = 0 \\ \left[L_n, G_r^{\pm}\right] &= \left(\frac{n}{2} - r\right) G_{n+r}^{\pm}, \quad [L_n, J_m] = -mJ_{m+n} \\ [J_n, J_m] &= \frac{c}{3} n\delta_{m+n,0}, \quad [J_n, G_r^{\pm}] = \pm G_{n+r}^{\pm}. \end{aligned} \tag{B.5}$$

The number c is the central charge of the algebra. Consistency requires $c = \frac{3d}{2}$ for a $d-$dimensional conformal field theory, $c = 1$ for each worldsheet boson and $c = 1/2$ for each worldsheet fermion. The conformal fields $\psi(z)$ of the worldsheet theory are in one-to-one correspondence to the states of the conformal algebra, $|\psi\rangle = lim_{z\to 0}\psi(z)|0\rangle$.

An isomorphic algebra can be obtained by transforming the above algebra under the spectral flow [49] generated by $\mathcal{U}_\Theta$

$$\mathcal{U}_\Theta L_n \mathcal{U}_\Theta^{-1} = L_n + \Theta J_n + \frac{c}{6}\Theta^2 \delta_{n,0}$$

[15] To describe physical couplings the vertex operators must not have any explicit dependence on the unphysical ghost sector of the conformal field theory, that is they are in the ghost number zero picture [2].

[16] In terms of the vertex operators they have half-integer ghost number and cannot be transformed to vertex operators with ghost number zero via picture changing [2].

$$\mathcal{U}_\Theta J_n \mathcal{U}_\Theta^{-1} = J_n + \frac{c}{3}\Theta\delta_{n,0}$$
$$\mathcal{U}_\Theta G_r^\pm \mathcal{U}_\Theta^{-1} = G^\pm_{r\pm\Theta}. \tag{B.6}$$

From the last equation it is clear that spectral flow with $\Theta = \mathbb{Z} + 1/2$ maps the R sector to the NS sector and vice versa, while $\Theta = \mathbb{Z}$ maps the NS to the NS and the R to the R sector. States transform as $|\psi\rangle \to \mathcal{U}_\Theta|\psi\rangle$.

We denote by $|\phi\rangle$ a primary state with $U(1)$ charge q, $J_0|\phi\rangle = q|\phi\rangle$, and with conformal weight h, $L_0|\phi\rangle = h|\phi\rangle$. Primary states are annihilated by the positive modes of the generators, $(L_n, G_r^\pm, J_n)|\phi\rangle = 0$, $n, r > 0$. Let us consider the ground states of the Ramond sector $|0\rangle_R$. Ground states are primary states that fulfill the additional condition

$$G_0^\pm|0\rangle_R = 0. \tag{B.7}$$

Using (B.5) we get

$$_R\langle 0|\{G_0^\pm, G_0^\mp\}|0\rangle_R = 2h - \frac{c}{12} = 0 \tag{B.8}$$

and thus $h = c/24$ for R ground states. Via spectral flow with $\Theta = \mp 1/2$ the ground states of the R sector are mapped to the chiral and anti-chiral primary fields of the NS sector $|i\rangle_{NS}$. The chiral and anti-chiral primary fields satisfy

$$G^\pm_{-1/2}|i\rangle_{NS} = 0 \tag{B.9}$$

and using again (B.5)

$$_{NS}\langle i|\{G^\pm_{-1/2}, G^\mp_{1/2}\}|i\rangle_{NS} = 2h \mp q = 0. \tag{B.10}$$

The chiral primary states have $h = q/2$ and the anti-chiral primary states have $h = -q/2$ [50].

The conformal fields ϕ_i associated to the chiral and anti-chiral states $|i\rangle_{NS}$ form a ring under multiplication [50]

$$\phi_i\phi_j = c_{ij}{}^k\phi_k, \tag{B.11}$$

with the structure constants $c_{ij}{}^k$. This ring is called the (anti-) chiral primary ring a respectively c. Considering both the left- and right-moving sector we get the rings (c, c), (c, a) and their conjugates (a, a) and (a, c).

Let us consider the compactification of type II theories on a Calabi-Yau threefold Y_3. The conformal charge of the interacting conformal field theory is $c = \bar{c} = 9$. The important point is that in the large radius limit of the threefold there is a one-to-one correspondence between the elements of the cohomology of the threefold and the primary states of the superconformal algebra [51]. In the R-sector the zero modes of the supercurrents are identified formally with the exterior holomorphic differentials, $G_0^\pm \sim \partial$. Via spectral flow to the NS-sector, the zero modes of the supercurrents are identified with the exterior differential as $G^\pm_{-1/2} \sim \partial$. It was shown in [50] that each state in the NS sector has a decomposition $|\chi\rangle = |i\rangle + G^\pm_{-1/2}|i_1\rangle + G^\mp_{1/2}|i_2\rangle$ with $|i\rangle$ an (anti-) chiral primary. For primary states $|\chi\rangle$ the last term vanishes of course. This decomposition corresponds to the hodge decomposition of the differential forms. In the NS-sector (r,s)-forms are mapped to the (anti-) chiral primary states of the superconformal $N=(2,2)$ algebra with $(h,\bar{h})=\left(\frac{1}{2}r,\frac{1}{2}s\right)$. Glueing together left- and right-movers, there are obviously two distinct ways of realizing this map, one can either identify the (c,a) ring with the elements of the cohomology $(G^+_{-1/2}\sim\partial,\,\bar{G}^-_{-1/2}\sim\bar{\partial})$ or one can identify the (c,c) rings with the elements of the cohomology $(G^+_{-1/2}\sim\partial,\,\bar{G}^+_{-1/2}\sim\bar{\partial}\,)$. [17] The first possibility results in a type IIA and the second possibility in a type IIB compactification.

$$\phi_{NS}\in H^{(r,s)}(Y,\mathbb{Z}) \;\;\leftrightarrow\;\; (h,\bar{h})=\left(\frac{1}{2}r,\frac{1}{2}s\right),(q,\bar{q})=(r,-s) \quad \text{for type IIA}$$
$$\phi_{NS}\in H^{(r,s)}(Y,\mathbb{Z}) \;\;\leftrightarrow\;\; (h,\bar{h})=\left(\frac{1}{2}r,\frac{1}{2}s\right),(q,\bar{q})=(r,s) \quad \text{for type IIB.} \quad \text{(B.12)}$$

It was shown in [52, 70, 71] that the moduli of the superconformal field theory are mapped to the (anti-) chiral primary states with $(h,\bar{h})=\left(\frac{1}{2},\frac{1}{2}\right)$.

Let us consider the NS-NS sector of Type IIA theory on a threefold. The decomposition of the 10-d metric is

$$g_{MN} \;=\; \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & g_{ab} \end{pmatrix}, \qquad \text{(B.13)}$$

where g_{ab} is the metric on the threefold. Expanding the metric around the background expectation value as $g_{ab}=g^0_{ab}+\delta g_{ab}$, one obtains the Laplace equation for the fluctuation, $\Delta\delta g_{ab}=0$. The solutions to Laplace equation are the massless modes of the spectrum. In complex coordinates $m,\bar{m}=1,2,3$, one obtains decoupled equations for the components $\delta g_{m\bar{n}}$ and $\delta g_{m,n}$. The first part, $\delta g_{m\bar{n}}$, can be expanded into harmonic $(1,1)$-forms and gives $h^{(1,1)}$ zero modes. The indices of $\delta g_{m,n}$ are not antisymmetrized, thus it is not a $(2,0)$-form. However, one can construct a $(1,2)$-form as $\tilde{g}_{m\bar{n}\bar{p}}=\delta g_{m,n}g^0_{m\bar{o}}\Omega_{\bar{o}\bar{n}\bar{p}}$, thus the number of zero modes of $\delta g_{m,n}$ is $h^{(1,2)}$. The equivalent procedure can be done for the complex conjugate component $\delta g_{\bar{m},\bar{n}}$, leading to another $h^{(2,1)}$ zero modes. The Laplace equation of the antisymmetric NS-NS two-form $b_{ab}=b^0_{ab}+\delta b_{ab}$ leads to decoupled equations for the components $\delta b_{m\bar{n}}$ and δb_{mn} in complex coordinates. The first term $\delta b_{m\bar{n}}$ can be expanded in $(1,1)$-forms the second one, δb_{mn}, in $(2,0)$-forms. The number of $(2,0)$-forms however vanishes on Calabi-Yau threefolds, $h^{(2,0)}=0$, thus δb_{mn} does not contribute any zero-modes. All in all, from the NS-NS sector one obtains $2h^{(1,1)}$ zero modes from the metric and the antisymmetric two-form and another $2h^{(1,2)}$ zero modes from the antisymmetric two-form.

The $2h^{(1,1)}+2h^{(1,2)}$ zero modes correspond to (anti-) chiral primary states with conformal charge $(h,\bar{h})=\left(\frac{1}{2},\frac{1}{2}\right)$. Thus the $(1,1)$-form $\delta b_{m\bar{n}}$ corresponds to the conformal state in the (c,a) ring with $(h,\bar{h})=\left(\frac{1}{2},\frac{1}{2}\right)$ and $(q,\bar{q})=(1,-1)$ (one can equivalently start with the $(1,1)$-form $\delta g_{m\bar{n}}$). In addition the spectrum contains the (c,c), (a,c) and (a,a) rings with $(q,\bar{q})=(1,1)$, $(q,\bar{q})=(-1,1)$ and $(q,\bar{q})=(-1,-1)$. Which elements of the cohomology the last three states correspond to can be found out by taking the spectral flow of the states by $\Theta=0,\pm1$ such that the states transformed via spectral flow are in some (c,a) ring. From (B.6) it is clear that a state in (c) is transformed into a state in

[17] Taking the conjugate, i.e. the (a,c) and the (a,a) ring is equivalent of course

(a) by spectral flow $\Theta = -1$ and an (a) state is transformed into a (c) state by spectral flow with $\Theta = 1$:

$$h_\Theta \;=\; h + \Theta q + \frac{3}{2}\Theta^2, \quad q_\Theta = q + 3\Theta. \tag{B.14}$$

From the (c, a) states with new conformal weights and charges $(h_\Theta, \bar{h}_\Theta)$, $(q_\Theta, \bar{q}_\Theta)$ we read off that the states are (p, k)-forms with $p = 2h_\Theta$ and $k = 2\bar{h}_\Theta$. The state with $(q, \bar{q}) = (1, 1)$ is mapped via spectral flow with $\Theta = -1$ to the (c, a) state $(h_0, \bar{h}_{-1}) = (\frac{1}{2}, 1)$ and $(q_0, \bar{q}_{-1}) = (1, -2)$, which is a $(1, 2)$ form. For the state $(q, \bar{q}) = (-1, 1)$ taking the spectral flow leads to $(h_1, \bar{h}_{-1}) = (1, 1)$ and $(q_1, \bar{q}_{-1}) = (2, -2)$ and thus to a $(2, 2)$-form. Finally from the state with $(q, \bar{q}) = (-1, -1)$ we get $(h_1, \bar{h}_0) = (1, \frac{1}{2})$ and $(q_1, \bar{q}_0) = (2, -1)$ which is a $(2, 1)$-form. Thus the moduli of the type IIA theory are identified with the conformal fields as

$$\begin{aligned}
h^{(1,1)} \text{ zero} - \text{modes with} \;&:\; (h, \bar{h}) = (\frac{1}{2}, \frac{1}{2}),\;\; (q, \bar{q}) = (1, -1) \\
h^{(1,2)} \text{ zero} - \text{modes with} \;&:\; (h, \bar{h}) = (\frac{1}{2}, \frac{1}{2}),\;\; (q, \bar{q}) = (1, 1) \\
h^{(2,1)} \text{ zero} - \text{modes with} \;&:\; (h, \bar{h}) = (\frac{1}{2}, \frac{1}{2}),\;\; (q, \bar{q}) = (-1, -1) \\
h^{(2,2)} = h^{(1,1)} \text{ zero} - \text{modes with} \;&:\; (h, \bar{h}) = (\frac{1}{2}, \frac{1}{2}),\;\; (q, \bar{q}) = (-1, 1).
\end{aligned} \tag{B.15}$$

An obvious question to ask is what changes if we do not identify the elements of the cohomology with the (c, a) rings but with the (c, c) rings. In this case one has $2h^{(1,1)}$ moduli with $(h, \bar{h}) = (\frac{1}{2}, \frac{1}{2})$ and $(q, \bar{q}) = (1, 1), (q, \bar{q}) = (-1, -1)$ and $2h^{(1,12)}$ moduli with $(h, \bar{h}) = (\frac{1}{2}, \frac{1}{2})$ and $(q, \bar{q}) = (1, -1), (q, \bar{q}) = (-1, 1)$. It is obvious that the theory is invariant if we in addition interchange $h^{(1,1)} \leftrightarrow h^{(1,2)}$.

This is just the statement of mirror symmetry [50, 52, 53, 54]: The conformal field theory is invariant under simultaneously exchanging

$$\begin{aligned}
(c, a) &\leftrightarrow (c, c) \\
h^{(1,1)} &\leftrightarrow h^{(1,2)}.
\end{aligned} \tag{B.16}$$

For general Calabi-Yau d-folds this reads $(c, a) \leftrightarrow (c, c)$, $h^{(p,p)} \leftrightarrow h^{(p,d-p)}$. The elements of the (c, a) rings are (p, p)-forms and the elements of the (c, c) rings are $(p, d-p)$-forms that form just the primary subspaces of the vertical cohomology $H^{(p,p)}(Y_d, \mathbb{Z})$ and the horizontal cohomology $H^{(p,d-p)}(Y_d, \mathbb{Z})$ introduced in A.5.

For Calabi-Yau threefolds it is sufficient to consider only $p = 1$ because $h^{(2,2)} = h^{(1,1)}$ and $h^{(3,3)} = h^{(3,0)} = h^{(0,3)} = h^{(0,0)} = 1$ and the primary cohomologies coincide with the full cohomologies $H^{(p,p)}(Y_3, \mathbb{Z})$ and $H^{(p,d-p)}(Y_3, \mathbb{Z})$. For Calabi-Yau fourfolds the $(2, 2)$-forms are independent from the $(1, 1)$-forms and the vertical and horizontal primary subspaces of $H^{(2,2)}(Y_4, \mathbb{Z})$ do not coincide with the full cohomology.

From the point of view of the conformal field theory mirror symmetry is a very simple observation, but geometrically it has highly non-trivial implications. For a Calabi-Yau threefold for example correlation functions of three Kähler moduli, usually called Yukawa couplings, are of great importance. They are determined by the cohomology ring of the compactification manifold only in the large radius limit. Taking into account stringy corrections the Yukawa couplings are determined by the deformed cohomology, sometimes also called quantum cohomology. It is really this deformed cohomology that coincides with the (anti-) chiral rings of the superconformal algebra. The calculation for determining the Yukawa couplings using mirror symmetry is reviewed in section B.3. The first application of mirror symmetry was given in [5]. Other examples can be found for example in [55, 56, 57, 58, 59].

By identifying elements of the cohomology with $(c,a)/(c,c)$-states we consider type IIA/type IIB compactifications. This implies that type IIA theory compactified on a Calabi-Yau threefold Y_3 is identical to type IIB theory compactified on the "mirror manifold" Y_3^*. The hodge numbers of the manifolds are $h^{(p,q)}(Y_3) = h^{(p,3-q)}(Y_3^*)$. Note that the mirror manifold of a given Calabi-Yau manifold is not necessarily a Calabi-Yau manifold. A Calabi-Yau threefold with $h^{(1,2)}(Y_3) = 0$ for example has a mirror manifold with $h^{(1,1)}(Y_3^*) = 0$, this is not a Calabi-Yau manifold because it has no Kähler structure.

B.2 Topological Field Theory

The main application of mirror symmetry in Calabi-Yau threefold compactifications is to derive the Yukawa couplings $\langle \phi_i \phi_j \phi_k \rangle$ of $(1,1)$-forms including all stringy corrections. The correlation functions of the (c,a) and (a,c) primary fields on the threefold Y can be derived in terms of the simpler correlation functions of the (c,c) and (a,a) fields on the mirror manifold Y^*.

Consider the correlation functions

$$c_{ijk} \;=\; \langle \phi_i \phi_j \phi_k \rangle = \int \mathcal{D}u \mathcal{D}\chi \mathcal{D}\rho \phi_i \phi_j \phi_k e^{-S(u,\chi,\rho)}, \tag{B.17}$$

where $S(u,\chi,\rho)$ is the sigma-model action with target space Y_3 with a field content consisting of the worldsheet bosons u and the fermions of the $N = (2,2)$ supersymmetry. The threepoint functions are fixed by the structure "constants" (B.11), which are not constant but an infinite series in the Kählermoduli t. The correlation functions are not evaluated using the above sigma model action but the "twisted" topological sigma model [48, 62, 63, 67] obtained from the original action by twisting the worldsheet fermions. The important point is that the threepoint functions containing the (anti-) chiral primary fields are invariant under the twisting, but can be derived explicitly in the twisted model.

There are two possible ways of twisting the sigma-model to obtain a topological field theory, these are called the A- and B-model. The observables of the A-model are the (c,a) rings and the observables of the B-model are the (c,c) rings. Mirror symmetry can be understood naturally in terms of the topological field theories, the correlation functions of the A-model with target space Y are mapped to the correlation functions of the B-model with target space Y^* via mirror symmetry.

Let us describe the twisting of the sigma-model. We follow the discussion of [67]. The starting point is the worldsheet action in complex coordinates of the target manifold Y_3 and left- and right-moving coordinates on the worldsheet $(z,\bar{z})$

$$S \;=\; T \int_\Sigma dz d\bar{z} \left(g_{m\bar{n}}(u)(\partial u^m \bar{\partial} u^{\bar{n}} + \bar{\partial} u^m \partial u^{\bar{n}}) - i B_{m\bar{n}}(u)(\partial u^m \bar{\partial} u^{\bar{n}} - \bar{\partial} u^m \partial u^{\bar{n}}) \right) \tag{B.18}$$

The action can be written in the form

$$S = 2T \int_\Sigma dz d\bar{z} G_{m\bar{n}}(u) \bar{\partial} u^m \partial u^{\bar{n}} - iT \int_\Sigma (u^* t), \tag{B.19}$$

where $t = B + iG$ is a complex Kählerform and

$$(u^* t) = (g_{m\bar{n}}(u) + i g_{m\bar{n}}(u))(\partial u^m \bar{\partial} u^{\bar{n}} - \bar{\partial} u^m \partial u^{\bar{n}}) \tag{B.20}$$

is the pullback of t from Y_3 to the worldsheet Σ. Including the fermions χ and ρ of the $N = 2$ supersymmetry the action reads [67]

$$\begin{aligned} S &= 2T \int_\Sigma dz d\bar{z} \left(g_{m\bar{n}} \bar{\partial} u^m \partial u^{\bar{n}} - \frac{i}{2} g_{m\bar{n}} \rho^m D \chi^{\bar{n}} - \frac{i}{2} g_{m\bar{n}} \rho^{\bar{m}} \bar{D} \chi^n - \frac{1}{4} R_{m\bar{m}n\bar{n}} \chi^m \chi^{\bar{m}} \rho^n \rho^{\bar{n}} \right) \\ &- iT \int_\Sigma (u^* t). \end{aligned} \tag{B.21}$$

The term in the second line depends only on the cohomology class of t and on the homology class of $u(\Sigma)$ in Y_3. The other part in the first line contains fermionic as well as bosonic degrees of freedom. This is the part that undergoes the twisting.

The twisting consists of transforming the world-sheet spinors $\chi^m, \chi^{\bar{m}}, \rho^m, \rho^{\bar{m}}$ into bosonic fields with spin one and spin zero. The twisted theory is not supersymmetric anymore as it contains no fermions. However, after the transformation, the supersymmetry is transformed into another symmetry, a BRST symmetry. The BRST symmetry is generated by the modified supercurrents $G^\pm$, which are bosonic and have spin $3/2 \pm 1/2$ after the twisting. Also, the twisted theory is BRST exact, that means the theory is the BRST variation of something. Being BRST invariant and exact is just the definition of a topological theory. Thus the twisting of the worldsheet theory generates a topological field theory. There are two inequivalent ways of doing the twisting: first, one assumes that the spinors χ^m and ρ^m are transformed to spin zero fields and the conjugates $\chi^{\bar{m}}$ and $\rho^{\bar{m}}$ are transformed to spin one fields on the worldsheet. This is the A-model. Second, one can do the twisting such that χ^m and $\chi^{\bar{m}}$ are transformed into spin one fields and $\rho^m, \rho^{\bar{m}}$ are transformed into spin zero fields. This is the B-model.

Being topological theories, the correlation functions of the A- and B-model are constant. The other part, containing $\int u^*(t)$, contributes moduli-dependend terms to the correlation functions but has no additional degrees of freedom. We denote the action for the topological A- and B-model by S_A and S_B in the following.

For the three-point function in the A-model we get

$$c_{ijk} = e^{iT \int \phi^*(t)} \int \mathcal{D}u \mathcal{D}\chi \mathcal{D}\rho \phi_i \phi_j \phi_k e^{-\mathcal{S}_A}. \tag{B.22}$$

The correlation functions are invariant under changing continuous parameters of the topological A-model. Thus we are free to choose the limit $T \to \infty$, which means that the A-model action only contributes by leading order $S_A = 0$ to the path integral.

From the equation of motion of the sigma model action (B.21), it follows that $\bar{\partial} u^i = 0$. This implies that that the maps u are $n-$fold covers of algebraic curves in Y_3. In other words, the worldsheet instantons are due to fundamental strings wrapped n times on holomorphic curves in the Calabi-Yau manifold.

We expand the Kählerform as $t = t^i e_i$, where e_i are a base of the $(1,1)$-forms, and denote the integers of the homology class of u by $d_i = \int_\Sigma (u^* e_i) = \frac{1}{2\pi}\int_\phi e_i$. The numbers d_i are also called instanton numbers or the degree of the holomorphic curves. Thus the second line of action (B.21) is

$$S = -2\pi i T \sum_i t^i d_i. \qquad \text{(B.23)}$$

Wrapping a fundamental string $n = 0$ times on the holomorphic curves means that $u(\Sigma)$ is topologically trivial. In this case all d_i vanish, leaving no instanton contribution.

For evaluating the correlation functions one expands them in classes

$$c_{ijk} = \sum_g c_{ijk}^{(g)}, \qquad \text{(B.24)}$$

where the sum over $g = (d_1, \ldots, d_n)$ takes into account all curves of multi-degree $(d_1, \ldots, d_n)$. Thus the path integral takes the form

$$c_{ijk}^{(g)} = e^{-S^{(g)}} \int^{(g)} \mathcal{D}u\mathcal{D}\rho\mathcal{D}\chi\ \phi_i\phi_j\phi_k e^{-\mathcal{S}_A}. \qquad \text{(B.25)}$$

Using $q_i = e^{2\pi i t^i}$ we finally have an expansion

$$c_{ijk} = N_{ijk}^{(0,\ldots,0)} + N_{ijk}^{(1,0,\ldots,0)} q_1 + N_{ijk}^{(0,1,0,\ldots,0)} q_2 + \ldots + N_{ijk}^{(2,1,0\ldots,0)} q_1^2 q_2 + \ldots, \qquad \text{(B.26)}$$

where the numbers N_{ijk} are

$$N_{ijk}^{(g)} = \int^{(g)} \mathcal{D}u\mathcal{D}\rho\mathcal{D}\chi\ \phi_i\phi_j\phi_k e^{-\mathcal{S}_A}. \qquad \text{(B.27)}$$

The correlation functions are an infinite series in q^i with integer coefficients given by the constant threepoint functions of the topological A-model. To lowest order $g = (0, \ldots, 0)$ the correlation function is given by the classical intersection numbers of three $(1,1)$-forms $d_{ijk} = \int_{Y_3} e_i \wedge e_j \wedge e_k$. In principle all higher-order terms can be derived explicitly for any given threefold Y_3. But as there are infinitely many terms this is impossible in practice. Mirror symmetry can solve this problem. We know that the correlation functions of the A-model on Y_3 are identical with those of the B-model on the mirror manifold Y_3^*. The correlation functions of the B-model are purely classical and do not have any instanton corrections. The calculation of the instanton corrections to the A-model threepoint function is subject of the next section.

Let us first consider other correlation functions of the topological A-model [18, 20, 62, 63], for example the two-point functions

$$\eta_{ij} = \langle \phi_i \phi_j \rangle. \qquad \text{(B.28)}$$

For a Calabi-Yau threefold this is the correlator of the i-th $(1,1)$-form and the j-th $(2,2)$-form in the (c,a), (a,c) rings. As $h^{(1,1)} = h^{(2,2)}$ it is correct to use the same index $i, j = 1, \ldots, h^{(1,1)}$ for both observables.

For general Calabi-Yau d-folds, the twopoint functions are the correlators of a (p,p)-form and a $(d-p, d-p)$-form, $p \leq d$, these two-point functions are usually called $\eta_{ij}^{(p)}$.

It was shown in [62] that the derivatives of all twopoint functions with respect to all Kählermoduli t^i vanish,

$$\frac{\partial}{\partial t^k}\eta_{ij} = 0. \tag{B.29}$$

Thus the two-point functions are constant and receive no worldsheet corrections.

For a Calabi-Yau threefold, the two- and three-point functions are the only correlation functions. A four-point function in the A-model is at least a $(4,4)$- form, which exists only for Calabi-Yau d-folds with $d \geq 4$. However, for Calabi-Yau d-folds of any dimension, the factorization properties of the topological field theory allow to compute all correlation functions in terms of the two- and the threepoint functions [18].

Let us call $\mathcal{O}^{(p)}_N$, $N = 1, \ldots, h^{(p,p)}(Y_d)$, a (p,p)-form and observable in the A-model with target space Y_d (for $d = 3$ we have just the one element $\mathcal{O}^{(1)}_i = \phi_i$ considered above). As explained for example in [18], the observables of the A-model are not all of the (p,p)-forms of Y_d with $p \leq d$, but only those that are the wedge product of p $(1,1)$-forms. Thus the observables of the A-model are exactly the elements of the vertical primary subspace of $H^{(p,p)}(Y_d)$ introduced in A.5. The observables of the B-model are the elements of the horizontal primary subspace of $H^{(d-p,p)}(Y_d)$. One can generalize (B.11) and introduce operator product coefficients $c^{(p,q)}$ for arbitrary dimensions d as

$$\mathcal{O}^{(p)}_N \mathcal{O}^{(q)}_M = c^{(p,q)\ \ k}_{MN}\mathcal{O}^{(p+q)}_O. \tag{B.30}$$

Thus any three-point function is of the form

$$c^{(p,q)}_{MNO} = \langle \mathcal{O}^{(p)}_M \mathcal{O}^{(q)}_N \mathcal{O}^{(d-p-q)}_O \rangle = c^{(p,q)\ \ P}_{MN}\ \eta^{(p+q)}_{PO} = c^{(q,d-p-q)\ \ P}_{NO}\ \eta^{(p)}_{MP} \tag{B.31}$$

The case $p = q = 1$, $d = 3$ reproduces $c^{(1,1)}_{ijk} = c_{ijk}$ of (B.22).

Using the above relation, the fourpoint functions on a Calabi-Yau fourfold ($d = 4$ and all fields $\mathcal{O}$ are $(1,1)$-forms) are

$$\langle \mathcal{O}^{(1)}_M \mathcal{O}^{(1)}_N \mathcal{O}^{(1)}_O \mathcal{O}^{(1)}_P \rangle = c^{(1,1)\ \ Q}_{MN}\ c^{(2,1)\ \ R}_{QO}\ \eta^{(3)}_{QP} = c^{(1,1)\ \ Q}_{MN}\ \eta^{(2)}_{QR} c^{(1,1)\ \ R}_{OP}. \tag{B.32}$$

For the metric we can use $\eta^{(3)} = \eta^{(1)T}$ to express the fourpoint function as [18, 20]

$$c^{(1,1)\ \ Q}_{MN}\ c^{(2,1)\ \ R}_{QO}\ \eta^{(1)}_{PR} = c^{(1,1)\ \ Q}_{NM}\ \eta^{(2)}_{QR} c^{(1,1)\ \ R}_{OP}. \tag{B.33}$$

Interesting conditions for correlation functions follow form the associativity of the operator products, for the fourpoint function we get for example

$$c^{(1,1)\ \ Q}_{MN}\ \eta^{(2)}_{QR} c^{(1,1)\ \ n}_{OP} = c^{(1,1)\ \ Q}_{MO}\ \eta^{(2)}_{QR} c^{(1,1)\ \ R}_{NO}. \tag{B.34}$$

From (B.31) we know that $c^{(1,1)\ \ Q}_{MN}\ \eta^{(2)}_{QO} = c^{(1,2)\ \ Q}_{NO}\ \eta^{(1)}_{MQ}$, and inserting this in the above equation leads to the relation

$$c^{(1,1)\ \ Q}_{MN}\ c^{(1,2)\ \ R}_{OQ} = c^{(1,1)\ \ Q}_{MO}\ c^{(1,2)\ \ R}_{NQ}, \tag{B.35}$$

which is highly non-trivial if one includes the worldsheet corrections.

B.3 Worldsheet Instantons

In this section, we review the derivation of the three-point function (B.26) of the topological A-model by using mirror symmetry. A more detailed review can be found in [60]. Special geometry, introduced in A.4, implies that the threepoint function c_{ijk} is the derivative of a holomorphic prepotential $\mathcal{F}$

$$c_{ijk} \quad = \quad \partial_i \partial_j \partial_k \mathcal{F}, \tag{B.36}$$

where $\partial_i = \partial/\partial t_i$. Mirror symmetry implies that the prepotential $\mathcal{F}$ is mapped to the prepotential F of the B-model on the mirror manifold Y_3^*. The prepotential F of the B-model depends on the $(1,2)$-forms of Y_3^* only and thus receives no worldsheet instanton corrections. Thus F can be derived explicitly. The mirror map gives the prepotential of the A-model on Y_3 including all instanton corrections.

Let us consider the mirror manifold Y_3^*. The threefold has a unique holomorphic $(3,0)$-form Ω that depends on the complex structure moduli. The derivatives of Ω with respect to the complex structure moduli have components in $H^{(3,0)} \oplus H^{(2,1)} \oplus H^{(1,2)} \oplus H^{(0,3)}$. More precisely, the first derivative is in $H^{(3,0)} \oplus H^{(2,1)}$, the second derivative is in $H^{(3,0)} \oplus H^{(2,1)} \oplus H^{(1,2)}$ and the third derivative in $H^{(3,0)} \oplus H^{(2,1)} \oplus H^{(1,2)} \oplus H^{(0,3)}$. Since the number of 3-forms $b^3 = 2(h^{(2,1)}+1)$ is finite there must be some linear relation between the derivatives, $\mathcal{L}\Omega = d\eta$ where $\mathcal{L}$ is some differential operator with moduli dependent coefficients. Integrating this equation over a closed threecycle $S^a \in H_3(Y_3^*, \mathbb{Z})$, $a = 1, \ldots, 2(h^{(2,1)}+1)$, we get a differential equation

$$\mathcal{L}\Pi_a = 0, \tag{B.37}$$

where $\Pi_a = \int_{S^a} \Omega$ are the period integrals of Ω. These equations are called the Picard-Fuchs equations. For a given threefold Y_3^* one can construct the differential operator $\mathcal{L}$ which fulfills the Picard-Fuchs equation and write down the explicit form of the period integrals Π_a. We do not perform such a calculation here but refer to [60] and to [5, 55, 56, 57, 58, 59].

Given an explicit expression for the periods one can write down the period integrals in a more convenient form for the mirror map. As explained in A.4, one can express the periods in terms of the special coordinates λ^i, $i = 1, \ldots, h^{(1,2)}(Y^*, \mathbb{Z})$, the periods are

$$\Pi(\lambda) = \left(1, \lambda^i, F_i, 2F - \lambda^i F_i\right), \tag{B.38}$$

with $F_i = \frac{\partial}{\partial \lambda^i} F$.

Mirror symmetry maps the prepotential $F(\lambda)$ of the complex structure moduli space of Y_3^* to the prepotential $\mathcal{F}(t)$ of the Kählermoduli space of Y_3. After the mirror map the above period vector is

$$\Pi(t) = \left(1, t^i, \mathcal{F}_i, 2F - t^i \mathcal{F}_\rangle\right), \tag{B.39}$$

with $\mathcal{F}_i = \frac{\partial}{\partial t^i}\mathcal{F}$. The index i labels the $(1,1)$-forms on Y_3, $i = 1, \ldots, h^{(1,1)}(Y_3, \mathbb{Z})$. The mirror map is $\lambda^i = t^i$ for the first two components, they do not contain worldsheet corrections on Y_3. For the last two components the mirror map is $\log(\lambda^i) = 2\pi i t^i + O(t^2)$. To perform the mirror map explicitly one keeps only the linear term in t^i. This leads to corrections in terms of the variables $q = e^{2\pi i t^i}$ defined in (B.27) [18]. The coordinates parametrizing the first two and the last two components of Π are called t-type and q-type coordinates in [18]. Let us stress again that the prepotential $F(z)$ on Y_3^* does not have worldsheet corrections of any kind and can be derived exactly. After the mirror map one obtains the full prepotential $\mathcal{F}(t)$ on Y_3 including all worldsheet corrections. $\mathcal{F}$ is of the form [5][19]

$$\mathcal{F} = \frac{1}{6} d_{ijk} t^i t^j t^k + a_{ij} t^i t^j + b_i t^i + \frac{1}{2} c - \frac{1}{(2\pi)^3} \sum_g n_g Li_3(e^{2\pi i \sum t^i d_i}), \quad \text{(B.40)}$$

where

$$Li_3(x) \equiv \sum_{j=1}^{\infty} \frac{x^j}{j^3}. \quad \text{(B.41)}$$

d_i is again the instanton number of the i-th $(1,1)$-form e_i, $d_i = \int_{\mathcal{C}} e_i$, and n_g is the number of isolated holomorphic curves $\mathcal{C}$ of multi-degree $g = (d_1, \ldots, d_{h^{(1,1)}})$. The sum over j takes into account multiple coverings of a fundamental string wrapped on a given curve.

The mirror symmetry result contains more information than the correlation function of the topological sigma model of the last section in the sense that one derives the full holomorphic prepotential, whereas the correlation functions are the third derivatives of the prepotential. The complex constants a_{ij}, b_i and c do not appear in the threepoint function (B.36).

The constants a_{ij}, b_i and c are determined in [59, 60]. The real parts of the constants are considered as being irrelevant because they neither contribute to the correlation functions c_{ijk} nor to the Kählerpotential $K = -\ln[2(\mathcal{F} - \bar{\mathcal{F}}) - (t^i - \bar{t}^i)(\mathcal{F}_i + \bar{\mathcal{F}}_i)]$ and thus have no impact on the theory. The Kählerpotential without the instanton corrections has a continuous Peccei-Quinn symmetry $t^i \to t^i + n^i$ [61]. This implies that the imaginary parts of a_{ij} and b_i are zero. Thus the only relevant contribution is the imaginary constant c,

$$c = \frac{1}{(2\pi i)^3} \chi(Y_3) \zeta(3), \quad \text{(B.42)}$$

where $\chi(Y_3)$ is the Euler number of Y_3 and $\zeta(3)$ is a constant. This term can be identified with a loop correction of the worldsheet theory.

B.4 Mirror Symmetry and D-branes

For completeness we consider type II theories including D-brane configurations on Calabi-Yau manifolds. D-branes wrapped on curved spaces have been studied first in [68] and in in the context of mirror symmetry in [69]. We follow the discussion of [69]. The question is whether mirror symmetry is applicable to D-branes in Calabi-Yau manifolds, or in

[18] For simplicity, the dependence of α' or equivalently the tension T is omitted

[19] In [5] models with $h^{(1,1)} = 1$ were considered. The generalization to arbitrary $h^{(1,1)}$ can be found for example in [59]

other words which kind of cycles the D-branes can be wrapped on. We only consider BPS states, which means that in the presence of D-branes half of the $N = (2, 2)$ sigma model supersymmetry is broken. Non-BPS branes break the whole supersymmetry.

The equations of motion from the sigma-model action for the worldsheet bosons and fermions give rise to Neumann and Dirichlet boundary conditions [3] for the worldsheet fields

$$\partial u^a = R^a{}_b \bar{\partial} u^b, \qquad (\chi, \rho)^a_L = \pm R^a{}_b (\chi, \rho)^b_R, \quad a = 1, \ldots 2d, \tag{B.43}$$

where $(\chi, \rho)_L$ are the two left-moving spinors, $(\chi, \rho)_R$ are the right-moving spinors of (B.2) and the matrix $R^a{}_b$ satisfies $g_{ab} R^a{}_c R^b{}_d = g_{bd}$. Note that we use real coordinates on the Calabi-Yau d-fold. The $\pm$ signs for the fermions are due to the NS- and R- sector. Dirichlet boundary conditions for u are given by the eigenvector of R with eigenvalue -1 and Neumann boundary conditions are given by eigenvalue $+1$. Dirichlet boundary conditions imply that a D-brane is located in the Calabi-Yau manifold perpendicular to the directions with the Dirichlet boundary conditions. As we want to consider D-branes wrapped on submanifolds of some Calabi-Yau manifold, we generically have mixed boundary conditions and the matrix R is not symmetric.

As half of the supersymmetry is broken on the boundaries, the boundary states describing the D-branes are invariant under a linear combination of the generators of the $N = (2, 2)$ superconformal algebra. It was shown in [69] that there are two ways of obtaining consistent boundary conditions breaking half of the supersymmetry, these are called A-type and B-type boundary conditions:

$$\begin{aligned} \mathrm{A-type:} \quad & T_F^+ = \pm i \bar{T}_F^+, \ T_F^- = \pm i \bar{T}_F^-, \ J = -\bar{J} \\ \mathrm{B-type:} \quad & T_F^+ = \pm i \bar{T}_F^-, \ T_F^- = \pm i \bar{T}_F^+, \ J = \bar{J} \end{aligned} \tag{B.44}$$

Both boundary conditions preserve the $N = (1, 1)$ superconformal algebra generated by the energy momentum tensor T and the supercurrent T_F. The $N = 1$ supercurrent is related to the $N = 2$ supercurrents as $T_F = T_F^+ + T_F^-$. The generators of the $N = (1, 1)$ algebra satisfy the relations $T = \bar{T}$ and $T_F = \pm i \bar{T}_F$ for both a A-and B-type boundary conditions as well as the boundary conditions (B.43). Note that $\bar{T}_F^+ \leftrightarrow \bar{T}_F^-$ and $\bar{J} \leftrightarrow -\bar{J}$ interchanges A- and B-type boundary conditions and also defines the mirror map $(c, c) \leftrightarrow (c, a)$ of B.1.

The $N = (2, 2)$ $U(1)$ currents (B.4) (in complex coordinates) are of the form $J = g_{i\bar{\jmath}} \chi^i \rho^{\bar{\jmath}}$, $\bar{J} = -g_{i\bar{\jmath}} \rho^i \chi^{\bar{\jmath}}$. Together with (B.43) this implies the following formulas for the Kählerform g_{ab} for B-type boundary conditions:

$$g_{ab} R^a{}_c R^b{}_d \ = \ -g_{cd}. \tag{B.45}$$

Denote by x^A, $A = 1, \ldots, p$, the coordinates in the directions with Neumann boundary conditions (tangential to the worldvolume of the brane) and by y^α, $\alpha = 1, \ldots, (2d - p)$, the coordinates in the directions with Dirichlet boundary conditions (normal to the worldvolume of the brane). With only Neumann indices the R is symmetric and has eigenvalues $+1$ only, and the above equation implies that

$$g_{CD} R^C{}_A R^D{}_B = g_{CD} \delta^C{}_A \delta^D{}_B = -g_{AB} \ \rightarrow \ g_{AB} = 0 \tag{B.46}$$

With only Dirichlet indices R is symmetric and has eigenvalues -1 only, thus the above equation implies that also $g_{\alpha\beta}$ vanishes

$$g_{\gamma\delta}R^{\gamma}{}_{\alpha}R^{\delta}{}_{\beta} = g_{\gamma\delta}(-)\delta^{\gamma}{}_{\alpha}(-)\delta^{\delta}{}_{\beta} = -g_{\alpha\beta} \;\to\; g_{\alpha\beta} = 0. \tag{B.47}$$

Only the components of the Kählerform with mixed components $g_{A\alpha}$ are non-vanishing. This implies that the number of directions with Dirichlet boundary conditions $(2d-p)$ is equal to the number of Neumann boundary conditions p, thus we have $p = d$. That means the brane is wrapped on a d-dimensional submanifold of the Calabi-Yau manifold. It was shown in [69] that this submanifold is a middle-dimensional cycle $S \in H_d(Y_d, \mathbb{Z})$ (a 3- cycle for a Calabi-Yau threefold) with volume

$$Vol(S) = \int_S \Omega, \tag{B.48}$$

where Ω is the holomorphic d-form of Y_d.

The A-type boundary conditions imply that R satisfies

$$g_{ab}R^{a}{}_{c}R^{b}{}_{d} \;\; = \;\; g_{cd}. \tag{B.49}$$

This implies that the Kählerform with one index in the directions with Neumann boundary conditions and one index in the directions with Dirichlet boundary conditions vanishes

$$k_{\gamma C}R^{\gamma}{}_{\alpha}R^{C}{}_{A} = k_{\gamma C}(-)\delta^{\gamma}{}_{\alpha}\delta^{C}{}_{A} = -k_{\alpha A} \;\to\; k_{\alpha A} = 0. \tag{B.50}$$

The Kählerform is bock diagonal with non-vanishing components g_{AB} and $g_{\alpha\beta}$. It was shown in [69] that the p-dimensional submanifold in the Neumann directions is an even-dimensional holomorphic submanifold of the Calabi-Yau d-fold. This means that the D-branes are wrapped on holomorphic p-cycles $C^{(p)} \in H_p(Y_d, \mathbb{Z})$ with p even. The volume of a holomorphic p-cycle is

$$Vol(C^{(p)}) = \int_{C^{(p)}} t \wedge \ldots \wedge t, \tag{B.51}$$

where the integral contains $p/2$ complex Kählerforms t.

It was mentioned above that the boundary states with A- and B-type boundary conditions (B.44) interchange right-moving (c) and (a) fields. If we denote the boundary states corresponding to the wrapped D-branes by $|D\rangle$, B-type boundary conditions imply

$$(T_F^{+} \mp i\bar{T}_F^{-})|D\rangle = 0, \quad (T_F^{-} \mp i\bar{T}_F^{+})|D\rangle = 0, \quad (J - \bar{J})|D\rangle = 0, \tag{B.52}$$

and A-type boundary conditions imply

$$(T_F^{+} \mp i\bar{T}_F^{+})|D\rangle = 0, \quad (T_F^{-} \mp i\bar{T}_F^{-})|D\rangle = 0, \quad (J + \bar{J})|D\rangle = 0. \tag{B.53}$$

The boundary states have $U(1)$ charge $q = \pm\bar{q}$. It was shown in [69] that the A-type boundary states can indeed be expanded in terms of the (c, a) rings of the $N = (2, 2)$ superconformal algebra,

$$|D\rangle_{\mathrm{A-type}} = \sum_s c_s|\phi_s\rangle, \quad \phi_s \in (c, a), \tag{B.54}$$

where $s = 1, \ldots, \dim(\sum_{p=0}^{d} H^{(p,p)})$. The coefficients depend on the Kählermoduli only and are given by the volumes of the p-cycles the branes are wrapped on,

$$c_s = \int_{C^s} t \wedge \ldots \wedge t + \text{stringy corrections.} \tag{B.55}$$

The cycles $\{C^s\} = \sum_p C^{(2p)}$, $p \leq d$, are the 2p-dimensional holomorphic cycles of the Calabi-Yau manifold. The B-type boundary states can be expanded in terms of the (c, c) rings,

$$|D\rangle_{\text{B-type}} = \sum_l d_l |\phi_l\rangle, \quad \phi_l \in (c, c), \tag{B.56}$$

with $l = 1, \ldots, \dim(\sum_{p=0}^{d} H^{(p,d-p)})$ and the coefficients depending on the complex structure moduli only,

$$d_l = \int_{S^l} \Omega, \tag{B.57}$$

where S^l are the middle-dimensional cycles in $\prod_{p=0}^{d} H^{(p,d-p)}$. Mirror symmetry demands that the coefficients c_s evaluated on a Calabi-Yau manifold Y_d are mapped to the coefficients d_l evaluated on the mirror manifold Y_d^*. Thus the volumes of the middle-dimensional and the holomorphic cycles are related by mirror symmetry. If we denote by S the sum of the middle-dimensional cycles S^l, we get the formula

$$\sum_p \left(\int_{C^{(p)}} t \wedge \ldots \wedge t + \text{stringy corrections} \right)_{Y_d} = \left(\int_S \Omega \right)_{Y_d^*}. \tag{B.58}$$

This equation is of great importance in chapter 2.

C Anomalies

Anomalies are the breakdown of classical conservation laws due to quantum corrections [1]. In the context of symmetry breaking one has to distinguish between local and global symmetries. A global symmetry broken by anomalies changes the physical content of a theory, but does not lead to inconsistencies. Local symmetries on the other hand must not be broken by anomalies because they make the theory inconsistent. Broken gauge symmetries for example lead to non-unitary theories. The unphysical longitudinal modes of the gauge fields no longer decouple from the physical states. On the level of scattering amplitudes, anomalies are caused by one-loop diagrams which cannot be cancelled by adding local counter terms to the theory. Only diagrams with massless particles circulating in the loop contribute to anomalies. Thus, it is sufficient in string or M-theory to derive the anomalies on the level of the low-energy effective theory. The infinitely many massive excitations of strings do not have to be taken into account. For a detailed review of anomalies see [1].

Anomalies occur in parity-violating loop diagrams only. Parity conserving amplitudes can be regularized in a covariant and gauge invariant way and do not lead to inconsistencies. Parity violating amplitudes require the existence of chiral, that means Weyl, spinors. Weyl spinors do not exist in odd dimensions, which implies that odd dimensional theories are always anomaly-free.

Anomalies are classified due to the external legs of the loop amplitudes. Anomalies from loop diagrams with external gauge fields only are called gauge anomalies, they are due to the breakdown of classical gauge invariance. Anomalies in loop diagrams with external gravitons are called gravitational anomalies and are due to a breakdown of general covariance. Parity violating gravitational interactions however are not possible for any even dimension. Chiral amplitudes exist only if the particles are in a complex representation of $SO(d)$, which possible for $d = 4k + 2$ for any integer k. These are the only dimensions with gravitational anomalies. Anomalies in loop diagrams with both gauge fields and gravitons as external legs are called mixed anomalies. They also exist in dimensions $d = 4k + 2$ only. As we consider theories in ten and six dimensions, we have to include gauge as well as gravitational and mixed anomalies in our considerations.

Anomalies in d dimensions can be expressed in a compact way in terms of a gauge invariant $(d+2)$-form I_{d+2}, which is called the anomaly polynomial. The gauge variation of the effective action Γ obtained by integrating out all fields except the gauge field and the graviton is given by the d-form I_d,

$$\delta\Gamma = \int I_d(F, R), \tag{C.1}$$

where F, R are the gauge field strength and the Riemann curvature two-form. The anomaly polynomial can be extracted from the variation of Γ via

$$dI_d = \delta I_{d+1}, \qquad I_{d+2} = dI_{d+1}. \tag{C.2}$$

Anomaly cancellation means $I_{d+2} = 0$.

C.1 Anomalies in $d = 10$

In this section, we consider the anomalies of ten-dimensional supergravity [1]. After a few remarks about N=2 type IIA and IIB supergravity, we concentrate on N=1 supergravity. This being the low energy effective theory of the heterotic string.

The massless spectrum of type IIA supergravity consists of two spin 1/2 fermions transforming as **8** and **8′** of $SO(8)$ and of two gravitinos transforming as **56** and **56′** of $SO(8)$. Thus the theory is non-chiral. The anomalies caused by the two spin 1/2 fermions cancel as well as those caused by the two gravitinos. This means that Type IIA supergravity is anomaly-free.

Type IIB supergravity has a massless chiral spectrum that contains two spin 1/2 spinors transforming as **8′**, two gravitinos transforming as **56** and one self-dual five-form field strength. Each chiral field gives rise to an anomaly. However, the sum of all anomalies vanishes and the theory is anomaly free.

The theory of interest is the strongly coupled heterotic string, thus we consider the low energy theory in greater detail. The ten-dimensional supergravity theory has $N = 1$ supersymmetry and a gauge group $G = E_8 \times E_8$ or $SO(32)$. The massless spectrum is chiral and contains a neutral fermion (neutral under the gauge group G) transforming as **8′** of $SO(8)$, one gravitino transforming as **56** of $SO(8)$ and a gaugino transforming as $(\mathbf{8}, \mathbf{496})$ of $SO(8) \times G$.

The gauge anomaly arises in a one-loop diagram with six external gauge fields F and chiral fermions circulating in the loop, this diagram is called the hexagon diagram. It is sufficient to use the linearized approximation of the Yang-Mills field strength $F = dA$, because we want to keep only the lowest number of external gauge fields for the hexagon diagram. The theory has a gauge anomaly if the effective action is not gauge invariant, i.e. $\delta_\Lambda \Gamma \neq 0$ for a gauge transformation $\delta_\Lambda A_M = \partial_M \Lambda$ for $M = 0, \ldots, 9$. For the hexagon diagram, the variation of the effective action is

$$\delta_\Lambda \Gamma = \int d^{10}x [c_1 Tr(\Lambda F^5) + c_2 Tr F^4 Tr(\Lambda F) + c_3 Tr(\Lambda F)(Tr F^2)^2], \tag{C.3}$$

with wedge products assumed and constants $c_{1,2,3}$ which depend on the gauge quantum numbers of the particles circulating in the loop.

The relation between the variation of the effective action and the anomaly polynomial I_{d+2} is given in (C.1) and (C.2). Inserting $d = 10$ leads to

$$I_{11} = [c_1 Tr(AF^5) + c_2 Tr F^4 Tr(AF) + c_3 Tr(AF(Tr F^2)^2)], \tag{C.4}$$

and from this one can extract the anomaly polynomial I_{12},

$$I_{12} = [c_1 Tr F^6 + c_2 Tr F^4 Tr F^2 + c_3 Tr(Tr F^2)^3]. \tag{C.5}$$

The gauge anomaly of the heterotic string arises from the chiral gaugino circulating in the loop and the explicit calculation fixes the constants [20] $c_1 = \frac{1}{2}\frac{1}{(2\pi)^5 6!}$, $c_2 = 0$, $c_3 = 0$ [1]. The first factor $\frac{1}{2}$ in c_1 is due to the fact that we have chiral Majorana-Weyl fermions

[20] Chiral fermions contribute to anomalies with a positive sign, anti-chiral fermions contribute with a negative sign.

in $d = 10$ which have half the degrees of freedom of complex chiral Weyl fermions. Thus the gauge anomaly polynomial of the heterotic string is

$$I_{12}^{\text{gauge},1/2} \quad = \quad \frac{1}{2(2\pi)^5 6!} TrF^6. \tag{C.6}$$

The index 1/2 indicates that a spin 1/2 fermion, the gaugino, circulating in the loop gives rise to the anomaly.

The gravitational anomaly arises from the hexagonal diagram with only gravitons as external legs and spin 1/2 as well as spin 3/2 fermions in the loop. To emphasize the similarities with the calculations of the gauge anomaly above, we treat gravitation as a gauge theory with the tangent space group $SO(10)$ as a gauge group. The curvature $R_{MN\alpha\beta}$ is a two-form with two space-time indices M, M and two tangent space indices α, β. Just as we restricted our considerations to the linear part of the field strength $F = dA$ for the gauge anomalies we take the linearized curvature tensor $R_{MN} = (dw)_{MN}$ for deriving the gravitational anomalies. The spin connection w_M plays the role of an $SO(10)$ gauge vector.

Gravitational anomalies occur if the action is not invariant under general infinitesimal diffeomorphisms

$$x^M \to x^M + \eta^M(x^N). \tag{C.7}$$

The variation of the effective action of the hexagon diagram is

$$\delta\Gamma \sim \int d^{10}x[d_1 tr(\Theta R^5) + d_2 trR^4 tr(\Theta R) + d_3 tr(\Theta R)(trR^2)^2], \tag{C.8}$$

with $\Theta_{MN} = D_M\eta_N - D_N\eta_M$ and three constants $d_{1,2,3}$ which have to be derived explicitly for the theory under consideration. Note that the variation of the action is of the same form as (C.3) with the curvature R interchanged with the gauge field strength F and Θ interchanged with the gauge parameter Λ.

The anomaly polynomial is

$$I_{12} = d_1 trR^6 + d_2 trR^4 trR^2 + d_3 tr(trR^2)^3. \tag{C.9}$$

Similar as in the case of the gauge anomaly, a part of the gravitational anomaly can be cancelled by the anomalous Feynman diagram which describes the exchange of the massless field B between two gravitons on one side and four gravitons on the other side. The calculations are in principal similar to those of the gauge anomaly. The gravitational anomaly has three different contributions though. The charged chiral spin 1/2 fermion (the gaugino), the neutral spin 1/2 fermion (the dilatino) and the chiral spin 3/2 gravitino all contribute to the anomaly.

Performing the calculation of d_1, d_2 and d_3 for the gravitino, gaugino and dilatino and adding the contributions leads to the gravitational anomaly of the heterotic string [1],

$$I_{12}^{\text{grav.}} \quad = I_{12}^{\text{grav.},3/2} + (n-1) I_{12}^{\text{grav.},1/2}, \tag{C.10}$$

where n is the dimension of the gauge group G and

$$\begin{aligned} I_{12}^{\text{grav.},3/2} &= \frac{1}{2(2\pi)^5 6!}\left(\frac{55}{56}trR^6 - \frac{77}{128}trR^4 trR^2 + \frac{35}{512}(trR^2)^3\right), \\ I_{12}^{\text{grav.},1/2} &= \frac{1}{2(2\pi)^5 6!}\left(-\frac{1}{504}trR^6 - \frac{1}{384}trR^4 trR^2 - \frac{5}{4608}(trR^2)^3\right). \end{aligned} \tag{C.11}$$

The contribution $I_{12}^{\rm grav.,3/2}$ arises from the spin 3/2 gravitino and $(n-1)I_{12}^{\rm grav.,1/2}$ are the contributions of the gaugino and the dilatino. The opposite signs are due to the chirality of the gaugino and the antichirality of the dilatino. We keep n unspecified for the moment in order to show later that the total anomaly cancels for $n = 496$ only.

The mixed anomaly comes from the one-loop hexagon diagram with both gravitons and gauge fields as external legs and spin 1/2 fermions in the loop. The variation of the effective action is of the same form as those of the pure gauge and pure gravitational anomalies. According to the possible distribution of the total six external gauge fields and gravitons in the hexagon diagram there are four contributions:

$$\begin{aligned}\delta\Gamma &= \int d^{10}x \left[e_1 Tr(\Lambda F) trR^4 + e_2 TrF^4 tr(\Theta R) + e_3 tr(\Theta R)(TrF^2)^2 \right.\\ &\quad \left. + e_4 Tr(\Lambda F)(trR^2)^2\right] . \end{aligned} \tag{C.12}$$

There are additional terms with different combinations of F and R in the hexagon diagram, but these terms can be cancelled by adding local counter terms [1] and do not contribute to the anomaly. The anomaly polynomial is

$$I_{12} = e_1 TrF^2 trR^4 + e_2 TrF^4 trR^2 + e_3 trR^2 (TrF^2)^2 + e_4 TrF^2 (trR^2)^2. \tag{C.13}$$

The mixed anomaly polynomial of the heterotic string is produced by the gaugino circulating in the loop. This leads to the anomaly polynomial [1]

$$I_{12}^{\rm mixed,1/2} = \frac{1}{2(2\pi)^5 6!}\left(\frac{1}{16}trR^4 TrF^2 + \frac{5}{64}(trR^2)^2 TrF^2 - \frac{5}{8}trR^2 TrF^4\right). \tag{C.14}$$

Adding the contributions of the gauge anomaly (C.6), the gravitational anomaly (C.10) and the mixed anomaly (C.14) leads to the total anomaly

$$\begin{aligned} I_{12} &= I_{12}^{\rm gauge,1/2} + I_{12}^{\rm grav.,3/2} + (n-1)I_{12}^{\rm grav.,1/2} + I_{12}^{\rm mixed,1/2} \\ &= \frac{1}{2(2\pi)^5 6!}\left(\frac{496-n}{504}trR^6 - \frac{224+n}{384}trR^4 trR^2 + \frac{5(64-n)}{4608}(trR^2)^3\right. \\ &+ \left. \frac{1}{16}trR^4 TrF^2 + \frac{5}{64}(trR^2)^2 TrF^2 - \frac{5}{8}trR^2 TrF^4 + TrF^6\right). \end{aligned} \tag{C.15}$$

This anomaly must be cancelled in order to keep the theory consistent. This can be done for a gauge group $G = E_8 \times E_8$ or $SO(32)$ via the Green-Schwarz mechanism [83] as we explain in the following. For simplicity we concentrate on the gauge anomaly first and assume an anomaly polynomial of the general form (C.5).

In order to cancel the gauge anomaly, the theory must contain another anomalous Feynman diagram with six external gauge bosons. Such a diagram indeed exists for the heterotic string. It consists of two vertices that are glued together. One is a three-point vertex with two gauge bosons and the antisymmetric NS two-form B_{MN} and the second one has five legs, four gauge bosons and one two-form B_{MN}. The vertices are glued together such that B_{MN} is the internal field exchanged between the gauge bosons. The action which contains both vertices is

$$S = \int d^{10}x\sqrt{g}\left(Tr|H|^2 + B(Tr|F|^4 + (Tr|F|^2)^2)\right), \tag{C.16}$$

again with wedge products assumed. The three-form field strength $H = H_0 - \omega_3 = dB - Tr(AdA + 2/3A^3)$ includes the Yang-Mills Chern-Simons term ω_3. Only the part of S of lowest order in the gauge field A contributes to the anomalous Feynman diagram,

$$S_0 = \int d^{10}x\sqrt{g}\left(H_0 TrAdA + B(TrF^4 + (TrF^2)^2)\right), \qquad (C.17)$$

where $F = dA$ is now the linearized field strength. This first term describes the coupling of B to two gauge bosons and the second one to four gauge bosons. The equation of motion for B is

$$D^M H_{MNR} = \epsilon_{NRP_1,\dots,P_8}(TrF^{P_1P_2}F^{P_3P_4}F^{P_5P_6}F^{P_7P_8} + Tr(F^{P_1P_2}F^{P_3P_4})Tr(F^{P_5P_6}F^{P_7P_8})). \qquad (C.18)$$

The gauge variation of the interaction term S_0 is

$$\delta_\Lambda S_0 = -\int d^{10}x\sqrt{g}Tr(\Lambda F^{MN})D^R H_{MNR}, \qquad (C.19)$$

where $\delta_\Lambda A_M = \partial_M \Lambda$ is again the linearized variation. Inserting the equation of motion (C.18) in (C.19) in the variation leaves

$$\delta_\Lambda S_0 = -\int d^{10}x\sqrt{g}\left(Tr(\Lambda F)(TrF^4 + (TrF^2)^2)\right). \qquad (C.20)$$

This leads to an anomaly polynomial

$$I_{12} \sim -TrF^2TrF^4 - (TrF^2)^3, \qquad (C.21)$$

which has exactly the correct form to cancel the second and third term of the anomaly (C.5).

The first term $\sim trF^6$ in (C.5) however, which is relevant in the heterotic gauge anomaly, cannot be cancelled in this way. The anomaly polynomial (C.5) needs to have some factorization property $I_{12} = TrF^2TrF^4 + (TrF^2)^3$ to be cancelled by an additional interaction of the form $\int B(TrF^4 + (TrF^2)^2)$. It is exactly for the gauge groups $E_8 \times E_8$ and $SO(32)$ that the gauge field strength has the factorization property[21]

$$TrF^6 = \frac{1}{48}TrF^4TrF^2 - \frac{1}{14400}(TrF^2)^2. \qquad (C.22)$$

Using this relation the anomaly polynomial (C.5) factorizes into the product of a four-form and an eight-form and can be cancelled. This statement is indeed true not only for the gauge anomaly. Anomalies can be cancelled in general by a Green-Schwarz mechanism if the anomaly poynomial factorizes into a four-form and an eight-form (for d=10).The contributions of the mixed and the gravitational anomaly to the total anomaly (C.15), except the term $\sim R^6$, have the required factorization and can be cancelled by the Green-Schwarz mechanism. The term $\sim R^6$ has to vanish, this fixes $n = 496$ as expected. Using

[21]Such a factorization also works for just one E_8 of course, but cancellation of the gravitational anomaly considered in the following determines the dimension n of the representation of the gauge group as $n = 496$ and thus leads to $E_8 \times E_8$ and $SO(32)$.

equation (C.22), the remaining contributions of the total anomaly can be brought to the form

$$I_{12} = -\frac{15}{2(2\pi)^5 6!} Y_4 Z_8, \tag{C.23}$$

with

$$\begin{aligned} Z_8 &= -\frac{1}{8} trR^4 + \frac{1}{32}(R^2)^2 - \frac{1}{240} trR^2 TrF^2 + \frac{1}{24} TrF^4 - \frac{1}{7200}(TrF^2)^2, \\ Y_4 &= trR^2 - \frac{1}{30} TrF^2. \end{aligned} \tag{C.24}$$

At this point we make a final remark. The operation trF used in the literature [1, 89, 90] and also in chapter 3 is defined by $TrF^2 = 30trF^2$. This notation originates from the gauge group $SO(32)$ where the trace in the adjoint representation is usually denoted by TrF and the trace in the fundamental representation by trF with a relation $TrF = 30trF$. The same relation is used to define the operation tr for the gauge group $E_8 \times E_8$.

C.2 Anomaly Cancellation in M-Theory Compactified on $I = S^1/\mathbb{Z}_2$ with Gauge Group $E_8 \times E_8$

In this section, we consider the anomalies of eleven-dimensional supergravity compactified on the interval $I = S^1/\mathbb{Z}_2$ [15] as described in section (3.1). The anomalies are also explained in detail in the first ref. of [90] and [75]. Being an odd-dimensional theory, the eleven-dimensional supergravity in the bulk is anomaly-free. The anomalies arise on the two fixed ten-planes of the $S^1/\mathbb{Z}_2$ orbifold. They have two origins. First, there are gravitational anomalies due to the projection of the eleven-dimensional gravitino to the ten-planes. Second, there are gravitational, gauge and mixed anomalies from the "twisted" gauge field and gaugino introduced on each ten-plane. We explain this in more detail in the following and review how the anomalies can be cancelled.

The compactification of the eleven-dimensional gravitino to ten dimensions results in a ten-dimensional chiral gravitino plus a ten-dimensional antichiral spin 1/2 fermion on each ten-plane. Both femions give rise to gravitational anomalies. To find out how these anomalies look like, it is instructive to consider the limit in which the length of the interval I approaches zero and the two ten-planes are pushed together. In this limit, the theory describes the weakly coupled heterotic string, or its low energy limit to be precise, considered in the last section. Thus, the gravitational anomaly in this limit is

$$I_{12} = I_{12}^{\text{grav},3/2} - I_{12}^{\text{grav.},1/2}, \tag{C.25}$$

where $I_{12}^{\text{grav},3/2}, I_{12}^{\text{grav.},1/2}$ are the anomaly polynomials of the dilatino and the gravitino of the heterotic string given in eqn. (C.11). Increasing the length of the interval and separating the two ten-planes leads to two separated anomalies, $I_{12} \to \hat{I}_{12}^1 + \hat{I}_{12}^2$, where the subscripts 1,2 label the ten-planes[22]. It is intuitive to assume that the anomalies are

[22]We use $\hat{I}_{12}$ for the anomaly polynomials on the ten-planes in order to distinguish them from those of the weakly coupled heterotic string denoted by I_{12} in the last section.

distributed evenly over the two ten-planes. Thus the anomaly on each ten-plane is equal to that of eqn. (C.25), multiplied by a factor 1/2. This leads to the anomaly

$$\hat{I}_{12}^{1,2} = \frac{1}{2}\left(I_{12}^{\text{grav.},3/2} - I_{12}^{\text{grav.},1/2}\right). \tag{C.26}$$

As mentioned above, the spectrum contains one gaugino on each ten-plane, which originates in ten dimensions rather than being the projection of some eleven-dimensional field. The gaugino couples to the ten-dimensional graviton as well as to the gauge field strength and thus gives rise to gauge, gravitational and mixed anomalies. Due to the ten-dimensional origin of the gaugino, the anomalies are identical to those of the heterotic string described in the last section. These are given by eqn. (C.6), the second equation of (C.11) and (C.14) and lead to

$$\hat{I}_{12}^{1,2} = I_{12}^{\text{gauge},1/2}(F_{1,2}) + n^{1,2} I_{12}^{\text{grav.},1/2}(R) + I_{12}^{\text{mixed},1/2}(R, F_{1,2}), \tag{C.27}$$

where $F_{1,2}$ are the gauge field strengths on the two ten-planes and $n^{1,2}$ are the dimensions of the gauge groups on the two ten-planes. Adding the two anomalies (C.26) and (C.27) gives the total anomaly

$$\begin{aligned}
\hat{I}_{12}^{1,2} &= \frac{1}{2}\left(I_{12}^{\text{grav.},3/2}(R) - I_{12}^{\text{grav.},1/2}(R)\right) \\
&+ \left(I_{12}^{\text{gauge},1/2}(F_{1,2}) + n^{1,2} I_{12}^{\text{grav.},1/2}(R) + I_{12}^{\text{mixed},1/2}(R, F_{1,2})\right) \\
&= \frac{1}{2(2\pi)^5 6!}\left(\frac{248 - n^{1,2}}{504} trR^6 - \frac{112 + n^{1,2}}{384} trR^4 trR^2 + \frac{5(32 - n^{1,2})}{4608}(R^2)^3\right. \\
&+ \left.\frac{1}{16} trR^4 TrF_{1,2}^2 + \frac{5}{64}(trR^2)^2 TrF_{1,2}^2 - \frac{5}{8} trR^2 trF_{1,2}^4 + TrF_{1,2}^6\right),
\end{aligned} \tag{C.28}$$

The term $\sim R^6$ has to vanish, which fixes $n^1 = n^2 = 248$. Also, to get rid of the term $\sim F_{1,2}^6$, we need some factorization property for the gauge field strengths. Both conditions are fulfilled by the gauge goup E_8. The factorization is

$$TrF_{1,2}^6 = \frac{1}{24} TrF_{1,2}^2 TrF_{1,2}^4 - \frac{1}{3600}(TrF_{1,2}^2)^3. \tag{C.29}$$

In addition, E_8 has the property $TrF^4 = \frac{1}{100}(TrF^2)^2$. Using these equations the anomaly polynomial factorizes as

$$\hat{I}_{12} = \hat{I}_{12}^1 + \hat{I}_{12}^2 = \left(\hat{Y}_4^1(R, F_1)\hat{X}_8^1(R, F_1) + \hat{Y}_4^2(R, F_2)\hat{X}_8^2(R, F_2)\right). \tag{C.30}$$

with

$$\hat{X}_8^{1,2} = \frac{1}{(2\pi)^3 4!}\left(\frac{1}{8} trR^4 + \frac{1}{32}(R^2)^2 - \frac{1}{4} trR^2 trF_{1,2}^2 + \frac{1}{4}(trF_{1,2}^2)^2\right) \tag{C.31}$$

and

$$\hat{Y_4}^{1,2} = \frac{1}{4(2\pi)^2}\left(trF_{1,2}^2 - \frac{1}{2} trR^2\right). \tag{C.32}$$

Note that we have used $trF = 1/30TrF$. The anomaly polynomial factorizes into a four- and an eight-form and can thus can be cancelled by the Green-Schwarz mechanism. Before we explain the cancellation in detail, we simplify the anomaly polynomial further. Using the eight-form introduced in (3.3),

$$X_8 = \frac{1}{(2\pi)^3 4!}\left(\frac{1}{8}trR^4 - \frac{1}{32}(trR^2)^2\right), \tag{C.33}$$

the anomaly reads

$$\begin{aligned} \hat{I}_{12} &= \frac{\pi}{3}\left(\hat{Y_4}^1(R,F_1)\right)^3 + X_8(R)\hat{Y_4}^1(R,F_1) \\ &+ \frac{\pi}{3}\left(\hat{Y_4}^2(R,F_2)\right)^3 + X_8(R)\hat{Y_4}^2(R,F_2). \end{aligned} \tag{C.34}$$

The two terms $\sim (\hat{Y}_4^{1,2})^3$ and $\sim X_8\hat{Y}_4^{1,2}$ can be cancelled seperately. The first term $\sim (\hat{Y}_4^{1,2})^3$ gets cancelled by the projection of the eleven-dimensional Chern-Simons term $\int C \wedge G \wedge G$ on the ten-planes and the second one $\sim X_8\hat{Y}_4^{1,2}$ gets cancelled by the projection of the higher order term $\int G \wedge X_7$ introduced in chapter 3.1. In the following, we review the calculation along the line of [75].

We first consider the cancellation of $\hat{I}_{12} = \frac{\pi}{3}(\hat{Y}_4^1)^3 + \frac{\pi}{3}(\hat{Y}_4^2)^3$ by the projection of the eleven-dimensional Chern-Simons term. The modified Bianchi-identity of the four-form field strength G as required by supersymmetry is related to the four-form $\hat{Y}_4$ as

$$dG = -\frac{(4\pi\kappa)^2}{\lambda^2}\left(\delta(x^{11})dx^{11} \wedge \hat{Y_4}^1(R,F_1) + \delta(x^{11}-\pi)dx^{11} \wedge \hat{Y_4}^2(R,F_2)\right), \tag{C.35}$$

where the length of the interval is set $l = \pi$, or in other words the radius of S^1 is equal to unity. The Bianchi-Identity is fulfilled if G is of the form

$$\begin{aligned} G &= dC - \frac{(4\pi\kappa)^2}{\lambda^2}\Big[(b-1)\left(\delta(x^{11})dx^{11} \wedge w_3^1(R,F_1) + \delta(x^{11}-\pi)dx^{11} \wedge w_3^2(R,F_2)\right) \\ &+\frac{b}{2}\left(\epsilon(x^{11})\hat{Y_4}^1(R,F_1) + \epsilon(x^{11}-\pi)\hat{Y_4}^2(R,F_2)\right) \\ &-\frac{b}{2\pi}dx^{11} \wedge \left(w_3^1(R,F_1) + w_3^2(R,F_2)\right)\Big], \end{aligned} \tag{C.36}$$

as can be easily seen by taking the derivative. We have introduced the three-form $dw_3 = \hat{Y}_4$. The "integration constant" b is not fixed by the Bianchi-identity and will be determined later. The zero-form $\epsilon(x)$ is defined by $\epsilon(x) = sign(x) - x/\pi$ and its derivative is $d\epsilon(x) = (2\delta(x) - \pi)dx^{11}$. The reason for using the function $\epsilon(x)$ instead of the step function $sign(x)$ only is that the step function alone is not periodic on the circle parametrized by x^{11} with $x^{11} \in [-\pi, \pi]$. The additional linear term makes the function well defined on the circle. This was realized in [75] and the result obtained by including the linear term in ϵ differs from those of earlier publications by the last term in (C.36). The periodicity of $\epsilon(x)$ was also taken into account correctly in [81] in a different context.

The next step is to find out the variation of C under gauge transformations. The four-form field strength G is invariant under gauge transformations, which means that

C transforms as. [23]

$$\delta C \;=\; \frac{(4\pi\kappa)^2}{\lambda^2}\Big[(b-1)\left(\delta(x^{11})dx^{11}\wedge w_2^1(R,F_1)+\delta(x^{11}-\pi)dx^{11}\wedge\; w_2^2(R,F_2)\right) -\frac{b}{2\pi}dx^{11}\wedge\left(w_2^1(R,F_1)+w_2^2(R,F_2)\right)\Big] \qquad (C.37)$$

with w_2 defined by $dw_2 = \delta w_3$. The third term in (C.36) $\sim \hat{Y}_4$ does not contribute because it is gauge invariant.

The variation of the Chern-Simons term under gauge transformations is

$$\delta\left(-\frac{1}{12\kappa^2}\int C\wedge G\wedge G\right) = -\frac{1}{12\kappa^2}\int \delta(C)\wedge G\wedge G \qquad (C.38)$$

and inserting (C.37) and (C.36) gives

$$-\frac{1}{12\kappa^2}\int\delta(C)\wedge G\wedge G = -\frac{1}{12\kappa^2}\frac{(4\pi\kappa)^6}{\lambda^6}\frac{b^2}{4}\int\Big[(b-1)\left(\delta(x^{11})dx^{11}\wedge w_2^1(R,F_1)\right.$$
$$\left.+\delta(x^{11}-\pi)dx^{11}\wedge\; w_2^2(R,F_2)\right)-\frac{b}{2\pi}dx^{11}\wedge\left(w_2^1(R,F_1)+w_2^2(R,F_2)\right)\Big]$$
$$\wedge\left[\epsilon(x^{11})\hat{Y_4}^1(R,F_1)+\epsilon(x^{11}-\pi)\hat{Y_4}^2(R,F_2)\right]$$
$$\wedge\left[\epsilon(x^{11})\hat{Y_4}^1(R,F_1)+\epsilon(x^{11}-\pi)\hat{Y_4}^2(R,F_2)\right)\Big].$$

Only the terms which to not contain x^{11} contribute to G as $G_{abc,11}=0$. To perform the integral in the compact eleventh direction we use the identities

$$\int dx^{11}\epsilon(x^{11})\epsilon(x^{11}-\pi) = -\frac{\pi}{3}, \qquad \int dx^{11}\epsilon(x^{11})\epsilon(x^{11}) = \pi \qquad (C.39)$$

and after a regularization

$$\delta(x^{11})\epsilon(x^{11})\epsilon(x^{11}-\pi)\simeq\frac{1}{3}\delta(x^{11}). \qquad (C.40)$$

For a detailed explanation of this see [75]. The factor 1/3 differs from earlier publications in the literature and was also noticed in [82]. Performing the integration in the eleventh direction we get

$$-\frac{1}{12\kappa^2}\int\delta(C)\wedge G\wedge G \;=\; -\frac{b^2(4\pi)^6}{144}\frac{\kappa^4}{\lambda^6}\int_{M^{10}}(w_2^1(R,F_1)+w_2^2(R,F_2)) \qquad (C.41)$$
$$\left(\left(\hat{Y_4}^1(R,F_1)\right)^2+\left(\hat{Y_4}^2(R,F_2)\right)^2+\left(\hat{Y_4}^1(R,F_1)\hat{Y_4}^2(R,F_2)\right)\right)$$

Using $dw_2=\delta w_3$, $dw_3=\hat{Y}_4$ and the formulas (C.2) leads to an anomaly polynomial of the form

$$I_{12} = -\frac{\pi}{3}\left(\frac{b^2(4\pi)^5}{12}\frac{\kappa^4}{\lambda^6}\right)\left(\left(\hat{Y_4}^1(R,F_1)\right)^3+\left(\hat{Y_4}^2(R,F_2)\right)^3\right) \qquad (C.42)$$

[23] By gauge transformation we always mean local Lorentz transformations acting on R and local gauge transformations acting on F.

The anomalies (3.8) and (C.42) cancel if

$$\frac{b^2(4\pi)^5}{12}\frac{\kappa^4}{\lambda^6} = 1. \tag{C.43}$$

We see that anomaly cancellation alone does not fix the constant b but leads to a quadratic equation. The fact that there is a one-parameter family of solutions to the Bianchi-identity has also been studied in [82, 84, 85, 86, 87, 90] [24]. Requiring the correct quantization condition of G [31] fixes $b = 1$ as was shown in [75]. The gauge and the gravitational coupling are related as $\frac{\kappa^4}{\lambda^6} = \frac{1}{12}(4\pi)^5$.

The same analysis applies for the higher order term $\int G \wedge X_7$, which cancels the anomaly $\sim X_8\hat{Y}_4^1 + X_8\hat{Y}_4^2$ of (C.34). The variation of the interaction $\int \delta(G \wedge X_7) = \int G \wedge \delta(X_7)$ is

$$-\frac{\lambda^2}{(4\pi\kappa)^2}\int G \wedge \delta(X_7) = -\frac{\lambda^2}{(4\pi\kappa)^2}\int G \wedge dX_6 = \frac{\lambda^2}{(4\pi\kappa)^2}\int dG \wedge X_6, \tag{C.44}$$

with $\delta X_7 = dX_6$. Inserting the modified Bianchi-identity (C.35) and performing the integral over the eleventh dimension we get

$$\begin{aligned}\frac{\lambda^2}{(4\pi\kappa)^2}\int dG \wedge X_6 &= -\int \Big(\delta(x^{11})dx^{11} \wedge \hat{Y_4}^1(R, F_1) + \delta(x^{11} - \pi)dx^{11} \wedge \hat{Y_4}^2(R, F_2)\Big) \wedge X_6 \\ &= -\int_{M^{10}} \Big(\hat{Y_4}^1(R, F_1) + \hat{Y_4}^2(R, F_2)\Big) \wedge X_6. \end{aligned} \tag{C.45}$$

Using again the formulas (C.2) and the relation $dX_7 = X_8$ defined below equation (3.3), the anomaly polynomial is

$$I_{12} = -\Big(\hat{Y_4}^1(R, F_1) + \hat{Y_4}^2(R, F_2)\Big) \wedge X_8 \tag{C.46}$$

This anomaly cancels exactly the remaining term in (C.34).

C.3 Anomalies in $d = 6$

We consider the heterotic string compactified on a $K3$ manifold. For anomaly cancellation in six-dimensions, see [102, 119] and in the context of Horava-Witten theory the appendices of [89] and the first ref. of [90]. The low-energy effective action of the heterotic string on $K3$ is $N = 1$ supergravity in $d = 6$. In six dimensions there are gravitational as well as gauge anomalies and mixed anomalies. The $N = 1$ supergravity multiplet contains a sechsbein, a self-dual two-form and a chiral spin 3/2 gravitino. In addition there is a vector multiplet which contains a charged gauge vector and chiral spin 1/2 gaugino, a hypermultiplet which contains four real scalars and an antichiral spin 1/2 fermion and a tensor multiplet which contains a real scalar, an anti self-dual two-form and an antichiral spin 1/2 fermion. The spin chiral 3/2 gravitino, the chiral spin 1/2 gaugino and the antichiral spin 1/2 fermions of the hyper- and tensor multiplet as well as the

[24] The calculation not taking into account the additional term in (C.36) leads to a cubic equation for b and to a different result

anti self-dual two-form contribute to the total anomaly. The anomaly can be expressed in terms of the eight-form anomaly polynomial I_8.

The number n_V of vector multiplets, the number n_H of hypermultiplets and the number n_T of tensor multiplets depend on the compactification manifold and the choice of the gauge bundle. Perturbative compactifications have one universal tensor multiplet. We keep these numbers arbitrary at this point and include non-perturbative compactifications, $n_T \geq 1$. The vector multiplet transforms in the adjoint representation of the gauge group and the tensor multiplets are gauge singlets. The representation of the hypermultiplets is not fixed.

Anomalies in six dimensions arise from one-loop diagrams with external gauge fields and gravitons, similar as in ten dimensions. In six dimensions the diagrams have only four external legs while the anomalous diagrams have six external legs in ten dimensions. To cancel the anomalies by a six-dimensional Green-Schwarz mechanism, the eight-form anomaly polynomial has to factorize as $I_8 = I_4 I_4'$. This leaves some constraints on the compactification manifold as we shall see below.

Consider the gauge anomaly first. The antichiral spin 1/2 fermion in the hypermultiplet and the chiral spin 1/2 fermion in the vector multiplet contribute to the gauge anomaly. As the representation of the hypermultiplet is not specified we call Tr_H the trace in the corresponding representation of the gauge group. The anomaly polynomial is

$$I_8^{\text{gauge},1/2} = \frac{-1}{4!(2\pi)^3}\left(TrF^4 - Tr_H F^4\right). \tag{C.47}$$

The gravitational anomaly of the one-loop diagram with four external gravitons has contributions from the spin 3/2 gravitino as well as from the chiral and anti-chiral spin 1/2 fermions of the vector, tensor and hypermultiplets. In addition the field strengths of the self-dual two-form field of the supergravity multiplet and the anti self-dual two-form of the tensor multiplet give rise to anomalies. As in the ten-dimensionsional case of the last chapter, the Riemann tensor is regarded as a two-form in spacetime and as a matrix of the tangent space group, in this case $SO(6)$. The trace in trR is taken over the tangent space indices. The chiral spin 1/2 fermion of the vector multiplet and the anti-chiral fermions of the tensor and hypermultiplet give a contribution

$$I_8^{\text{grav.},1/2} = \frac{1}{4!(2\pi)^3}(n_V - n_H - n_T)\left(-\frac{1}{240}trR^4 - \frac{1}{192}(trR^2)^2\right). \tag{C.48}$$

The gravitino gives rise to the anomaly polynomial

$$I_8^{\text{grav.},3/2} = \frac{1}{4!(2\pi)^3}\left(-\frac{49}{48}trR^4 + \frac{43}{192}(trR^2)^2\right). \tag{C.49}$$

The contribution of the three-form field strengths of the supergravity and the tensor multiplets is

$$I_8^{\text{grav.},3-\text{form}} = \frac{1}{4!(2\pi)^3}(1 - n_T)\left(-\frac{7}{60}trR^4 + \frac{1}{24}(trR^2)^2\right). \tag{C.50}$$

Finally the mixed anomaly arising from the one-loop diagram with two external gravitons and two external gauge fields has contributions from the chiral and antichiral spin 1/2 fermions of the vector- and the hypermultiplets is

$$I_8^{\text{mixed},1/2} = \frac{1}{4!(2\pi)^3}\left(\frac{1}{4}trR^2TrF^2 - \frac{1}{4}trR^2Tr_HF^2\right). \tag{C.51}$$

Adding all the contributions gives the resulting anomaly polynomial

$$\begin{aligned} I_8 &= \frac{1}{4!(2\pi)^3}\left(\frac{1}{240}(n_H - n_V + 29n_T - 273)trR^4 + \frac{1}{192}(n_H + n_V - 7n_T + 51)(trR^2)^2 \right.\\ &\left. + \frac{1}{4}(trR^2TrF^2 - trR^2Tr_HF^2) - TrF^4 + Tr_HF^4\right). \end{aligned} \tag{C.52}$$

Cancelling the anomaly by a Green-Schwarz mechanism requires a factorization property of the anomaly polynomial $I_8 = I_4I_4'$. As the term $\sim R^4$ cannot possibly factorize in this way, the factor of that term has to vanish,

$$n_H - n_V + 29n_T - 273 = 0. \tag{C.53}$$

This equation has to be satisfied for every consistent heterotic $K3$ compactification. The terms $\sim F^4$ have to cancel or factorize as TrF^2TrF^2 and $Tr_HF^2Tr_HF^2$.

C.4 Anomaly Cancellation in M-Theory Compactified on $T^4/\mathbb{Z}_2\times S^1/\mathbb{Z}_2$ with Gauge Group $SO(16)\times[E_7\times SU(2)]$

In this section we review the anomalies of M-theory compactified on $T^4/\mathbb{Z}_2\times S^1/\mathbb{Z}_2$ with gauge group $SO(16)\times[E_7\times SU(2)]$[90, 89]. The field content of the theory is explained in chapter 3.3. The spectrum contains in total 500 hypermultiplets, 256 vector multiplets and one tensor multiplet, thus eqn. (C.53) is fulfilled. The anomalies arise on the fixed six-planes of the compactification manifold. We follow the discussion of [89] and use the notation

$$I_{12} = \frac{1}{(2\pi)^3 4!}\mathcal{A} \tag{C.54}$$

to express the anomalies in terms of $\mathcal{A}$ and keep the formulas simple. There are several contributions to the one-loop anomaly on the six-planes resulting from the chiral projection of fields which live in the bulk, on the fixed ten-planes of $S^1/\mathbb{Z}_2$ and on the fixed seven-planes of $T^4/\mathbb{Z}_2$ and from fields which live on the six-planes. We classify the contributions according to the planes on which the chiral fields contributing to the anomaly have their origin.

The supergravity multiplet, the tensor multiplet and the four moduli hypermultiplets give rise to a gravitational anomaly which is distributed over the thirty-two fixed planes. The supergravity, tensor and hypermultiplets live in the bulk and thus contribute to the six-dimensional anomaly via projection on the thirty-two six-planes. Thus the result is 1/32 times the six-dimensional standard anomaly derived by the corresponding anomaly calculation taking into account the same spectrum. Adding the contributions of the supergravity, tensor and hypermultiplets of the last chapter leads to the bulk anomaly $I_{\text{bulk}} = I_8^{\text{grav.},3/2} - 5I_8^{\text{grav.},1/2}$ and to

$$\mathcal{A}_{\text{bulk}} = -\frac{1}{32}trR^4 + \frac{1}{128}(trR^2)^2 \tag{C.55}$$

on each fixed six-plane.

The other multiplets give rise to gravitational as well as mixed and gauge anomalies. The decomposition of the gauge fields is

$$trF_{E_8}^2 = \frac{1}{30}Tr_{\mathbf{248}}F_{E_8}^2 = \frac{1}{30}\left(Tr_{\mathbf{133}}F_{E_7}^2 - 2Tr_{\mathbf{56}}F_{E_7}^2 + Tr_{\mathbf{3}}F_{SU(2)}^2 - 56Tr_{\mathbf{2}}F_{SU(2)}^2\right), \tag{C.56}$$

where the negative signs of the hypermultiplets are due to the anti-chirality of the hyperinos, and

$$trF_{E_8}^2 = \frac{1}{30}Tr_{\mathbf{248}}F_{E_8}^2 = \frac{1}{30}\left(Tr_{\mathbf{120}}F_{SO(16)}^2 - Tr_{\mathbf{128}}F_{SO(16)}^2\right). \tag{C.57}$$

We follow the notation of [89] by introducing the operation $trF^2 = \sum_a F^aF^a$ and normalizing the long roots to length one[25]. If Tr_R denotes the trace in the representation R the relations are are

$$\begin{aligned}
Tr_{\mathbf{2}}F_{SU(2)}^2 &= 1/2trF_{SU(2)}^2, & Tr_{\mathbf{2}}F_{SU(2)}^4 &= 1/8tr(F_{SU(2)}^2)^2,\\
Tr_{\mathbf{3}}F_{SU(2)}^2 &= 2trF_{SU(2)}^2, & Tr_{\mathbf{3}}F_{SU(2)}^4 &= 2(trF_{SU(2)}^2)^2,\\
Tr_{\mathbf{120}}F_{SO(16)}^2 &= 14trF_{SO(16)}^2, & Tr_{\mathbf{120}}F_{SO(16)}^4 &= 3(trF_{SO(16)}^2)^2 + 12trF^4,\\
Tr_{\mathbf{128}}F_{SO(16)}^2 &= 16trF_{SO(16)}^2, & Tr_{\mathbf{128}}F_{SO(16)}^4 &= 6(trF_{SO(16)}^2)^2 - 8trF^4,\\
Tr_{\mathbf{133}}F_{E_7}^2 &= 18trF_{E_7}^2, & Tr_{\mathbf{133}}F_{E_7}^4 &= 6(trF_{E_7}^2)^2\\
Tr_{\mathbf{56}}F_{E_7}^2 &= 6trF_{E_7}^2, & Tr_{\mathbf{56}}F_{E_7}^4 &= 3/2(trF_{E_7}^2)^2.
\end{aligned} \tag{C.58}$$

The untwisted vector and hypermultiplets on the ten-planes charged under the perturbative gauge groups $SO(16)$ and $[E_7 \times SU(2)]$ contribute 1/16 of the six-dimensional standard anomaly result on each of the six-planes which are embedded into the same ten-plane. According to the two different perturbative gauge groups on the ten-planes we have

$$\begin{aligned}
\mathcal{A}_{[E_7\times SU(2)]}^{10-\text{plane}} &= -\frac{1}{160}trR^4 - \frac{1}{128}(trR^2)^2 + trR^2(\frac{3}{32}trF_{E_7}^2 - \frac{13}{32}trF_{SU(2)}^2)\\
&- \frac{3}{16}(trF_{E_7}^2)^2 + \frac{5}{16}(trF_{SU(2)}^2)^2 + \frac{9}{8}trF_{SU(2)}^2trF_{E_7}^2
\end{aligned} \tag{C.59}$$

and

$$\begin{aligned}
\mathcal{A}_{SO(16)}^{10-\text{plane}} &= \frac{1}{480}trR^4 + \frac{1}{384}(trR^2)^2 - \frac{1}{32}trR^2trF_{SO(16)}^2 + \frac{3}{16}(trF_{SO(16)}^2)^2\\
&- trF_{SO(16)}^4
\end{aligned} \tag{C.60}$$

on the six-planes with the corresponding perturbative gauge group.

The fields on the fixed seven-planes charged under the non-perturbative SU(2) contribute to the anomaly by a hypermultiplet on the intersection six-planes with the perturbative $[E_7 \times SU(2)]$ gauge group and by a vector multiplet on each intersection six-plane

[25] [90] have a different notation and express all traces in terms of the fundamental representation.

with a perturbative $SO(16)$. As the fields are localized on the seven-planes the contribution on each of the two six-planes is 1/2 of that of the six-dimensional standard result.

$$\mathcal{A}^{7-\text{plane}}_{[E_7\times SU(2)]} = \frac{1}{160}trR^4 + \frac{1}{128}(trR^2)^2 - \frac{1}{4}trR^2trF^2_{SU(2)} + (trF^2_{SU(2)})^2, \quad \text{(C.61)}$$

$$\mathcal{A}^{7-\text{plane}}_{SO(16)} = -\frac{1}{160}trR^4 - \frac{1}{128}(trR^2)^2 + \frac{1}{4}trR^2trF^2_{SU(2)} - (trF^2_{SU(2)})^2. \quad \text{(C.62)}$$

Another contribution to the anomaly comes from the twisted states living on the six-planes. Only the six-planes with the gauge group $SO(16)$ have twisted states. There is no contribution on the other intersection six-planes.

$$\begin{aligned}\mathcal{A}^{6-\text{plane}}_{[SO(16)\times SU(2)]} &= \frac{1}{15}trR^4 + \frac{1}{12}(trR^2)^2 - trR^2(trF^2_{SU(2)} - \frac{1}{4}trF^2_{SO(16)}) \\ &+ trF^4_{SO(16)} + (trF^2_{SU(2)})^2 + \frac{3}{2}trF^2_{SU(2)}trF^2_{SO(16)}, \quad \text{(C.63)}\end{aligned}$$

where the $SU(2)$ gauge group is the non-perturbative gauge group on the seven-plane intersecting the six-plane.

All contributions so far are one-loop anomalies. Just as on the ten-plane in the compactification of M-theory on $R^{10}\times I$ there are additional contributions via inflow of the Chern-Simons term $\int C\wedge G\wedge G$ and the higher order term $\int G\wedge X_7$ on the six-planes. These contributions are referred to as inflow anomalies in [89] and depend on the perturbative gauge groups on the ten-planes.

$$\mathcal{A}^{\text{inflow}}_{[E_7\times SU(2)]} = -g_1(\frac{1}{8}trR^4 - \frac{1}{32}(trR^2)^2) + \frac{3g_1}{4}(trF^2_{E_7} + trF^2_{SU(2)} - \frac{1}{2}trR^2)^2 \quad \text{(C.64)}$$

$$\mathcal{A}^{\text{inflow}}_{SO(16)} = -g_2(\frac{1}{8}trR^4 - \frac{1}{128}(trR^2)^2) + \frac{3g_2}{4}(trF^2_{SO(16)} - \frac{1}{2}trR^2)^2, \quad \text{(C.65)}$$

where the numbers g_1 and g_2 are the magnetic charges on the six-planes.

The last contributions are called intersection anomalies and have their origin in a coupling of the seven-plane fields to the three-form C. The coupling has the form of a Chern Simons term $\int C\wedge I_4$ with $I_4 = 3/2(\eta trR^2 - \rho trF^2_{SU(2)})$. The label $SU(2)$ refers the seven-plane gauge group and ρ and η are free parameters so far and will be determined by the vanishing of the resulting anomaly. The contribution of the intersection anomaly is

$$\mathcal{A}^{\text{intersection}}_{[E_7\times SU(2)]} = (trF^2_{E_7} + trF^2_{SU(2)} - \frac{1}{2}trR^2)I_4, \quad \text{(C.66)}$$

$$\mathcal{A}^{\text{intersection}}_{SO(16)} = (trF^2_{SO(16)} - \frac{1}{2}trR^2)I_4 \quad \text{(C.67)}$$

Adding up all contributions on the planes with perturbative gauge $[E_7\times SU(2)]$ leads to

$$\begin{aligned}\mathcal{A}^{\text{resulting}}_{[E_7\times SU(2)]} &= (-\frac{1}{32} - \frac{g_1}{8})trR^4 + (\frac{1}{128} + \frac{g_1}{32} + \frac{3g_1}{16} - \frac{3\eta}{4})(trR^2)^2 \\ &+ (\frac{3}{32} + \frac{3g_1}{4} + \frac{3\eta}{2})trR^2trF^2_{E_7} + (-\frac{21}{32} + \frac{3g_1}{4} + \frac{3\rho}{4} + \frac{3\eta}{2})trR^2trF^2_{SU(2)}\end{aligned}$$

$$
\begin{aligned}
&+ \quad (-\frac{3}{16}-\frac{3g_1}{4})(trF^2_{E_7})^2+(\frac{21}{16}-\frac{3g_1}{4}-\frac{3\rho}{2})(trF^2_{SU(2)})^2 \\
&+ \quad (\frac{9}{8}-\frac{6g_1}{4}-\frac{3\rho}{2})trF^2_{E_7}trF^2_{SU(2)}. \qquad \text{(C.68)}
\end{aligned}
$$

The anomalies on the six-planes which are the intersection of seven-planes and the $SO(16)$ ten-planes are

$$
\begin{aligned}
\mathcal{A}^{\text{resulting}}_{SO(16)} &= (\frac{1}{32}-\frac{g_2}{8})trR^4+(\frac{33}{384}+\frac{g_2}{32}-\frac{3g_2}{16}-\frac{3\eta}{4})(trR^2)^2 \\
&+ \quad (-\frac{9}{32}+\frac{3g_2}{4}-\frac{3\eta}{2})trR^2trF^2_{SO(16)}+(-\frac{3}{4}+\frac{3\rho}{4})trR^2trF^2_{SU(2)} \\
&+ \quad (\frac{3}{16}-\frac{3g_2}{4})(trF^2_{SO(16)})^2+(-1+1)trF^4_{SO(16)}+(-1+1)(trF^2_{SU(2)})^2 \\
&+ \quad (\frac{3}{2}-\frac{3\rho}{2})trF^2_{SO(16)}trF^2_{SU(2)}. \qquad \text{(C.69)}
\end{aligned}
$$

Vanishing of both anomalies fixes

$$
g_1=-1/4, \quad g_2=1/4, \quad \eta=1/16 \;\; \text{and} \;\; \rho=1. \qquad \text{(C.70)}
$$

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Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt, und die den benutzten Werken wörtlich oder inhaltlich entnommenen Stellen als solche kenntlich gemacht habe.

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